Lecture Notes for Math 210 – Monday, 26 Nov. 2007

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Chapter 6: Martingales

The Black-Scholes Formula for the Lattice Model

In the last lecture we proved the Black-Scholes formula:

$$\lim_{N\to\infty} C_0^{(N)}(0) \;=\; \mathbf{E}\left[\max(0, e^{-(1/2)\sigma^2 T + \sigma\sqrt{T}\,\mathsf{Z}}\, S_0 - \tilde{K})\right],$$

where Z is the standard normal distribution.

Therefore,

$$C_0 \;=\; \int_{-\infty}^{\infty} \max\left(0\,,\; e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\,z} S_0 - \tilde{K}\right) \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz\,.$$

Part of your homework is to prove that this gives

$$C_0 \;=\; S_0 \cdot \Phi(d_1) - \tilde{K} \cdot \Phi(d_2) \qquad \text{where,}$$

$$d_1 \;=\; \frac{\ln(S_0/\tilde{K}) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad \text{and}$$

$$d_2 \;=\; \frac{\ln(S_0/\tilde{K}) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\,.$$

Since this is homework, we will not derive this explicit formula.

But, let us instead consider another option, called a "cash or nothing" call option.

In this option, we have a payoff at expiration which is

$$D_T = \begin{cases} A & \text{if } S_T > K, \\ A/2 & \text{if } S_T = K, \\ 0 & \text{if } S_T < K, \end{cases}$$

where K is a strike price and A is an amount of money, to be paid at time T.

The case of equality, $S_T = K$ is not really important because, assuming the stock price is a continuous random variable, the probability of exact equality is 0.

This option is useful for us in class, because it gives an easy way to infer the volatility, as well will see.

Following the same pattern as before, we would consider an N-step binomial tree model.

This results in a martingale $(\tilde{\mathsf{S}}_0, \tilde{\mathsf{S}}_{\Delta t}, \dots, \tilde{\mathsf{S}}_{N\Delta t})$.

We define

$$f(x) = \begin{cases} A & \text{if } x > K, \\ A/2 & \text{if } x = K, \\ 0 & \text{if } x < K. \end{cases}$$

We define $(\mathsf{D}_0, \mathsf{D}_{\Delta t}, \dots, \mathsf{D}_{N\Delta t})$ to be a stochastic process where D_{t_n} is a function of S_{t_n}.

Namely,

$$\mathsf{D}_{t_n} = e^{r(t_n - T)}\, \mathbf{E}[f(\mathsf{S}_{N\Delta t}) \,|\, \mathsf{S}_{t_n}]\,.$$

By Δ-hedging, it is easy to see that the correct price for the cash-or-nothing bond is $D_0^{(N)}(0) =$

$\mathbf{E}[\mathsf{D}_0]$.

We put the superscript N to remind ourselves that this is the price of the "cash-or-nothing" call option in the N-step Binomial tree model.

But, defining $\tilde{\mathsf{D}}_{t_n} = e^{-rt_n}\mathsf{D}_{t_n}$, we have

$$\tilde{\mathsf{D}}_{t_n} \;=\; \mathbf{E}[e^{-rT}\, f(\mathsf{S}_{N\Delta t}) \,|\, \mathsf{S}_{t_n}]\,.$$

Then this is, by definition, a "Doob martingale".

Note that

$$\mathbf{E}[\tilde{\mathsf{D}}_{t_n+\Delta t} \,|\, \mathsf{S}_{t_n}] \;=\; \mathbf{E}[\mathbf{E}[e^{-rT} f(\mathsf{S}_{N\Delta t}) \,|\, \mathsf{S}_{t_n+\Delta t}] \,|\, \mathsf{S}_{t_n}]\,.$$

But, since $(\mathsf{S}_0, \mathsf{S}_{\Delta t}, \ldots, \mathsf{S}_{N\Delta t})$ is a Markov process, we have

$$\mathbf{E}[e^{-rT} f(\mathsf{S}_{N\Delta t}) \,|\, \mathsf{S}_{t_n+\Delta t}] \;=\; \mathbf{E}[e^{-rT} f(\mathsf{S}_{N\Delta t}) \,|\, \mathsf{S}_{t_n+\Delta t}, \mathsf{S}_{t_n}]\,,$$

since knowing S_{t_n} does not give any new information over knowing $\mathsf{S}_{t_n+\Delta t}$ for predicting the value of $f(\mathsf{S}_{N\Delta t})$ for a Markov process.

Then by the tower rule (law of the iterated conditional expectation again), we have

$$\begin{aligned}
\mathbf{E}[\tilde{\mathsf{D}}_{t_n+\Delta t} \,|\, \mathsf{S}_{t_n}] \;&=\; \mathbf{E}[\mathbf{E}[e^{-rT} f(\mathsf{S}_{N\Delta t}) \,|\, \mathsf{S}_{t_n+\Delta t}] \,|\, \mathsf{S}_{t_n}] \\
&=\; \mathbf{E}[\mathbf{E}[e^{-rT} f(\mathsf{S}_{N\Delta t}) \,|\, \mathsf{S}_{t_n+\Delta t}, \mathsf{S}_{t_n}] \,|\, \mathsf{S}_{t_n}] \\
&=\; \mathbf{E}[e^{-rT} f(\mathsf{S}_{N\Delta t}) \,|\, \mathsf{S}_{t_n}] \\
&=\; \tilde{\mathsf{D}}_{t_n}\,.
\end{aligned}$$

The important point is that, all we have done is manipulate symbols.

The real reason that $(\tilde{\mathsf{D}}_0, \tilde{\mathsf{D}}_{\Delta t}, \ldots, \tilde{\mathsf{D}}_{N\Delta t})$ is a martingale is that we defined it that way.

So, finally, we have

$$D_0^{(N)}(0) \;=\; \mathbf{E}[\tilde{\mathsf{D}}_0] \;=\; \mathbf{E}[\tilde{\mathsf{D}}_{N\Delta t}] \;=\; \mathbf{E}[e^{-rT}\, f(\mathsf{S}_{N\Delta t})]\,.$$

Now we want to take the $N \to \infty$ limit.

By the same type of calculations that we did before, we find that a good model for $\tilde{\mathsf{S}}_{N\Delta t}$ is $e^{-\sigma^2 T/2} e^{\sigma\sqrt{T}\mathsf{Z}} S_0$, where Z is a standard normal random variable.

This is the "limit in distribution" that we would get by taking $N \to \infty$.

Therefore, we have that $\mathsf{S}_{N\Delta t}$ converges to $e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\,\mathsf{Z}} S_0$ in the $N \to \infty$ limit.

So

$$D_0 \;:=\; \lim_{N\to\infty} D_0^{(N)}(0) \;=\; \mathbf{E}\left[e^{-rT} f\left(e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\,\mathsf{Z}} S_0\right)\right]\,.$$

Now let us calculate what we get.

Note that $f(x)$ is zero unless $x \geq K$ and it is essentially equal to A if x is $\geq K$.

Therefore,

$$\mathbf{E}\left[e^{-rT} f\left(e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\,\mathsf{Z}} S_0\right)\right] \;=\; e^{-rT} A\,\mathbf{P}\{e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\,\mathsf{Z}} S_0 \geq K\}\,.$$

But

$$\begin{aligned}
\left\{e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\,\mathsf{Z}} S_0 \geq K\right\} \;&\Leftrightarrow\; \left\{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,\mathsf{Z} \geq \ln(K/S_0)\right\} \\
&\Leftrightarrow\; \left\{\sigma\sqrt{T}\,\mathsf{Z} \geq \ln(K/S_0) - (r-\frac{1}{2}\sigma^2)T\right\} \\
&\Leftrightarrow\; \left\{\mathsf{Z} \geq \frac{\ln(K/S_0) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right\}\,.
\end{aligned}$$

So

$$
\begin{aligned}
D_0 &= Ae^{-rT}\,\mathbf{P}\left\{\mathsf{Z} \ge \frac{\ln(K/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right\} \\
&= Ae^{-rT}\left[1 - \Phi\left(\frac{\ln(K/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right],
\end{aligned}
$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz$ is the cumulative distribution function for a standard normal random variable.

But an important property of $\Phi(x)$ is

$$
\Phi(-x) = 1 - \Phi(x)\,.
$$

So, a simpler formula is

$$
\begin{aligned}
D_0 &= Ae^{-rT}\,\Phi\left(-\frac{\ln(K/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&= Ae^{-rT}\,\Phi\left(\frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&= Ae^{-rT}\,\Phi(d_2)\,,
\end{aligned}
$$

where

$$
d_2 = \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\,.
$$

In particular, suppose that $K = S_0 e^{rT}$. Then in this special situation, we have just that

$$
D_0^* = Ae^{-rT}\,\Phi(-\sigma\sqrt{T}/2)\,.
$$

Therefore, if we know what value D_0^* is trading at, for example by offering such a cash-or-nothing call option for sale and seeing how much it trades for, then we can easily solve for σ by solving the equation above.

This requires us to consult a table for $\Phi(x)$.