Lecture Notes for Math 210 – Wednesday, 21 Nov. 2007

Shannon Starr

21 November 2007

Chapter 6: Martingales

The Black-Scholes Formula for the Lattice Model

Let us take a closer look at the derivation of the Black-Scholes model. We start with the call-option price in the Binomial tree model:

$$C_0(0) \;=\; \mathbf{E}\left[\max\left(0, e^{(a-r)\mathsf{X}_N\,\Delta t+(b-r)(N-\mathsf{X}_N)\,\Delta t}\,S_0(0)-\tilde{K}\right)\right]$$

where X_N is Binomial, with parameters N and $\tilde{P}_+$.

The de Moivre, Laplace Limit Theorem: Fix $0 < p < 1$, and for each n, let X_n be a binomial random variable with parameters n and p. Then, for any numbers x_1 and x_2 such that $-\infty \leq x_1 < x_2 < \infty$,

$$\lim_{n\to\infty} \mathbf{P}\left\{ x_1 \leq \frac{\mathsf{X}_n - np}{\sqrt{np(1-p)}} \leq x_2 \right\} = \Phi(x_2) - \Phi(x_1)\,,$$

where $\Phi(z)$ is the cumulative distribution function for a standard, normal random variable:

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz\,.$$

We chose

$$\begin{aligned} a &= r + \frac{1}{\Delta t} \ln\left[1 + \sigma(\Delta t)^{1/2}\right]\,, \\ b &= r + \frac{1}{\Delta t} \ln\left[1 - \sigma(\Delta t)^{1/2}\right]\,. \end{aligned}$$

for some $\sigma > 0$.

This means

$$\begin{aligned} (a-r)\,\Delta t &= \ln\left[1 + \sigma(\Delta t)^{1/2}\right]\,, \\ (b-r)\,\Delta t &= \ln\left[1 - \sigma(\Delta t)^{1/2}\right]\,. \end{aligned}$$

So,

$$\begin{aligned} e^{(a-r)\,\Delta t} &= 1 + \sigma(\Delta t)^{1/2}\,, \\ e^{(b-r)\,\Delta t} &= 1 - \sigma(\Delta t)^{1/2}\,. \end{aligned}$$

From this it was easy to deduce that $\tilde{P}_+ = 1/2$.

That is important because $\tilde{P}_+$ needs to be constant as $N \to \infty$ and $\Delta t \to 0$ to use the de Moivre, Laplace limit theorem.

L'Hospital's Rule and the $N \to \infty$ limit

We return to the formula for the call option again:

$$C_0^{(N)}(0) \;=\; \mathbf{E}\left[\max\left(0, e^{(a-r)\mathsf{X}_N \Delta t + (b-r)(N-\mathsf{X}_N)\Delta t}\, S_0(0) - \tilde{K}\right)\right]$$

where X_N is Binomial with parameter N and $\tilde{P}_+$, which equals 1/2.

We have written $C_0^{(N)}(0)$ to remind ourselves that this is the result of the N-step binomial tree model.

We can also write this as

$$C_0^{(N)}(0) \;=\; \mathbf{E}[\tilde{\mathsf{C}}_N]\,,$$

where

$$\tilde{\mathsf{C}}_N \;=\; \max\left(0, \tilde{\mathsf{S}}_N - \tilde{K}\right)\,,$$

and

$$\tilde{\mathsf{S}}_N \;=\; e^{(a-r)(\Delta t)\mathsf{X}_N + (b-r)(\Delta t)(N-\mathsf{X}_N)}\,.$$

We have decomposed the problem this way in order to simplify notation, not for any deep reason.

Plugging-in for $(a-r)(\Delta t)$ and $(b-r)(\Delta t)$, we get

$$\tilde{\mathsf{S}}_N \;=\; \exp\left[\ln\left[1+\sigma(\Delta t)^{1/2}\right]\mathsf{X}_N + \ln\left[1-\sigma(\Delta t)^{1/2}\right](N-\mathsf{X}_N)\right]\; S_0(0)\,,$$

where $\exp(x) = e^x$.

Let us define a random variable

$$\mathsf{L}_N \;=\; \ln\left[1+\sigma(\Delta t)^{1/2}\right]\mathsf{X}_N + \ln\left[1-\sigma(\Delta t)^{1/2}\right](N-\mathsf{X}_N)\,,$$

so that we just have

$$\tilde{\mathsf{S}}_N \;=\; e^{\mathsf{L}_N}\, S_0(0)\,.$$

Finally, let us define the "standardized" version of X_N.

This is the random variables Z_N that we get if we first subtract the mean of X_N and then divide by the standard deviation (which is the square-root of the variance).

So

$$\mathsf{Z}_N \;=\; \frac{\mathsf{X}_N - \mathbf{E}[\mathsf{X}_N]}{\sqrt{\mathrm{Var}(\mathsf{X}_N)}} \;=\; \frac{\mathsf{X}_N-(N/2)}{\sqrt{N(1/2)(1-1/2)}} \;=\; \frac{\mathsf{X}_N-(N/2)}{\sqrt{N/4}}\,.$$

So

$$\mathsf{X}_N \;=\; \frac{N+N^{1/2}\,\mathsf{Z}_N}{2}\,.$$

Note that it is Z_N which comes into the de Moivre, Laplace limit theorem.

So,

$$\mathsf{L}_N \;=\; A_N + B_N\mathsf{Z}_N\,,$$

where

$$A_N \;=\; \frac{N}{2}\left(\ln\left[1+\sigma(\Delta t)^{1/2}\right]+\ln\left[1-\sigma(\Delta t)^{1/2}\right]\right)\,,$$

and

$$B_N \;=\; \frac{N^{1/2}}{2}\left(\ln\left[1+\sigma(\Delta t)^{1/2}\right]-\ln\left[1-\sigma(\Delta t)^{1/2}\right]\right)\,.$$

The de Moivre, Laplace limit theorem says that, for every $-\infty \le x \le y \le \infty$,

$$\lim_{N\to\infty} \mathbf{P}\{x < \mathsf{Z}_N \le y\} = \int_x^y \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz\,.$$

Also

$$f_{\mathsf{Z}}(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}} \qquad \text{for } -\infty < z < \infty,$$

is the probability density function for a standard, normal random variable Z.

Therefore, we say that the sequence of random variables $(\mathsf{Z}_N)_{N=1}^{\infty}$ converges to the standard, normal random variable "in distribution".

This is a weak mode of convergence, but one which is satisfactory for our purposes.

If we can also prove that the sequences of numbers $(A_N)_{N=1}^{\infty}$ and $(B_N)_{N=1}^{\infty}$ converge to limits, A and B, then we will obtain the result that

$$\lim_{N\to\infty} \mathsf{L}_N = A + B\mathsf{Z}\,,$$

"in distribution".

So the next thing we check is the limit of the A_N and B_N.

In order to calculate the limits

$$\begin{aligned} A &= \lim_{N\to\infty} \frac{N}{2}\left(\ln\left[1+\sigma(\Delta t)^{1/2}\right] + \ln\left[1-\sigma(\Delta t)^{1/2}\right]\right), \quad \text{and} \\ B &= \lim_{N\to\infty} \frac{N^{1/2}}{2}\left(\ln\left[1+\sigma(\Delta t)^{1/2}\right] - \ln\left[1-\sigma(\Delta t)^{1/2}\right]\right), \end{aligned}$$

we will use L'Hospital's rule.

But we will treat the two limits separately.

For A, let us first use the property of the logarithm that $\ln(x) + \ln(y) = \ln(xy)$ to write

$$
\begin{aligned}
A &= \lim_{N\to\infty} \frac{N}{2} \ln\left[\left(1 + \sigma(\Delta t)^{1/2}\right)\left(1 - \sigma(\Delta t)^{1/2}\right)\right] \\
&= \lim_{N\to\infty} \frac{N}{2} \ln[1 - \sigma^2 \Delta t]\,.
\end{aligned}
$$

Now remember that $\Delta t = T/N$ to get

$$
A = \lim_{N\to\infty} \frac{N}{2} \ln\left[1 - \frac{\sigma^2 T}{N}\right]\,.
$$

Finally, let us write $x = 1/N$ to obtain

$$
A = \lim_{x\to 0^+} \frac{1}{2x} \ln[1 - \sigma^2 T x] = \lim_{x\to 0+} \frac{\ln(1 - \sigma^2 T x)}{2x}\,.
$$

Now, $\ln(1) = 0$ so both the numerator and denominator converge to 0.

So we are justified in using L'Hospital's rule

$$
\begin{aligned}
A &= \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln(1 - \sigma^2 T x)}{\frac{d}{dx}(2x)} \\
&= \lim_{x\to 0^+} \frac{(1 - \sigma^2 T x)^{-1}(-\sigma^2 T)}{2} \\
&= \frac{(1 - \sigma^2 T \cdot 0)^{-1}(-\sigma^2 T)}{2} \\
&= \frac{-\sigma^2 T}{2}\,.
\end{aligned}
$$

For B we leave the logarithms as they are, but we set $x = N^{-1/2}$, which also goes to 0 from the right, as $N \to \infty$.

Note that $(\Delta t)^{1/2} = (T/N)^{1/2} = T^{1/2} x$.

So,

$$\begin{aligned}
B &= \lim_{N\to\infty} \frac{N^{1/2}}{2} \left(\ln\left[1+\sigma(\Delta t)^{1/2}\right] - \ln\left[1-\sigma(\Delta t)^{1/2}\right]\right) \\
&= \lim_{x\to 0^+} \frac{1}{2x} \left(\ln\left[1+\sigma T^{1/2}x\right] - \ln\left[1-\sigma T^{1/2}x\right]\right) \\
&= \lim_{x\to 0^+} \frac{\ln\left[1+\sigma T^{1/2}x\right] - \ln\left[1-\sigma T^{1/2}x\right]}{2x} \,.
\end{aligned}$$

Now, again, both the numerator and denominator vanish if we plug-in $x = 0$.

So we are justified in using L'Hospital's rule:

$$\begin{aligned}
B &= \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln\left[1+\sigma T^{1/2}x\right] - \frac{d}{dx}\ln\left[1-\sigma T^{1/2}x\right]}{\frac{d}{dx}(2x)} \\
&= \lim_{x\to 0^+} \frac{(1+\sigma T^{1/2}x)^{-1}(\sigma T^{1/2}) - (1-\sigma T^{1/2}x)^{-1}(-\sigma T^{1/2})}{2} \\
&= \frac{(1+\sigma T^{1/2}\cdot 0)^{-1}(\sigma T^{1/2}) + (1-\sigma T^{1/2}\cdot 0)^{-1}(\sigma T^{1/2})}{2} \\
&= \frac{2\sigma T^{1/2}}{2} \\
&= \sigma T^{1/2} \,.
\end{aligned}$$

Putting these together, we get that

$$\mathsf{L}_N \to -\frac{\sigma^2 T}{2} + \sigma T^{1/2}\,\mathsf{Z}\,,$$

"in distribution", as $N \to \infty$.

So that means that

$$\tilde{\mathsf{S}}_N \to e^{-(1/2)\sigma^2 T + \sigma\sqrt{T}\,\mathsf{Z}}\, S_0\,,$$

"in distribution", where S_0 is the spot price for the stock at time 0.

Hence,

$$\tilde{\mathsf{C}}_N \to \max(0, e^{-(1/2)\sigma^2 T + \sigma\sqrt{T}\,\mathsf{Z}}\, S_0 - \tilde{K})\,,$$

"in distribution", where $\tilde{K} = Ke^{-rT}$ is discounted strike price.

Therefore, under the assumption that convergence "in distribution" implies convergence of the expectation, we deduce that

$$\lim_{N\to\infty} C_0^{(N)}(0) \;=\; \lim_{N\to\infty} \mathbf{E}[\tilde{\mathsf{C}}_N] \;=\; \mathbf{E}\left[\max(0, e^{-(1/2)\sigma^2 T + \sigma\sqrt{T}\,\mathsf{Z}}\, S_0 - \tilde{K})\right]\,,$$

where, as before, Z is a standard, normal random variable.

Actually, convergence in distribution is *not* strong enough to imply convergence of the expectation for unbounded functions such as this.

But that is really a mathematical technicality.

After all, we did not really prove the de Moivre, Laplace limit theorem (although you will prove it in your homework).

It is true that for this problem we do have convergence of the expectation.

So let us simply take that for granted as the alternative version of the de Moivre, Laplace limit theorem.

(In fact, there is a mathematical trick in asymptotic analysis know as Laplace's method, and that does allow us to determine convergence of the expectations.

In fact, the simplest method of proving the de Moivre, Laplace limit theorem is to combine Laplace's method, with the Laplace transform to deduce the convergence in-distribution version from convergence of the moment generating functions.

It is easy to see why this is called the de Moivre, Laplace limit theorem, even though de Moivre had discovered a special case (with a small input from Stirling) long before Laplace entered the picture.

It is because of the fundamental contributions Laplace made to simplifying and generalizing the theorem.)