Lecture Notes for Math 210 – Monday, 19 Nov. 2007

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Chapter 6: Martingales

The Black-Scholes Formula for the Lattice Model

Last time we looked at the N-step lattice model.

We constructed a stochastic process $(\tilde{\mathsf{C}}_0, \tilde{\mathsf{C}}_{\Delta t}, \dots, \tilde{\mathsf{C}}_{N\Delta t})$, and showed that, with respect to the risk-neutral probabilities, it is a martingale.

We then used this fact to obtain a formula for the call option price at time 0:

$$C_0(0) \,=\, \mathbf{E}\left[\max\left(0, e^{(a-r)\mathsf{X}_N\,\Delta t + (b-r)(N-\mathsf{X}_N)\,\Delta t}\, S_0(0) - \tilde{K}\right)\right]$$

where X_N is Binomial, with parameters N and $\tilde{P}_+$:

$$\mathbf{P}\{\mathsf{X}_N = k\} \,=\, \binom{N}{k} (\tilde{P}_+)^k\, (1-\tilde{P}_+)^{N-k}\,.$$

This gives a concise probabilistic formula for the price of a call option within the binomial-tree/lattice model.

Now we want to take a continuum limit, wherein we have $N \to \infty$ and $\Delta t \to 0$, related in such a way that $N\Delta t$ is fixed, and equals the expiration date T.

The de Moivre, Laplace Limit Theorem: Fix $0 < p < 1$, and for each n, let X_n be a binomial random variable with parameters n and p. Then, for any numbers x_1 and x_2 such that $-\infty \leq x_1 < x_2 < \infty$,

$$\lim_{n\to\infty} \mathbf{P}\left\{ x_1 \leq \frac{\mathsf{X}_n - np}{\sqrt{np(1-p)}} \leq x_2 \right\} = \Phi(x_2) - \Phi(x_1)\,,$$

where $\Phi(z)$ is the cumulative distribution function for a standard, normal random variable:

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz\,.$$

It turns out that, to take the limits $N \to \infty$, $\Delta t \to 0$ (so that $N\Delta t = T$ is fixed) and use the de Moivre, Laplace limit theorem, we will need to make the constants a and b actually become variables, that depend on Δt.

The precise choice we will make is:

$$a = r + \frac{1}{\Delta t} \ln\left[1 + \sigma(\Delta t)^{1/2}\right]\,,$$
$$b = r + \frac{1}{\Delta t} \ln\left[1 - \sigma(\Delta t)^{1/2}\right]\,.$$

Here $\sigma > 0$ is a constant that we will interpret later.

Note that with these choices, we have

$$
\begin{aligned}
(a-r)\,\Delta t \;&=\; \ln\left[1+\sigma(\Delta t)^{1/2}\right]\,, \\
(b-r)\,\Delta t \;&=\; \ln\left[1-\sigma(\Delta t)^{1/2}\right]\,.
\end{aligned}
$$

So,

$$
\begin{aligned}
e^{(a-r)\,\Delta t} \;&=\; 1+\sigma(\Delta t)^{1/2}\,, \\
e^{(b-r)\,\Delta t} \;&=\; 1-\sigma(\Delta t)^{1/2}\,.
\end{aligned}
$$

The reasons we chose these precise forms for a and b are as follows:

1) We would like $\tilde{P}_+$ to be constant in N.

With the precise choice we have made,

$$
\begin{aligned}
\tilde{P}^+ \;&=\; \frac{e^{r\Delta t}-e^{b\Delta t}}{e^{a\Delta t}-e^{b\Delta t}} \\
&=\; \frac{1-e^{(b-r)\Delta t}}{e^{(a-r)\Delta t}-e^{(b-r)\Delta t}} \\
&=\; \frac{1-\left[1-\sigma(\Delta t)^{1/2}\right]}{\left[1+\sigma(\Delta t)^{1/2}\right]-\left[1-\sigma(\Delta t)^{1/2}\right]} \\
&=\; \frac{\sigma(\Delta t)^{1/2}}{2\sigma(\Delta t)^{1/2}} \\
&=\; \frac{1}{2}\,.
\end{aligned}
$$

Therefore $\tilde{P}_+$ is constant in N.

2) We want the variance to be well-controlled.

We define the conditional variance as follows:

$$\mathrm{Var}(\tilde{\mathsf{S}}_{t_n+\Delta t} \,|\, \mathsf{S}_{t_n}) \;=\; \mathbf{E}[(\tilde{\mathsf{S}}_{t_n+\Delta t} - \tilde{\mathsf{S}}_{t_n})^2 \,|\, \mathsf{S}_{t_n}]\,.$$

This definition conforms to the usual formula:

$$\mathrm{Var}(\mathsf{X} \,|\, \mathcal{F}) = \mathbf{E}[(\mathsf{X} - \mathbf{E}[\mathsf{X}|\mathcal{F}])^2 \,|\, \mathcal{F}]\,,$$

because $\mathbf{E}[\tilde{\mathsf{S}}_{t_n+\Delta t} \,|\, \mathsf{S}_{t_n}] = \tilde{\mathsf{S}}_{t_n}$, according to the martingale formula for $(\tilde{\mathsf{S}}_0, \tilde{\mathsf{S}}_{\Delta t}, \ldots, \tilde{\mathsf{S}}_{N\Delta t})$.

According to our choices for a and b, we have

$$\begin{aligned} \tilde{S}_{t_n+\Delta t}(m+1) \;&=\; e^{(a-r)\Delta t}\, \tilde{S}_{t_n}(m) \\ &=\; \left[1 + \sigma(\Delta t)^{1/2}\right]\, \tilde{S}_{t_n}(m)\,, \qquad \text{and} \\ \tilde{S}_{t_n+\Delta t}(m-1) \;&=\; e^{(b-r)\Delta t}\, \tilde{S}_{t_n}(m) \\ &=\; \left[1 - \sigma(\Delta t)^{1/2}\right]\, \tilde{S}_{t_n}(m)\,. \end{aligned}$$

Thererfore,

$$\begin{aligned} \tilde{S}_{t_n+\Delta t}(m+1) - \tilde{S}_{t_n}(m) \;&=\; \sigma(\Delta t)^{1/2}\, \tilde{S}_{t_n}(m)\,, \qquad \text{and} \\ \tilde{S}_{t_n+\Delta t}(m-1) - \tilde{S}_{t_n}(m) \;&=\; -\sigma(\Delta t)^{1/2}\, \tilde{S}_{t_n}(m)\,. \end{aligned}$$

So, no matter whether m steps up by 1 or down by 1, either way we have

$$[\tilde{S}_{t_n+\Delta t}(m \pm 1) - \tilde{S}_{t_n}(m)]^2 \;=\; \sigma^2\, [\tilde{S}_{t_n}(m)]^2\, \Delta t\,.$$

Therefore,

$$\mathrm{Var}(\tilde{\mathsf{S}}_{t_n+\Delta t}\,|\,\mathsf{S}_{t_n}) \;=\; \sigma^2\,[\tilde{\mathsf{S}}_{t_n}]^2\,\Delta t\,.$$

This formula is good because it tells us that the variance is growing proportionally to t.

We think of the conditional variance as the step size for the variance, which we want to be proportional to Δt.

We will return to this point in a later lecture.

Taking the $N \to \infty$ limit

We now want to rewrite the formula for $\tilde{S}_{N\Delta t}$ in terms of k, using our formulas for a and b.

In order to do this, we will want to use the Taylor series for the natural logarithm function.

Recall the Geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for } |x| < 1.$$

Integrating this, we obtain

$$\begin{aligned}
\ln(1+x) &= \int_0^{-x} \left(-\frac{1}{1-y}\right) dy \\
&= \int_0^{-x} \left(-\sum_{n=0}^{\infty} y^n\right) dy \\
&= -\sum_{n=0}^{\infty} \int_0^{-x} y^n\,dy \\
&= -\sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}\,.
\end{aligned}$$

Careful students will be somewhat worried about the fact that we interchanged the sum and the integral above.

This is perfectly justified, however, if $|x| < 1$ because then we are inside the radius of convergence for the Geometric power series, and the series converges absolutely and uniformly over the interval we are integrating.

So we obtain the power series:

$$\ln(1+x) \;=\; \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} \qquad \text{for } |x| < 1.$$

Using this formula, we have

$$\begin{aligned}
(a-r)\,\Delta t \;&=\; \ln\left[1+\sigma(\Delta t)^{1/2}\right] \\
&=\; \sigma(\Delta t)^{1/2} - \frac{\sigma^2}{2}\,\Delta t + O((\Delta t)^{3/2})\,, \qquad \text{and} \\
(b-r)\,\Delta t \;&=\; \ln\left[1-\sigma(\Delta t)^{1/2}\right] \\
&=\; -\sigma(\Delta t)^{1/2} - \frac{\sigma^2}{2}\,\Delta t + O((\Delta t)^{3/2})\,,
\end{aligned}$$

where we use the notation $O((\Delta t)^{3/2})$ to indicate that the remainder terms can all be bounded by a fixed constant C time $(\Delta t)^{3/2}$.

For example, both terms can be bounded by $12\sigma^3(\Delta t)^{3/2}[1-\sigma(\Delta t)^{1/2})]^{-3}$, which is small as soon as N is large enough to have $\sigma^2\Delta t < 1$. (We will not bother to prove this bound, though. You can prove it using integration-by-parts 3 times.)

But, by this formula, we obtain:

$$
\begin{aligned}
\tilde{\mathsf{S}}_{N\Delta t} &= e^{(a-r)\mathsf{X}_N\,\Delta t+(b-r)(N-\mathsf{X}_N)\,\Delta t}\,S_0(0) \\
&= e^{-\frac{1}{2}\sigma^2 N\Delta t}\,e^{2\sigma(\mathsf{X}_N-\frac{N}{2})(\Delta t)^{1/2}+O(N(\Delta t)^{3/2})}\,S_0(0) \\
&= e^{-\frac{1}{2}\sigma^2 T}\,e^{\sigma\left(\frac{\mathsf{X}_N-(N/2)}{\sqrt{N/4}}\right)}\,e^{O(N^{-1/2})}\,S_0(0)\,.
\end{aligned}
$$

Combining this formula and the de Moivre, Laplace limit theorem, we find that

$$
\lim_{\substack{N\to\infty\\ \Delta t\to 0\\ N\Delta t=T}} \mathbf{P}\left\{e^{-\sigma^2T/2}e^{\sigma x_1}S_0(0) \,<\, \tilde{\mathsf{S}}_{N\Delta t} \,\leq\, e^{-\sigma^2T/2}e^{\sigma x_2}\right\} \,=\, \Phi(x_2)-\Phi(x_1)\,.
$$

In order to apply this to the problem of determining $C_0(0)$, we need a different version of the de Moivre, Laplace limit theorem.

The following fact is easily obtained by combining that theorem with Laplace's method in asymptotic analysis:

The de Moivre, Laplace Limit Theorem, version II: Fix $0 < p < 1$, and for each n, let X_n be a binomial random variable with parameters n and p. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, such that

$$
\lim_{x\to\pm\infty} |g(x)|e^{-\epsilon x^2} \,=\, 0\,,
$$

for all $\epsilon > 0$. Then

$$
\lim_{n\to\infty} \mathbf{P}\left\{g\left(\frac{\mathsf{X}_n - np}{\sqrt{np(1-p)}}\right)\right\} \,=\, \int_{-\infty}^{\infty} g(z)\,\frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz\,.
$$

Note that the function $g(z) = \max(0, e^{-\sigma^2T/2}e^{\sigma z}S_0(0) - \tilde{K})$ does satisfy the hypotheses because it is continuous, and it only grows exponentially at ∞, while $e^{-\epsilon z^2}$ is decaying much faster than exponentially.

Therefore, the de Moivre, Laplace limit theorem does apply to give

$$C_0(0) \;=\; \int_{-\infty}^{\infty} \max(0, e^{-\sigma^2 T/2} e^{\sigma z} S_0(0) - \tilde{K})\, \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz\,.$$

This is the celebrated Black-Scholes formula for option pricing.

A remarkable fact is that the integral can be evaluated explicitly in terms of $\Phi(x)$ to give

$$C_0(0) \;=\; S_0(0)\,\Phi(d_1) - \tilde{K}\,\Phi(d_2)\,,$$

where

$$d_1 \;=\; \frac{\ln(S_0(0)/\tilde{K}) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 \;=\; \frac{\ln(S_0(0)/\tilde{K}) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\,.$$

Rewriting in terms of K, instead of $\tilde{K} = e^{-rT}K$, we obtain

$$C_0(0) \;=\; S_0(0)\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2) \qquad \text{where}$$

$$d_1 \;=\; \frac{\ln(S_0(0)/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 \;=\; \frac{\ln(S_0(0)/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\,.$$