Lecture Notes for Math 210 – Wednesday, 14 November 2007

Shannon Starr

14 November 2007

Chapter 6: Martingales

The lattice model (Binomial tree model) as a martingale (continued)

Expectation and Conditional Expectation

Last time we constructed stochastic processes determined by the *risk-neutral* model of the stock and call option price.

These were called $(\mathsf{S}_0, \mathsf{S}_{\Delta t}, \dots, \mathsf{S}_{N\Delta t})$ and $(\mathsf{C}_0, \mathsf{C}_{\Delta t}, \dots, \mathsf{C}_{N\Delta t})$.

We then defined the discounted version of the stochastic processes:

$(\tilde{\mathsf{S}}_0, \tilde{\mathsf{S}}_{\Delta t}, \dots, \tilde{\mathsf{S}}_{N\Delta t})$ and $(\tilde{\mathsf{C}}_0, \tilde{\mathsf{C}}_{\Delta t}, \dots, \tilde{\mathsf{C}}_{N\Delta t})$, where $\tilde{\mathsf{S}}_{t_n} = e^{-rt_n}\mathsf{S}_{t_n}$ and $\tilde{\mathsf{C}}_{t_n} = e^{-rt_n}\mathsf{C}_{t_n}$.

We showed that these were martingales.

This time we begin to show how to use this fact to find a simpler expression for $C_0(0)$.

We have said before that the expectation of the conditional expectation equals the expectation:

$$\mathbf{E}[\mathbf{E}[\mathsf{X} \,|\, \mathcal{F}]] \;=\; \mathbf{E}[\mathsf{X}]\,.$$

Heuristically, this says the following: Suppose somebody is nice enough to tell you that they will give you some inside information indirectly related to X. More specifically, they will answer all the questions contained in the event space $\mathcal{F}$. So you do some work and find out what your best guess for X should be, conditional on knowing the information in $\mathcal{F}$, in anticipation of your friend's information. But then suppose you find out that the person is unreliable, so that you cannot count on his information. So, instead if you want to know something about $\mathcal{F}$ then you have to just guess. Then your end-result guess for X is exactly what you would have got if that unreliable person had never suggested they could give you inside information to begin with.

Let us start with calculating $\mathbf{E}[\tilde{\mathsf{C}}_{\Delta t}]$. We have

$$\mathbf{E}[\tilde{\mathsf{C}}_{\Delta t} \,|\, \mathsf{S}_0] \;=\; \tilde{\mathsf{C}}_0\,.$$

But notice that everybody actually knows $\mathsf{S}_0 = S_0(0)$ with probability 1, because that is the current spot price for the stock.

When you condition $\mathbf{E}[\tilde{\mathsf{C}}_{\Delta t} \,|\, \mathsf{S}_0]$, you might think of what you are conditioning on as some kind of "inside information".

If the inside information is vacuous information that everybody already knows, then your conditional expectation might as well be just the usual expectation.

This also fits with the current formula, because $\tilde{\mathsf{C}}_0$ is also non-random.

Rather, $\mathsf{C}_0 = C_0(0)$, with probability 1.

So, $\tilde{\mathsf{C}}_0 = e^{-r\cdot 0}\mathsf{C}_0 = \mathsf{C}_0 = C_0(0)$, with probability 1 as well.

Therefore, putting these facts together, we obtain.

$$\mathbf{E}[\tilde{\mathsf{C}}_{\Delta t}] \;=\; C_0(0)\,.$$

We have replaced the conditional expectation by the unconditional expectation, because what we were conditioning on was vacuous, and we replaced $\tilde{\mathsf{C}}_0$ by $C_0(0)$, which it is equal to with probability 1.

Next we notice that, by the martingale condition,

$$\mathbf{E}[\tilde{\mathsf{C}}_{2\Delta t}\,|\,\mathsf{S}_{\Delta t}] \;=\; \tilde{\mathsf{C}}_{\Delta t}\,.$$

All of the random variables in this formula are really random.

So there is no reduction to nonrandom variables we can do here.

But, by the "tower rule", we do have

$$\mathbf{E}[\mathbf{E}[\tilde{\mathsf{C}}_{2\Delta t}\,|\,\mathsf{S}_{\Delta t}]] \;=\; \mathbf{E}[\tilde{\mathsf{C}}_{2\Delta t}]\,.$$

Therefore, putting these two together, we get:

$$\begin{aligned} \mathbf{E}[\tilde{\mathsf{C}}_{2\Delta t}] \;&=\; \mathbf{E}[\mathbf{E}[\tilde{\mathsf{C}}_{2\Delta t}\,|\,\mathsf{S}_{\Delta t}]] && \text{by the tower rule}\\ &=\; \mathbf{E}[\tilde{\mathsf{C}}_{\Delta t}] && \text{by the martingale property}\\ &=\; C_0(0) && \text{by the equation we showed, just before.} \end{aligned}$$

Similarly,

$$\mathbf{E}[\tilde{\mathsf{C}}_{3\Delta t}] \;=\; \mathbf{E}[\mathbf{E}[\tilde{\mathsf{C}}_{3\Delta t} \,|\, \mathsf{S}_{2\Delta t}]] \;=\; \mathbf{E}[\tilde{\mathsf{C}}_{2\Delta t}] \;=\; C_0(0)\ .$$

Continuing inductively, we finally find

$$\mathbf{E}[\tilde{\mathsf{C}}_{N\Delta t}] \;=\; C_0(0)\ .$$

We actually want to turn this equation around!

This gives a useful formula for $C_0(0)$:

$$C_0(0) \;=\; \mathbf{E}[\tilde{\mathsf{C}}_{N\Delta t}]\ .$$

Since $T = N\Delta t$ is the expiration date, we know a formula for $\tilde{\mathsf{C}}_{N\Delta t}$:

$$\tilde{\mathsf{C}}_{N\Delta t} \;=\; e^{-rN\Delta t}\ \max(0, \mathsf{S}_{N\Delta t} - K) \;=\; \max(0, \tilde{\mathsf{S}}_{N\Delta t} - \tilde{K})\,,$$

where, remember $\tilde{K} = e^{-rT}K = e^{-rN\Delta t}K$.

So,

$$C_0(0) \;=\; \mathbf{E}[\max(0, \tilde{\mathsf{S}}_{N\Delta t} - \tilde{K})]\ .$$

Recall that, since $\tilde{S}_{t_n+\Delta t}(m+1) = e^{(a-r)\Delta t}\tilde{S}_{t_n}(m)$ and $\tilde{S}_{t_n+\Delta t}(m-1) = e^{(b-r)\Delta t}\tilde{S}_{t_n}(m)$, we have

$$\tilde{S}_{N\Delta t}(m) \;=\; e^{(a-r)k\Delta t + (b-r)(N-k)\Delta t}\, S_0(0)\,,$$

where k is the number of up-steps and $N-k$ is the number of down-steps required to arrive at m at time $N\Delta t$: $k = (N+m)/2$ and $N-k = (N-m)/2$.

In describing $\tilde{\mathsf{S}}_{N\Delta t}$, let $\mathsf{X}_{N\Delta t}$ be the number of times the stock stepped up, so that $N - \mathsf{X}_{N\Delta t}$ is the number of times the stock stepped down.

Then

$$\tilde{\mathsf{S}}_{N\Delta t} \;=\; e^{(a-r)\Delta t \cdot \mathsf{X}_{N\Delta t} + (b-r)\Delta t \cdot (N - \mathsf{X}_{N\Delta t})}\, S_0(0) \;.$$

But $\mathsf{X}_{N\Delta t}$ is a Binomial random variable, with parameters $n = N$ and $p = \tilde{P}_+$, because there are N total time steps (each of size Δt) and the probability of stepping up is $p = \tilde{P}_+$, each time (independently of all other steps).

So

$$\mathbf{P}\{\mathsf{X}_N = k\} \;=\; \binom{N}{k} (\tilde{P}_+)^k\, (\tilde{P}_-)^{N-k} \;.$$

Therefore, we obtain one of the first major results associated to the lattice (Binomial tree) model:

$$C_0(0) \;=\; \sum_{k=0}^{N} \binom{N}{k} (\tilde{P}_+)^k\, (\tilde{P}_-)^{N-k}\, \max\left(0, e^{(a-r)k\Delta t + (b-r)(N-k)\Delta t} - \tilde{K}\right) \;.$$

This is a great formula.

In particular, it would allow for a greatly simplified approach to the binomial tree model, to work out by hand.

If you want to calculate the call option price at time 0, simply first calculate the stock prices according to:

$$\begin{aligned}[\tilde{S}_{N\Delta t}(N), \tilde{S}_{N\Delta t}(N-2), \ldots, \tilde{S}_{N\Delta t}(-N)]^T \\ \;=\; [e^{(a-r)N\Delta t}, e^{(a-r)(N-2)\Delta t + 2(b-r)\Delta t}, \ldots, e^{(b-r)N\Delta t}]^T \;.\end{aligned}$$

Then take the dot-product of the two vectors:

$$[\max(0, \tilde{S}_{N\Delta t}(N) - \tilde{K}), \max(0, \tilde{S}_{N\Delta t}(N-2) - \tilde{K}), \ldots, \max(0, \tilde{S}_{N\Delta t}(-N) - \tilde{K})]\,,$$

and

$$\left[\binom{N}{N}(\tilde{P}_+)^N(\tilde{P}_-)^0, \binom{N}{N-1}(\tilde{P}_+)^{N-1}(\tilde{P}_-)^1, \dots, \binom{N}{0}(\tilde{P}_-)^0(\tilde{P}_-)^N\right].$$

This would be trivial to implement in Matlab, for example.

(Actually knowing how to do numerical implementation of the binomial model could potentially be even more useful to a future financial analyst than fully understanding the SDE approach to the Black-Scholes formula. But it is not the main point of the present course.)

However, the way we will want to use the formula is to go back one step, and rewrite the formula as

$$C_0(0) \;=\; \mathbf{E}\left[\max\left(0, e^{(a-r)\mathsf{X}_N\,\Delta t+(b-r)(N-\mathsf{X}_N)\,\Delta t} - \tilde{K}\right)\right]$$

where X_N is Binomial, with parameters N and $\tilde{P}_+$.

The reason is that, if we can fixe $\tilde{P}_+$ as we take $N \to \infty$ and $\Delta t \to 0$, so that $N\Delta t = T$, then we can use the de Moivre, Laplace limit theorem, which says that the *standardized* version of the Binomial random variable converges to a standard, normal random variable.