Lecture Notes for Math 210 – 10 September 2007

Shannon Starr

9 September 2007

1 Put-call parity

Call option: A contract which gives the holder the *right* (but not the obligation) to buy a stock at time T for a predetermined price K, called the *strike price*.

Put option: A contract which gives the holder the *right* (but not the obligation) to sell a stock at time T for a predetermined price K.

Put-call parity:

Let C_t = the price for a call option, with $T = 1$ year in the future.

Let P_t = the price for a put option with same T and K as the call option.

Then

$$C_t - P_t \;=\; S_t - (1+r)^{-1}K\,,$$

where S_t = spot price for the stock, at present time t

and r = the 1 year borrowing and lending rate.

2 Proof of Put-call parity

Step 1: Payoff at expiration

$$C_T = \max(S_T - K, 0)\,; \quad (1)$$

$$P_T = \max(K - S_T, 0)\,. \quad (2)$$

Proof of (2):

- If $S_T > K$, holder won't exercise.

  (Better to sell stock on open market.)

  Payoff = 0.

- If $S_T < K$, holder will exercise.

  Buy on market for S_T. Sell for K.

  Net payoff = $K - S_T$.

Suppose you are long 1 call option, and short 1 put option.

Your payoff at time T is

$$\begin{aligned} C_T - P_T &= \max(S_T - K, 0) - \max(K - S_T, 0) \\ &= \begin{cases} (S_T - K) - 0 & \text{if } S_T > K, \\ 0 - (K - S_T) & \text{if } K > S_T, \end{cases} \\ &= S_T - K\,. \end{aligned}$$

Step 2: equivalent portfolio

1st portfolio: Long 1 call option and short 1 put option.

Payoff at time $T = S_T - K$.

2nd portfolio: Long 1 forward.

At time T you pay F_t for stock.

But stock has actual value S_T at time T.

(Q1) What is the payoff for the forward at time T?

(Q2) Supposing that F_t is the *fair foward price*, what is the value of the forward at time $t = 0$?

The payoff for the foward at time T is $S_T - F_t$.

The value of the forward at time $t = 0$ is 0 because we used the *fair forward price*.

3rd portfolio: Long 1 call option, short 1 put option, short 1 forward option.

Present value $= C_t - P_t - 0$.

Payoff at time $T = (S_T - K) - (S_T - F_t) = F_t - K$.

The present value for $F_t - K$ at time T is

$$\text{P.V.}(F_t - K) \;=\; \frac{F_t - K}{1 + r}\,,$$

because $(F_t - K)/(1 + r)$ collecting interest has value $F_t - K$ after 1 year.

(Q) Is there any risk in the payoff of the 3rd portfolio?

Answer: No. The risks involved in the payoff for the call, put and forward offset each other.

Since the payoff at time T is $F_t - K$, risk-free,

it is just like investing $(F_t - K)/(1 + r)$ in the bank, risk-free.

But we know the present value of the third portfolio is $C_t - P_t$.

This leads to the following.

$$C_t - P_t = \frac{F_t - K}{1 + r}\,.$$

(Q) What is the formula for F_t?

Answer: Last time we determined the fair forward price

$$F_t = (1 + r)S_t\,.$$

So

$$C_t - P_t = \frac{F_t - K}{1 + r} = \frac{(1 + r)S_t - K}{1 + r} = S_t - (1 + r)^{-1}K\,.$$

⋆<u>Another intuitive derivation:</u>

At time T, the payoff for portfolio 1 is

$$C_T - P_T \;=\; S_T - K\,.$$

The present value of K in cash is $(1+r)^{-1}K$.

The present value of S_T is S_t: people who want the stock at time T are willing to pay exactly S_t for that privilege, today.

Similarly:

The present value of C_T is C_t.

The present value of P_T is P_t.

Therefore,

$$C_t - P_t \;=\; S_t - (1+r)^{-1}K\,.$$

<u>Long derivation:</u>

Prove that $C_t - P_t = S_t - (1+r)^{-1}K$ by no-arbitrage contradiction arguments, just like we did to find $F_t = (1+r)S_t$.

3 Chapter 2: A Primer on the Arbitrage Theorem

Subtitle: The *binomial tree model.*

Notation: t will represent time. We now write a stock price as $S(t)$ instead of S_t.

We sometimes still mean t to mean the time today.

$B(t)$ = price (or value) of a bond at time t.

$S(t)$ = price of a stock at time t.

$C(t)$ = price of a call option at time t.

Suppose at time t we know the price of all these assets, but at a later time $t+\Delta$, we do not.

- If we know the interest rate r, then $B(t+\Delta) = (1+r\Delta)\,B(t)$, assuming $\Delta \leq 1$.

- Suppose for the sake of argument that we do know that $S(t+\Delta)$ must take one of exactly two possible values $S_1(t+\Delta)$ and $S_2(t+\Delta)$.

- Suppose the call option has expiration date $T = t+\Delta$. Then the payoff at expiration is one of two possible values:

$$C_1(T) \;=\; \max(S_1(T) - K, 0) \qquad \text{or} \qquad C_2(T) \;=\; \max(S_2(T) - K, 0)\,.$$

Theorem

Suppose $S_1(t+\Delta)$ and $S_2(t+\Delta)$ both have positive probability to occur, and no other outcome for $S(t+\Delta)$ is possible. Then the principle of *no-arbitrage* is equivalent to the fact that there are two positive constants ψ_1 and ψ_2 such that

$$\begin{bmatrix} B(t) \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} B(t+\Delta) & B(t+\Delta) \\ S_1(t+\Delta) & S_2(t+\Delta) \\ C_1(t+\Delta) & C_2(t+\Delta) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} .$$

Assuming $S_1(t+\Delta) \neq S_2(t+\Delta)$, the constants ψ_1 and ψ_2 are unique.

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Note: We said before to assume that we knew $B(t)$, $S(t)$ and $C(t)$ at present. But actually we do not know a formula for $C(t)$ yet. But this theorem gives us a way to calculate $C(t)$ for this simple "binomial" model.

To be continued next time . . .