# Lecture Notes for Math 210 – Wednesday, 14 November 2007

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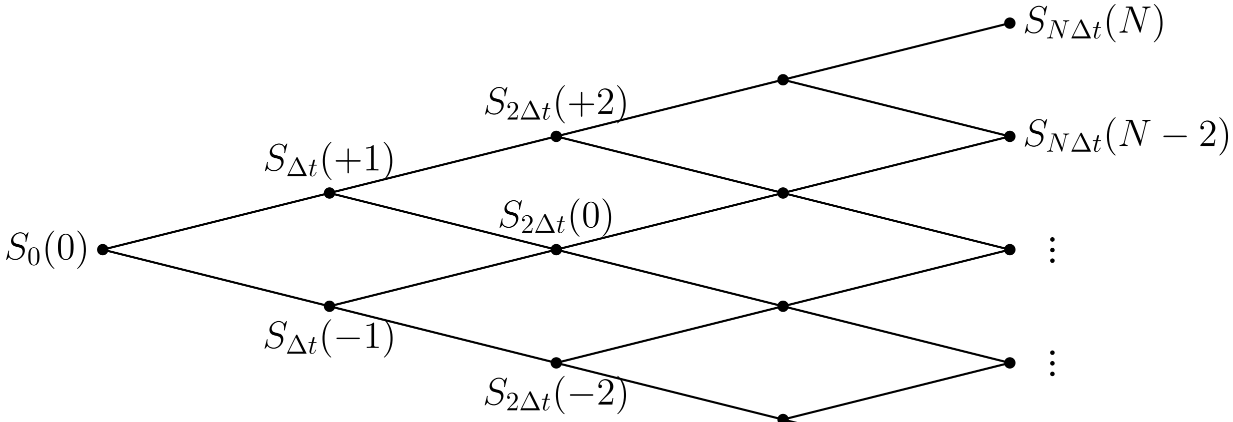
## Chapter 6: Martingales

### The lattice model (Binomial tree model) as a martingale (continued)

We want to begin by considering the binomial tree model when $r > 0$.

Consider an $N$-step binomial tree model.



Now assume

$$S_{t_n+\Delta t}(m+1) \;=\; e^{a\Delta t}\, S_{t_n}(m) \quad \text{and} \quad S_{t_n+\Delta t}(m-1) \;=\; e^{b\Delta t}\, S_{t_n}(m)$$

for $t_n = n\Delta t$ and $n = 0, 1, \ldots, N-1$ and $m = n, n-2, \ldots, -n$.

There is an assumption we have to make which is that $b \leq r \leq a$.

We do not want the the stock to do better than the bond, even when it goes down. Nor do we want the bond to outperform the stock, even when the stock goes up. Or else there would be an arbitrage portfolio.

Because of our assumptions on $S_{t_n+\Delta t}(m+1)$ and $S_{t_n+\Delta t}(m-1)$, we have

$$S_{t_n}(m) \;=\; e^{ak\Delta t + b(n-k)\Delta t}\, S_0(0)\ ,$$

where $k - (n-k) = m$, because after stepping up by $+1$ a total of $k$ times, and stepping down by $-1$ a total of $n-k$ times, you have to be at $m$.

So $2k - n = m$ or $k = \frac{n+m}{2}$.

Then $n - k = \frac{n-m}{2}$.

Our ultimate goal will be to refer to a Binomial random variable, and then use the de Moivre, Laplace limit theorem.

We consider a call option on this stock, with a given strike price $K$, and we suppose that the expiration date is $T = N\Delta t$.

At the very outset we are going to define $\tilde{K} = e^{-rT}K$.

**Q:** What does $\tilde{K}$ represent?

**A:** It is the present value of the strike price, after discounting.

Let $\tilde{S}_{t_n}(m) = e^{-rt_n} S_{t_n}(m)$.

Then

$$\begin{aligned}
\tilde{S}_{t_n+\Delta t}(m+1) &= e^{-r(t_n+\Delta t)} S_{t_n+\Delta t}(m+1) \\
&= e^{-r(t_n+\Delta t)}\, e^{a\Delta t}\, S_{t_n}(m) \\
&= e^{-r\Delta t}\, e^{a\Delta t}\, \tilde{S}_{t_n}(m) \\
&= e^{(a-r)\Delta t}\, \tilde{S}_{t_n}(m) \ .
\end{aligned}$$

and

$$\tilde{S}_{t_n+\Delta t}(m-1) \ = \ e^{-r\Delta t} e^{b\Delta t}\, \tilde{S}_{t_n}(m) \ = \ e^{-(r-b)\Delta t}\, \tilde{S}_{t_n}(m) \ .$$

Note that we also have, in analogy with a previous formula,

$$\tilde{S}_{t_n}(m) \ = \ e^{(a-r)k\Delta t + (b-r)(n-k)\Delta t}\, \tilde{S}_0(0) \ = \ e^{(a-r)k\Delta t + (b-r)(n-k)\Delta t}\, S_0(0) \ ,$$

where $k = \frac{n+m}{2}$ and $n-k = \frac{n-m}{2}$.

Note that we used the fact that $\tilde{S}_0(0) = e^{-r\cdot 0} S_0(0) = S_0(0)$.

We define $C_{N\Delta t}(m)$ via the usual payoff function:

$$C_{N\Delta t}(m) \ = \ \max(0, S_{N\Delta t}(m) - K) \ ,$$

for $m = N, N-2, \ldots, -N$.

But now we also define $\tilde{C}_{t_n}(m) = e^{-rt_n} C_{t_n}(m)$ for all $t_n$ and $m$.

So,

$$\begin{aligned}\tilde{C}_{N\Delta t}(m) &= e^{-rN\Delta t}\,\max(0, S_{N\Delta t}(m) - K)\\ &= \max(e^{-rN\Delta t}\cdot 0, e^{-rN\Delta t}S_{N\Delta t}(m) - e^{-rN\Delta t}K)\\ &= \max(0, \tilde{S}_{N\Delta t}(m) - \tilde{K})\ .\end{aligned}$$

Also, for all $n = 0, 1, \ldots, N-1$, we inductively solve for $C_{t_n}(m)$ backwards, using the formula

$$C_{t_n}(m) \;=\; e^{-r\Delta t}\,[\tilde{P}_+ \cdot C_{t_n+\Delta t}(m+1) + \tilde{P}_- \cdot C_{t_n+\Delta t}(m-1)]\ ,$$

where

$$\tilde{P}_+ \;=\; \frac{e^{r\Delta t} - e^{b\Delta t}}{e^{a\Delta t} - e^{b\Delta t}}\ .$$

and

$$\tilde{P}_- \;=\; \frac{e^{a\Delta t} - e^{r\Delta t}}{e^{a\Delta t} - e^{b\Delta t}}\ .$$

But now we have

$$\begin{aligned}\tilde{C}_{t_n}(m) &= e^{-rt_n}e^{-r\Delta t}\,[\tilde{P}_+ \cdot \tilde{C}_{t_n+\Delta t}(m+1) + \tilde{P}_- \cdot \tilde{C}_{t_n+\Delta t}(m-1)]\\ &= \tilde{P}_+ \cdot \tilde{C}_{t_n+\Delta t}(m+1) + \tilde{P}_- \cdot \tilde{C}_{t_n+\Delta t}(m-1)\ .\end{aligned}$$

Ultimately, we obtain $\tilde{C}_0(0)$, this way.

But $\tilde{C}_0(0) = e^{-r\cdot 0}\,C_0(0) = C_0(0)$, which is our real goal.

We define the random stock prices, $(\mathsf{S}_0, \mathsf{S}_{\Delta t}, \ldots, \mathsf{S}_{N\Delta t})$, and the random call option prices, $(\mathsf{C}_0, \mathsf{C}_{\Delta t}, \ldots, \mathsf{C}_{N\Delta t})$, as before.

We set

$$\mathbf{P}\{\mathsf{S}_0 = S_0(0)\} \;=\; 1 \;,$$

and

$$\mathbf{P}(\{\mathsf{S}_{t_n+\Delta t} = S_{t_n+\Delta t}(m+1)\} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}) \;=\; \tilde{P}_+$$

and

$$\mathbf{P}(\{\mathsf{S}_{t_n+\Delta t} = S_{t_n+\Delta t}(m-1)\} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}) \;=\; \tilde{P}_- \;,$$

for $n = N-1, N-2, \ldots, 0$ and $m = n, n-2, \ldots, -n$.

We also set

$$\mathsf{C}_{t_n} \;=\; C_{t_n}(m) \quad \Leftrightarrow \quad \mathsf{S}_{t_n} \;=\; S_{t_n}(m)\,.$$

But now we also define $\tilde{\mathsf{S}}_{t_n} = e^{-rt_n}\,\mathsf{S}_{t_n}$ and $\tilde{\mathsf{C}}_{t_n} = e^{-rt_n}\,\mathsf{C}_{t_n}$.

We first want to check that these are martingales.

$$\begin{aligned}
&\mathbf{E}[\tilde{\mathsf{S}}_{t_n+\Delta t} \,|\, \{\tilde{\mathsf{S}}_{t_n} = \tilde{S}_{t_n}(m)\}] \\
&\qquad\qquad = \; \mathbf{E}[\tilde{\mathsf{S}}_{t_n+\Delta t} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}] \\
&\qquad\qquad = \; [\tilde{S}_{t_n+\Delta t}(m+1)\cdot\tilde{P}_+ + \tilde{S}_{t_n+\Delta t}(m-1)\cdot\tilde{P}_-] \\
&\qquad\qquad = \; e^{(a-r)\Delta t}\tilde{S}_{t_n}(m)\,\frac{e^{r\Delta t} - e^{b\Delta t}}{e^{a\Delta t} - e^{b\Delta t}} + e^{(b-r))\Delta t}\tilde{S}_{t_n}(m)\,\frac{e^{a\Delta t} - e^{r\Delta t}}{e^{a\Delta t} - e^{b\Delta t}} \\
&\qquad\qquad = \; \left(\frac{e^{(a-r)\Delta t}e^{r\Delta t} - e^{(a-r)\Delta t}e^{b\Delta t}}{e^{a\Delta t} - e^{b\Delta t}} + \frac{e^{(b-r)\Delta t}e^{a\Delta t} - e^{(b-r)\Delta t}e^{r\Delta t}}{e^{a\Delta t} - e^{b\Delta t}}\right)\tilde{S}_{t_n}(m) \\
&\qquad\qquad = \; \tilde{S}_{t_n}(m)\;.
\end{aligned}$$

It is important to note that this calculation looks like the one from last time.

But the one last time only worked because $r$ was set to be 0.

When $r \neq 0$, it is not the stock price itself, but the discounted stock price $(\tilde{\mathsf{S}}_0, \ldots, \tilde{\mathsf{S}}_{N\Delta t})$ that forms a martingale.

Making the usual replacement and doing the usual thing for the conditional expectation, we have

$$\mathbf{E}[\tilde{\mathsf{S}}_{t_n+\Delta t} \,|\, \tilde{\mathsf{S}}_{t_n}] \;=\; \tilde{\mathsf{S}}_{t_n} \ .$$

Actually, it is equally acceptable to have simply said that

$$\mathbf{E}[\tilde{\mathsf{S}}_{t_n+\Delta t} \,|\, \mathsf{S}_{t_n}] \;=\; \tilde{\mathsf{S}}_{t_n} \ .$$

But in any case, we have checked that $(\tilde{\mathsf{S}}_0, \tilde{\mathsf{S}}_{\Delta t}, \ldots, \tilde{\mathsf{S}}_{N\Delta t})$ is a martingale.

Next we calculate

$$\begin{aligned}
&\mathbf{E}[\tilde{\mathsf{C}}_{t_n+\Delta t} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}] \\
&\qquad = \tilde{C}_{t_n+\Delta t}(m+1) \cdot \mathbf{P}(\{\mathsf{S}_{t_n+\Delta t} = S_{t_n+\Delta t}(m+1)\} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}) \\
&\qquad\qquad + \tilde{C}_{t_n+\Delta t}(m-1) \cdot \mathbf{P}(\{\mathsf{S}_{t_n+\Delta t} = S_{t_n+\Delta t}(m-1)\} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}) \\
&\qquad = \tilde{C}_{t_n+\Delta t}(m+1) \cdot \tilde{P}_+ + \tilde{C}_{t_n+\Delta t}(m-1) \cdot \tilde{P}_- \ ,
\end{aligned}$$

and this equals $\tilde{C}_{t_n}(m)$.

So we do have

$$\mathbf{E}[\tilde{\mathsf{C}}_{t_n+\Delta t} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}] \;=\; \tilde{C}_{t_n}(m) \ .$$

Saying that $\mathsf{S}_{t_n} = S_{t_n}(m)$ means that $\mathsf{C}_{t_n} = C_{t_n}(m)$, hence that $\tilde{\mathsf{C}}_{t_n} = \tilde{C}_{t_n}(m)$.

Therefore, replacing $\tilde{C}_{t_n}(m)$ by $\tilde{\mathsf{C}}_{t_n}$, we have

$$\mathbf{E}[\tilde{\mathsf{C}}_{t_n+\Delta t} \,|\, \mathsf{S}_{t_n}] \;=\; \tilde{\mathsf{C}}_{t_n} \ .$$

So $(\tilde{\mathsf{C}}_0, \ldots, \tilde{\mathsf{C}}_{t_N})$ is a martingale (with respect to the filtration generated by $(\mathsf{S}_0, \ldots, \mathsf{S}_{N\Delta t})$).