Lecture Notes for Math 210 – Wednesday, 14 November 2007

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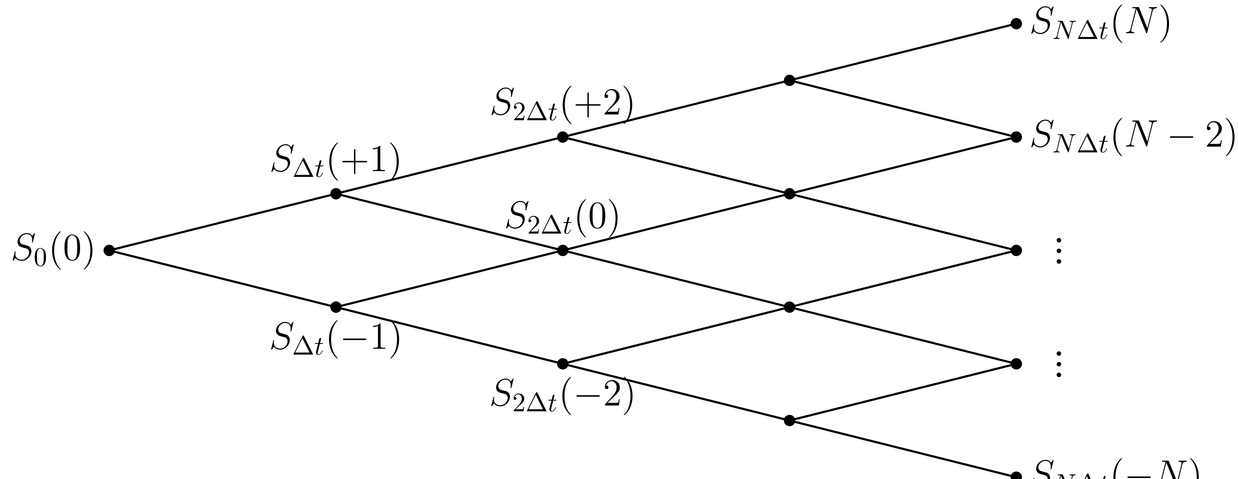
Chapter 6: Martingales

The lattice model (Binomial tree model) as a martingale (continued)

We want to begin by considering the binomial tree model when $r > 0$.

Consider an N-step binomial tree model.



Now assume

$$S_{t_n+\Delta t}(m+1) \;=\; e^{a\Delta t}\, S_{t_n}(m) \quad \text{and} \quad S_{t_n+\Delta t}(m-1) \;=\; e^{b\Delta t}\, S_{t_n}(m)$$

for $t_n = n\Delta t$ and $n = 0, 1, \ldots, N-1$ and $m = n, n-2, \ldots, -n$.

There is an assumption we have to make which is that $b \le r \le a$.

We do not want the the stock to do better than the bond, even when it goes down. Nor do we want the bond to outperform the stock, even when the stock goes up. Or else there would be an arbitrage portfolio.

Because of our assumptions on $S_{t_n+\Delta t}(m+1)$ and $S_{t_n+\Delta t}(m-1)$, we have

$$S_{t_n}(m) \;=\; e^{ak\Delta t + b(n-k)\Delta t}\, S_0(0) \ ,$$

where $k - (n-k) = m$, because after stepping up by $+1$ a total of k times, and stepping down by -1 a total of $n-k$ times, you have to be at m.

So $2k - n = m$ or $k = \frac{n+m}{2}$.

Then $n - k = \frac{n-m}{2}$.

Our ultimate goal will be to refer to a Binomial random variable, and then use the de Moivre, Laplace limit theorem.

We consider a call option on this stock, with a given strike price K, and we suppose that the expiration date is $T = N\Delta t$.

At the very outset we are going to define $\tilde{K} = e^{-rT}K$.

Q: What does $\tilde{K}$ represent?

A: It is the present value of the strike price, after discounting.

Let $\tilde{S}_{t_n}(m) = e^{-rt_n} S_{t_n}(m)$.

Then

$$\begin{aligned}
\tilde{S}_{t_n+\Delta t}(m+1) &= e^{-r(t_n+\Delta t)} S_{t_n+\Delta t}(m+1) \\
&= e^{-r(t_n+\Delta t)}\, e^{a\Delta t}\, S_{t_n}(m) \\
&= e^{-r\Delta t}\, e^{a\Delta t}\, \tilde{S}_{t_n}(m) \\
&= e^{(a-r)\Delta t}\, \tilde{S}_{t_n}(m) \ .
\end{aligned}$$

and

$$\tilde{S}_{t_n+\Delta t}(m-1) \;=\; e^{-r\Delta t} e^{b\Delta t}\, \tilde{S}_{t_n}(m) \;=\; e^{-(r-b)\Delta t}\, \tilde{S}_{t_n}(m) \ .$$

Note that we also have, in analogy with a previous formula,

$$\tilde{S}_{t_n}(m) \;=\; e^{(a-r)k\Delta t+(b-r)(n-k)\Delta t}\, \tilde{S}_0(0) \;=\; e^{(a-r)k\Delta t+(b-r)(n-k)\Delta t}\, S_0(0) \ ,$$

where $k = \frac{n+m}{2}$ and $n-k = \frac{n-m}{2}$.

Note that we used the fact that $\tilde{S}_0(0) = e^{-r\cdot 0} S_0(0) = S_0(0)$.

We define $C_{N\Delta t}(m)$ via the usual payoff function:

$$C_{N\Delta t}(m) \;=\; \max(0, S_{N\Delta t}(m) - K) \ ,$$

for $m = N, N-2, \ldots, -N$.

But now we also define $\tilde{C}_{t_n}(m) = e^{-rt_n} C_{t_n}(m)$ for all t_n and m.

So,

$$\begin{aligned}\tilde{C}_{N\Delta t}(m) &= e^{-rN\Delta t}\,\max(0, S_{N\Delta t}(m) - K)\\ &= \max(e^{-rN\Delta t}\cdot 0, e^{-rN\Delta t}S_{N\Delta t}(m) - e^{-rN\Delta t}K)\\ &= \max(0, \tilde{S}_{N\Delta t}(m) - \tilde{K})\ .\end{aligned}$$

Also, for all $n = 0, 1, \ldots, N-1$, we inductively solve for $C_{t_n}(m)$ backwards, using the formula

$$C_{t_n}(m) \;=\; e^{-r\Delta t}\,[\tilde{P}_+ \cdot C_{t_n+\Delta t}(m+1) + \tilde{P}_- \cdot C_{t_n+\Delta t}(m-1)]\ ,$$

where

$$\tilde{P}_+ \;=\; \frac{e^{r\Delta t} - e^{b\Delta t}}{e^{a\Delta t} - e^{b\Delta t}}\ .$$

and

$$\tilde{P}_- \;=\; \frac{e^{a\Delta t} - e^{r\Delta t}}{e^{a\Delta t} - e^{b\Delta t}}\ .$$

But now we have

$$\begin{aligned}\tilde{C}_{t_n}(m) &= e^{-rt_n}e^{-r\Delta t}\,[\tilde{P}_+ \cdot \tilde{C}_{t_n+\Delta t}(m+1) + \tilde{P}_- \cdot \tilde{C}_{t_n+\Delta t}(m-1)]\\ &= \tilde{P}_+ \cdot \tilde{C}_{t_n+\Delta t}(m+1) + \tilde{P}_- \cdot \tilde{C}_{t_n+\Delta t}(m-1)\ .\end{aligned}$$

Ultimately, we obtain $\tilde{C}_0(0)$, this way.

But $\tilde{C}_0(0) = e^{-r\cdot 0}\, C_0(0) = C_0(0)$, which is our real goal.

We define the random stock prices, $(\mathsf{S}_0, \mathsf{S}_{\Delta t}, \ldots, \mathsf{S}_{N\Delta t})$, and the random call option prices, $(\mathsf{C}_0, \mathsf{C}_{\Delta t}, \ldots, \mathsf{C}_{N\Delta t})$, as before.

We set

$$\mathbf{P}\{\mathsf{S}_0 = S_0(0)\} \;=\; 1 \;,$$

and

$$\mathbf{P}(\{\mathsf{S}_{t_n+\Delta t} = S_{t_n+\Delta t}(m+1)\} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}) \;=\; \tilde{P}_+$$

and

$$\mathbf{P}(\{\mathsf{S}_{t_n+\Delta t} = S_{t_n+\Delta t}(m-1)\} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}) \;=\; \tilde{P}_- \;,$$

for $n = N-1, N-2, \ldots, 0$ and $m = n, n-2, \ldots, -n$.

We also set

$$\mathsf{C}_{t_n} \;=\; C_{t_n}(m) \quad \Leftrightarrow \quad \mathsf{S}_{t_n} \;=\; S_{t_n}(m)\,.$$

But now we also define $\tilde{\mathsf{S}}_{t_n} = e^{-rt_n}\,\mathsf{S}_{t_n}$ and $\tilde{\mathsf{C}}_{t_n} = e^{-rt_n}\,\mathsf{C}_{t_n}$.

We first want to check that these are martingales.

$$\begin{aligned}
&\mathbf{E}[\tilde{\mathsf{S}}_{t_n+\Delta t} \,|\, \{\tilde{\mathsf{S}}_{t_n} = \tilde{S}_{t_n}(m)\}] \\
&\qquad = \mathbf{E}[\tilde{\mathsf{S}}_{t_n+\Delta t} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}] \\
&\qquad = [\tilde{S}_{t_n+\Delta t}(m+1)\cdot \tilde{P}_+ + \tilde{S}_{t_n+\Delta t}(m-1)\cdot \tilde{P}_-] \\
&\qquad = e^{(a-r)\Delta t}\tilde{S}_{t_n}(m)\,\frac{e^{r\Delta t}-e^{b\Delta t}}{e^{a\Delta t}-e^{b\Delta t}} + e^{(b-r))\Delta t}\tilde{S}_{t_n}(m)\,\frac{e^{a\Delta t}-e^{r\Delta t}}{e^{a\Delta t}-e^{b\Delta t}} \\
&\qquad = \left(\frac{e^{(a-r)\Delta t}e^{r\Delta t}-e^{(a-r)\Delta t}e^{b\Delta t}}{e^{a\Delta t}-e^{b\Delta t}} + \frac{e^{(b-r)\Delta t}e^{a\Delta t}-e^{(b-r)\Delta t}e^{r\Delta t}}{e^{a\Delta t}-e^{b\Delta t}}\right)\tilde{S}_{t_n}(m) \\
&\qquad = \tilde{S}_{t_n}(m)\;.
\end{aligned}$$

It is important to note that this calculation looks like the one from last time.

But the one last time only worked because r was set to be 0.

When $r \neq 0$, it is not the stock price itself, but the discounted stock price $(\tilde{\mathsf{S}}_0, \ldots, \tilde{\mathsf{S}}_{N\Delta t})$ that forms a martingale.

Making the usual replacement and doing the usual thing for the conditional expectation, we have

$$\mathbf{E}[\tilde{\mathsf{S}}_{t_n+\Delta t} \,|\, \tilde{\mathsf{S}}_{t_n}] \;=\; \tilde{\mathsf{S}}_{t_n} \;\; .$$

Actually, it is equally acceptable to have simply said that

$$\mathbf{E}[\tilde{\mathsf{S}}_{t_n+\Delta t} \,|\, \mathsf{S}_{t_n}] \;=\; \tilde{\mathsf{S}}_{t_n} \;\; .$$

But in any case, we have checked that $(\tilde{\mathsf{S}}_0, \tilde{\mathsf{S}}_{\Delta t}, \ldots, \tilde{\mathsf{S}}_{N\Delta t})$ is a martingale.

Next we calculate

$$\begin{aligned}
&\mathbf{E}[\tilde{\mathsf{C}}_{t_n+\Delta t} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}] \\
&\qquad = \tilde{C}_{t_n+\Delta t}(m+1) \cdot \mathbf{P}(\{\mathsf{S}_{t_n+\Delta t} = S_{t_n+\Delta t}(m+1)\} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}) \\
&\qquad\qquad + \tilde{C}_{t_n+\Delta t}(m-1) \cdot \mathbf{P}(\{\mathsf{S}_{t_n+\Delta t} = S_{t_n+\Delta t}(m-1)\} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}) \\
&\qquad = \tilde{C}_{t_n+\Delta t}(m+1) \cdot \tilde{P}_+ + \tilde{C}_{t_n+\Delta t}(m-1) \cdot \tilde{P}_- \;\; ,
\end{aligned}$$

and this equals $\tilde{C}_{t_n}(m)$.

So we do have

$$\mathbf{E}[\tilde{\mathsf{C}}_{t_n+\Delta t} \,|\, \{\mathsf{S}_{t_n} = S_{t_n}(m)\}] \;=\; \tilde{C}_{t_n}(m) \;\; .$$

Saying that $\mathsf{S}_{t_n} = S_{t_n}(m)$ means that $\mathsf{C}_{t_n} = C_{t_n}(m)$, hence that $\tilde{\mathsf{C}}_{t_n} = \tilde{C}_{t_n}(m)$.

Therefore, replacing $\tilde{C}_{t_n}(m)$ by $\tilde{\mathsf{C}}_{t_n}$, we have

$$\mathbf{E}[\tilde{\mathsf{C}}_{t_n+\Delta t} \,|\, \mathsf{S}_{t_n}] \;=\; \tilde{\mathsf{C}}_{t_n} \;\; .$$

So $(\tilde{\mathsf{C}}_0, \ldots, \tilde{\mathsf{C}}_{t_N})$ is a martingale (with respect to the filtration generated by $(\mathsf{S}_0, \ldots, \mathsf{S}_{N\Delta t})$).

Expectation and Conditional Expectation

We have said before that the expectation of the conditional expectation equals the expectation:

$$\mathbf{E}[\mathbf{E}[\mathsf{X} \,|\, \mathcal{F}]] \;=\; \mathbf{E}[\mathsf{X}]\,.$$

Heuristically, this says the following: Suppose somebody is nice enough to tell you that they will give you some inside information indirectly related to X. More specifically, they will answer all the questions contained in the event space $\mathcal{F}$. So you do some work and find out what your best guess for X should be, conditional on knowing the information in $\mathcal{F}$, in anticipation of your friend's information. But then suppose you find out that the person is unreliable, so that you cannot count on his information. So, instead if you want to know something about $\mathcal{F}$ then you have to just guess. Then your end-result guess for X is exactly what you would have got if that unreliable person had never suggested they could give you inside information to begin with.

Because of this, we see that

$$\mathbf{E}[\tilde{\mathsf{C}}_{2\Delta t}] \;=\; \mathbf{E}[\mathbf{E}[\tilde{\mathsf{C}}_{2\Delta t} \,|\, \mathsf{S}_{\Delta t}]] \;=\; \mathbf{E}[\tilde{\mathsf{C}}_{\Delta t}] \;=\; C_0(0)\ .$$

(Note that S_0 must equal $S_0(0)$, with probability 1, so that also $\tilde{\mathsf{C}}_0$ must equal $C_0(0)$ with probability 1. But we also know that $\mathbf{E}[\tilde{\mathsf{C}}_{\Delta t} \,|\, \mathsf{S}_0] = \tilde{\mathsf{C}}_0(0) = C_0(0)$. But on the other hand knowing the outcome for $\mathsf{S}_0(0)$ really has no information content because everybody knows that $\mathsf{S}_0(0) = S_0(0)$ because that happens with probability 1. So the conditional expectation $\mathbf{E}[\tilde{\mathsf{C}}_{\Delta t} \,|\, \mathsf{S}_0]$ is really just the same as the unconditional expectation $\mathbf{E}[\tilde{\mathsf{C}}_{\Delta t}]$.)

Similarly,

$$\mathbf{E}[\tilde{\mathsf{C}}_{3\Delta t}] \;=\; \mathbf{E}[\mathbf{E}[\tilde{\mathsf{C}}_{3\Delta t} \,|\, \mathsf{S}_{2\Delta t}]] \;=\; \mathbf{E}[\tilde{\mathsf{C}}_{2\Delta t}] \;=\; C_0(0) \ .$$

Continuing inductively, we finally find

$$\mathbf{E}[\tilde{\mathsf{C}}_{N\Delta t}] \;=\; C_0(0) \ .$$

We actually want to turn this equation around!

This gives a useful formula for $C_0(0)$:

$$C_0(0) \;=\; \mathbf{E}[\tilde{\mathsf{C}}_{N\Delta t}] \ .$$

Since $T = N\Delta t$ is the expiration date, we know a formula for $\tilde{\mathsf{C}}_{N\Delta t}$:

$$C_0(0) \;=\; \mathbf{E}[\max(0, \tilde{\mathsf{S}}_{N\Delta t} - \tilde{K})] \ .$$

In turn, let $\mathsf{X}_{N\Delta t}$ be the number of times the stock stepped up, so that $N - \mathsf{X}_{N\Delta t}$ is the number of times the stock stepped down.

Then

$$\tilde{\mathsf{S}}_{N\Delta t} \;=\; e^{(a-r)\Delta t\cdot \mathsf{X}_{N\Delta t} + (b-r)\Delta t\cdot (N - \mathsf{X}_{N\Delta t})}\, S_0(0) \ .$$

But $\mathsf{X}_{N\Delta t}$ is a Binomial random variable, with parameters $n = N$ and $p = \tilde{P}_+$. So

$$\mathbf{P}\{\mathsf{X}_N = k\} \;=\; \binom{N}{k} \, (\tilde{P}_+)^k \, (\tilde{P}_-)^{N-k} \ .$$

Therefore, we obtain one of the first major results associated to the lattice (Binomial tree) model:

$$C_0(0) \;=\; \sum_{k=0}^{N} \binom{N}{k} \, (\tilde{P}_+)^k \, (\tilde{P}_-)^{N-k} \, \max\left(0, e^{(a-r)k\Delta t+(b-r)(N-k)\Delta t} - \tilde{K}\right) \; .$$

Next time we will discuss the de Moivre, Laplace limit theorem, and see how to get an exact formula for $C_0(0)$ in the $N \to \infty$ limit, if T stays fixed, and $\Delta t = T/N$. That formula *is* the Black-Scholes formula.