Lecture Notes for Math 210 – Friday, 9 November 2007

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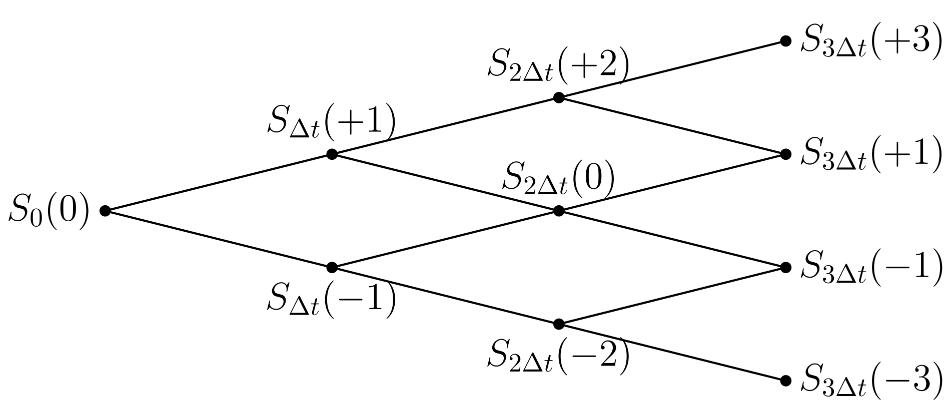
Chapter 6: Martingales

The lattice model (Binomial tree model) as a martingale

Let us consider a simple problem that belongs more in Chapter 2 than Chapter 6.

(But I will explain why I want to consider this now, momentarily.)

Consider a 3-step lattice model, in the case that $r = 0$.

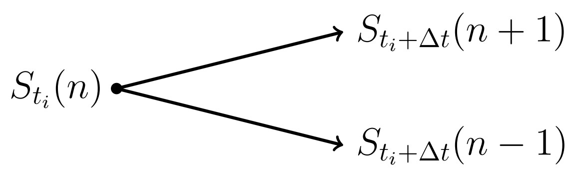


We have set $r = 0$ so it will not really matter what Δt is.

We assume

$$S_{t_i+\Delta t}(n+1) \;=\; 1.5\, S_{t_i}(n) \quad \text{and} \quad S_{t_i+\Delta t}(n-1) \;=\; 0.75\, S_{t_i}(n)$$

for $t_i = 0, \Delta t, 2\Delta t$, and $n = 0$, or $n = +1, -1$ or $n = +2, 0, -2$, respectively.



Let $S_0(0) = \$100$.

Then we have:

$$\begin{bmatrix} S_{\Delta t}(+1) \\ S_{\Delta t}(-1) \end{bmatrix} = \begin{bmatrix} \$150 \\ \$75 \end{bmatrix}, \quad \begin{bmatrix} S_{2\Delta t}(+2) \\ S_{2\Delta t}(0) \\ S_{2\Delta t}(-2) \end{bmatrix} = \begin{bmatrix} \$225 \\ \$112.50 \\ \$56.25 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} S_{3\Delta t}(+3) \\ S_{3\Delta t}(+1) \\ S_{3\Delta t}(-1) \\ S_{3\Delta t}(-3) \end{bmatrix} = \begin{bmatrix} \$337.50 \\ \$168.75 \\ \$84.375 \\ \$42.1875 \end{bmatrix}$$

Q: Consider a call-option with strike price $K = \$100$ and $T = 3\Delta t$.

What is C_0?

A: First of all $C_T = \max(0, S_T - K)$, as always.

So:

$$\begin{bmatrix} C_{3\Delta t}(+3) \\ C_{3\Delta t}(+1) \\ C_{3\Delta t}(-1) \\ C_{3\Delta t}(-3) \end{bmatrix} = \begin{bmatrix} \$237.50 \\ \$68.75 \\ 0 \\ 0 \end{bmatrix} \quad .$$

Next, the synthetic, risk-neutral probabilities are

$$\tilde{P}_{t_i,t_i+\Delta t}(n, n{+}1) \;=\; \frac{e^{r\Delta t} S_{t_i}(n) - S_{t_i+\Delta t}(n-1)}{S_{t_i+\Delta t}(n+1) - S_{t_i+\Delta t}(n-1)} \;=\; \frac{S_{t_i}(n) - 0.75\, S_{t_i}(n)}{1.5\, S_{t_i}(n) - 0.75\, S_{t_i}(n)} \;=\; \frac{1}{3}\,,$$

for all t_i and n.

So, also,

$$\tilde{P}_{t_i,t_i+\Delta t}(n, n-1) \;=\; 1 - \tilde{P}_{t_i,t_i+\Delta t}(n, n+1) \;=\; \frac{2}{3}\;.$$

For simplicity, let us just write $\tilde{P}_+ = 1/3$ and $\tilde{P}_- = 2/3$ for these two.

Then we calculate:

$$\begin{bmatrix} C_{2\Delta t}(+2) \\ \\ C_{2\Delta t}(0) \\ \\ C_{2\Delta t}(-2) \end{bmatrix} \;=\; \begin{bmatrix} \frac{1}{3}(237.50) + \frac{2}{3}(68.75) \\ \\ \frac{1}{3}(68.75) + \frac{2}{3}(0) \\ \\ +\frac{1}{3}(0) + \frac{2}{3}(0) \end{bmatrix} \;=\; \begin{bmatrix} 125 \\ \\ 22.91\bar{6} \\ \\ 0 \end{bmatrix}\;.$$

Similarly:

$$\begin{bmatrix} C_{\Delta t}(+1) \\ \\ C_{\Delta t}(-1) \end{bmatrix} \;=\; \begin{bmatrix} \frac{1}{3}(125) + \frac{2}{3}(22.91\bar{6}) \\ \\ \frac{1}{3}(22.91\bar{6}) + \frac{2}{3}(0) \end{bmatrix} \;=\; \begin{bmatrix} 56.9\bar{4} \\ \\ 7.63\bar{8} \end{bmatrix}\;.$$

Finally:

$$C_0 \;=\; \frac{1}{3}(56.9\bar{4}) + \frac{2}{3}(7.63\bar{8}) \;=\; 34.0\bar{7}4\;.$$

So, approximating to cents, we have $C_0 \approx \$24.07$.

The point of the exercise is not merely to prove that we could solve a 3-step lattice model.

In fact, we can solve much longer models without difficulty.

But the point is to show how to construct 2 new martingales.

An exponential random walk for stocks

Define a new finite sequence of random variables, $\mathsf{S}_0, \mathsf{S}_{\Delta t}, \mathsf{S}_{2\Delta t}, \mathsf{S}_{3\Delta t}$ such that

$$\mathbf{P}\{\mathsf{S}_0 = S_0(0)\} \;=\; 1\,,$$

and

$$\mathbf{P}(\{\mathsf{S}_{t_i+\Delta t} = S_{t_i+\Delta t}(n+1)\}\,|\,\{\mathsf{S}_{t_i} = S_{t_i}(n)\}) \;=\; \tilde{P}_{t_i,t_i+\Delta t}(n,n+1) \;=\; \frac{1}{3}\;.$$

Then, $(\mathsf{S}_0, \mathsf{S}_{\Delta t}, \mathsf{S}_{2\Delta t}, \mathsf{S}_{3\Delta t})$ forms a martingale.

To see this, we just check:

$$\begin{aligned}
&\mathbf{E}[\mathsf{S}_{t_i+\Delta t}\,|\,\{\mathsf{S}_{t_i} = S_{t_i}(n)\}] \\
&\qquad := \; S_{t_i+\Delta t}(n+1)\cdot\tilde{P}_{t_i,t_i+\Delta t}(n,n+1) + S_{t_i+\Delta t}(n-1)\cdot\tilde{P}_{t_i,t_i+\Delta t}(n,n-1) \\
&\qquad = \; 1.5\; S_{t_i}(n)\cdot\frac{1}{3} + 0.75\; S_{t_i}(n)\cdot\frac{2}{3} \\
&\qquad = \; (0.5+0.5)\,S_{t_i}(n) \\
&\qquad = \; S_{t_i}(n)\;.
\end{aligned}$$

So, replacing $S_{t_i}(n)$ by S_{t_i} (because the latter equals the former, under the conditioning) we have

$$\mathbf{E}[\mathsf{S}_{t_i+\Delta t}\,|\,\mathsf{S}_{t_i}] \;=\; \mathsf{S}_{t_i}\;.$$

This is the condition to have a martingale.

(Technically this is the condition to have a martingale if we already know that the stochastic variables form a “Markov process”, which I haven’t defined, yet.)

This is not a coincidence.

The formulae,

$$\tilde{P}_{t_i,t_i+\Delta t}(n,n+1) \;=\; \frac{e^{r\Delta t}S_{t_i}(n) - S_{t_i+\Delta t}(n-1)}{S_{t_i+\Delta t}(n+1) - S_{t_i+\Delta t}(n-1)} \quad \text{and}$$
$$\tilde{P}_{t_i,t_i+\Delta t}(n,n-1) \;=\; \frac{S_{t_i+\Delta t}(n+1) - e^{r\Delta t}S_{t_i}(n)}{S_{t_i+\Delta t}(n+1) - S_{t_i+\Delta t}(n-1)} \;,$$

were chosen precisely so that

$$S_{t_i+\Delta t}(n+1)\cdot \tilde{P}_{t_i,t_i+\Delta t}(n,n+1) + S_{t_i+\Delta t}(n-1)\cdot \tilde{P}_{t_i,t_i+\Delta t}(n,n-1) \;=\; S_{t_i}(n) \;,$$

which is what guarantees that $\mathbf{E}[\mathsf{S}_{t_i+\Delta t}\,|\,\mathsf{S}_{t_i}] = \mathsf{S}_{t_i}$.

Something which is worthwhile to note is the following:

$$\mathbf{P}\{\mathsf{S}_{n\Delta t} = S_{n\Delta t}(m)\} \;=\; \binom{n}{(n+m)/2}\,(\tilde{P}_+)^{(n+m)/2}\,(\tilde{P}_-)^{(n-m)/2} \;.$$

These are the binomial probabilities.

But note that $\mathsf{S}_{n\Delta t}$ is not a binomial random variable.

Rather, defining X_n to be a Binomial random variable with parameter n and $p = \tilde{P}_+$, we have

$$\mathsf{S}_{n\Delta t} \;\stackrel{\mathcal{D}}{=}\; (1.5)^{\mathsf{X}_n}\,(0.75)^{n-\mathsf{X}_n}\,S_0 \;=\; \left(\frac{3}{4}\right)^n\,2^{\mathsf{X}_n}\,S_0 \;,$$

where the funny equal sign means that the two random variables are equal, “in distribution”.

We will return to this fact a bit later, when we discuss the de Moivre, Laplace limit theorem for Binomial random variables.

The call-option is a trivial (Doob) martingale

Let us define another finite sequence of random variables $(\mathsf{C}_0, \mathsf{C}_{\Delta t}, \mathsf{C}_{2\Delta t}, \mathsf{C}_{3\Delta t})$.

We define C_{t_i} to be a function of S_{t_i}, wherein

$$\mathsf{C}_{t_i} \ = \ C_{t_i}(n) \ \Leftrightarrow \ \mathsf{S}_{t_i} \ = \ S_{t_i}(n) \ .$$

Checking that $(\mathsf{C}_0, \mathsf{C}_{\Delta t}, \mathsf{C}_{2\Delta t}, \mathsf{C}_{3\Delta t})$ is a martingale is even easier than checking that $(\mathsf{S}_0, \mathsf{S}_{\Delta t}, \mathsf{S}_{2\Delta t}, \mathsf{S}_{3\Delta t})$ is a martingale.

Q: Does anybody know why the call option is a martingale?

A: It is by definition.

We defined

$$C_{t_i}(n) \ = \ \tilde{P}_+ \cdot C_{t_i+\Delta t}(n+1) + \tilde{P}_- \cdot C_{t_i+\Delta t}(n-1) \ .$$

So,

$$\mathbf{E}[\mathsf{C}_{t_i+\Delta t} \,|\, \{\mathsf{S}_{t_i} = S_{t_i}(n)\}] \ = \ \tilde{P}_+ \cdot C_{t_i+\Delta t}(n+1) + \tilde{P}_- \cdot C_{t_i+\Delta t}(n-1) \ = \ C_{t_i}(n) \ .$$

Replacing $S_{t_i}(n)$ by S_{t_i} amounts to replacing $C_{t_i}(n)$ by C_{t_i}.

So

$$\mathbf{E}[\mathsf{C}_{t_i+\Delta t} \,|\, \mathsf{S}_{t_i}] \ = \ \mathsf{C}_{t_i} \ .$$

Again this is enough to check that $(\mathsf{C}_0, \mathsf{C}_{\Delta t}, \mathsf{C}_{2\Delta t}, \mathsf{C}_{3\Delta t})$ is a martingale, adapted to $(\mathsf{S}_0, \mathsf{S}_{\Delta t}, \mathsf{S}_{2\Delta t}, \mathsf{S}_{3\Delta t})$, (because it is a Markov process).

Next we will see how to make martingales out of the risk-neutral random stock price and random call option price, when the interest rate r is not equal to 0.