Lecture Notes for Math 210 – 7 September 2007

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1 Arbitrage

John Hull's definition of "arbitrage": A trading strategy that takes advantage of two or more securities being mispriced relative to each other.

My informal definition of "arbitrage opportunity": An opportunity to make a risk-free profit.

Running assumption: There are no arbitrage opportunities.

(We will always assume this, no matter what.)

2 Problem

For a stock,

$$S_t = \$100.00 = \text{spot price today.}$$

Forward contract: Agreement to buy (sell) 1 share of this stock at time T, 1 year in the future.

$$T = 1 \text{ year} = \text{maturity date.}$$

Further assumptions:

- **No yield.** The stock provides no yield during the period between now and time T. I.e., there is no dividend paying out during that period.
- **No bid-ask spread.** You can borrow from the bank at the same interest rate you collect on savings.

  More specifically, assume $r = 5\%$:

  $$N \text{ today} \implies \text{has value } (1+r)N = 1.05\,N \text{ at time } T\,.$$

  (Also means N' at time T has present value of $(1+r)^{-1}N'$ today, which is $0.9524\,N'$.)

Question: What is the fair forward price F_t that we would agree on, today?

Answer: $F_t = (1+r)S_t = \$105.00.$

Proof:

Part 1.

Suppose $F_t > (1+r)S_t$: we will show this leads to an arbitrage opportunity.

(Q1) Is price too high or too low?

(Q2) Do we want to take the long position or the short position in this contract?

(Q3) What do "long position" and "short position" mean?

The price is too high.

So we want to take the short position in the contract.

The "short position" is the position of selling the asset. The "long position" is the position of buying the asset.

Strategy at time t, now.

- Take short position in forward contract.
- Borrow S_t from bank, to be repaid at T.
- Buy 1 share of stock.

Payoff at time T.

- Sell 1 share of stock for F_t, by forward agreement.

- Payoff your debt to bank of $(1+r)S_t$.

$$\text{Net profit} \;=\; F_t - (1+r)S_t \;>\; 0 \quad \text{because we assumed } F_t > (1+r)S_t.$$

(Q) Was there any risk involved in this strategy? (Open-ended question)

Answer: No. In our model of the universe there was no risk. We assume both parties of the contract will fulfill their obligation, that the bank will give repay our investment with interest, and that no default ever occurs.

Logical contradiction: We assumed that there are no arbitrage opportunities, yet we just made one.

That is a contradiction.

When we reach a mathematical contradiction, we have to re-examine our hypotheses: one of them must be wrong.

There were only two hypotheses:

(1) No arbitrage opportunities exist.

(2) $F_t \;>\; (1+r)S_t\,.$

We refuse to give up (1), so (2) must be wrong.

Conclusion: $F_t \;\leq\; (1+r)S_t\,.$

Part 2.

Suppose $F_t < (1+r)S_t$: we will show this also leads to an arbitrage opportunity.

Because the price is too low, we now want to take the long position in the contract. We should reverse our strategy from before.

- Sell 1 share of stock.
- Invest S_t from selling the stock in the bank.
- Take the long position in 1 forward.

Obstacle: What if we don't have 1 share of the stock to sell at time t?

(Q) What should we do? (Open-ended question)

Answer: *We can imagine at least 2 things to do.*

One is to sell the stock at time T instead, since forward means we can buy it at time T for F_t. But this has a problem, because the stock's value could plummet between now and time T. We do not expect it to, but that introduces some risk, and we want a risk-free profit.

The correct answer is to sell somebody else's stock.

John Hull's definition of "short selling": Selling in the market shares that have been borrowed from another investor.

My expanded definition of "short selling": To short sell a stock today means that you borrow a stock from another investor, to be repaid at a future date (say T), then you sell that stock for the spot price S_t. (You still owe the original investor his stock at time T.)

Revised strategy, at time t now:

- Sell 1 share for S_t.
- Invest S_t in the bank until T, 1 year.
- Take the long position in 1 forward.

Payoff at time T.

- Collect $(1+r)S_t$ from bank.
- Pay F_t for 1 share, by forward agreement.
- Repay stock to investor you borrowed from for the short-sale.

Net profit $= (1+r)S_t - F_t > 0$ because now we assumed $F_t < (1+r)S_t$.

This is also an arbitrage opportunity. So by contradiction we know that it is impossible to have $F_t < (1+r)S_t$.

Conclusion: $F_t \geq (1+r)S_t$.

Completion of proof.

Combine Parts 1 and 2. Since

$$F_t \leq (1+r)S_t \,,$$

and

$$F_t \geq (1+r)S_t \,,$$

it must be that

$$F_t = (1+r)S_t \,.$$