Lecture Notes for Math 210 – 22 October 2007

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Chapter 5: Tools in Probability Theory

Some probability distributions.

Binomial distribution: Typically, in basic probability courses, we learn about binomial distributions with parameters $p \in [0,1]$ and $n \in \{1,2,3,\dots\}$:

X is a "discrete" random variable.

That is, it only takes values in a discrete set, rather than a continuous set.

A discrete set is a set which is either finite or countably infinite, like $\{0,1,2,\dots\}$.

The binomial random variable only takes values in the set $\{0,1,2,\dots,n\}$.

Moreover,

$$P\{\mathsf{X} = k\} = \binom{n}{k} p^k (1-p)^{n-k} .$$

Then we have

$$\mathbf{E}[\mathsf{X}] = np ,$$

and

$$\mathrm{Var}(\mathsf{X}) = np(1-p) .$$

For us, the binomial distribution is related to the binomial tree model.

But, unlike what I told several people in office hours, they are not the same!

The binomial tree model uses an exponentiated version of the binomial distribution, which we probably will not discuss.

Poisson distribution:

The Poisson random variable with parameter λ is another discrete random variable X, this time taking values in $\{0, 1, 2, \dots\}$:

$$\mathbf{P}\{\mathsf{X} = n\} \;=\; e^{-\lambda}\,\frac{\lambda^n}{n!}$$

Note that the power series for e^x is

$$e^x \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}\,.$$

So

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbf{P}\{\mathsf{X} = n\} \;&=\; \sum_{n=0}^{\infty} e^{-\lambda}\,\frac{\lambda^n}{n!} \\
&=\; e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \\
&=\; e^{-\lambda} \cdot e^{\lambda} \\
&=\; 1\,.
\end{aligned}$$

So this random variable does have total probability equal to 1, as is required.

For this random variable $\mathbf{E}[\mathsf{X}] = \lambda$ and $\mathrm{Var}(\mathsf{X}) = \lambda$, too.

This random variable models "rare events".

Normal distribution:

The most ubiquitous distribution is the normal random variable.

This has parameters μ and σ^2.

This is a continuous random variable X, such that

$$g_{\mathsf{X}}(x) \;=\; \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,.$$

It has mean $\mu_{\mathsf{X}} = \mu$ and variance $\sigma_{\mathsf{X}}^2 = \sigma^2$.

It comes up a lot because of the "Central Limit Theorem".

We will talk about these things more next time.

When we talk about a "standard" normal random variable, then we set $\mu = 0$ and $\sigma^2 = 1$.

Conditional Probability and Conditional Expectation

Remember that for one sample space Ω, there can be different event spaces (σ-fields) $\mathcal{F}$.

A random variable $\mathsf{X} : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$-measurable if all the sets $\{\omega \in \Omega : \mathsf{X}(\omega) = x\}$ are in $\mathcal{F}$ for all $x \in \mathbb{R}$.

This is an abstract idea, but consider the following simple example.

Example 1. Let X be the outcome of rolling 1 fair die, and let Y be the outcome of rolling another fair die.

So we could say

$$\Omega \;=\; \{(i,j) \,:\, i,j = 1,2,\ldots,6\}\,.$$

Let $\mathcal{F}_1$ be the full event space consisting of all possible subsets of Ω.

Let $\mathcal{F}_2$ be the event space generated by X, by itself.

So, some events in $\mathcal{F}_2$ are

$$\begin{aligned} E_1 &= \{(1,j) \,:\, j = 1,\ldots,6\}\,, \\ E_2 &= \{(2,j) \,:\, j = 1,\ldots,6\}\,, \\ &\vdots \\ E_6 &= \{(6,j) \,:\, j = 1,\ldots,6\}\,. \end{aligned}$$

All other events in $\mathcal{F}_2$ are unions of some of these sets, or the empty set $\emptyset$.

(The empty set $\emptyset$ is always in every event space).

Note that using events in $\mathcal{F}_2$ we can ask the question "Is $\mathsf{X} = 4$?" or even "What is X?" but we cannot specify anything about Y.

So X is $\mathcal{F}_2$-measurable, but Y is not $\mathcal{F}_2$-measurable.

But both X and Y are $\mathcal{F}_1$-measurable.

Fact: Suppose Ω is some sample space

and that $\mathcal{F}_1$ and $\mathcal{F}_2$ are two event spaces

such that $\mathcal{F}_2 \subseteq \mathcal{F}_1$.

If X is a random variable which is $\mathcal{F}_1$-measurable,

then there is always another random variable, called $\mathbf{E}[\mathsf{X} \,|\, \mathcal{F}_2]$,

which is $\mathcal{F}_2$-measurable, and such that for any other $\mathcal{F}_2$ measurable function, Y, we have

$$\mathbf{E}[\mathsf{Y} \cdot \mathbf{E}[\mathsf{X} \,|\, \mathcal{F}_2]] \;=\; \mathbf{E}[\mathsf{Y} \cdot \mathsf{X}]\,.$$

This is also a somewhat abstract definition, but in particular cases calculating

$\mathbf{E}[\mathsf{X} \,|\, \mathcal{F}_2]$ is usually a very concrete problem.

In case $\mathcal{F}_2$ is the event space generated by some other random variable Y, we usually write $\mathbf{E}[\mathsf{X} \,|\, \mathsf{Y}]$ instead of $\mathcal{F}_2$.

Usually, when we calculate $\mathbf{E}[\mathsf{X} \,|\, \mathsf{Y}]$, we find that it is an explicit function of Y.