Lecture Notes for Math 210 – 19 October 2007

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Chapter 5: Tools in Probability Theory

Probability

Sample space: Ω.

Outcome: ω.

New idea: Event space $\mathcal{F}$.

(The symbol $\mathcal{F}$ stands for "σ-field". We will not really define what that means.)

$\mathcal{F}$ consists of "measurable events", which are subsets of Ω.

This is *never* introduced in undergraduate courses on probability.

But it sometimes *is* introduced in undergraduate courses on stochastic processes, especially when it comes to martingales.

The event space $\mathcal{F}$ is really telling you which questions you can ask.

Probability measure: a function $P : \mathcal{F} \to [0, 1]$ satisfying the axioms of probability.

Neftci has some *nasty* notation.

If we ever integrate over P it will be something like

$$\int_{\Omega} dP(\omega) \;=\; 1\,,$$

NOT WHAT HE WROTE FOR EQUATION (2) ON PAGE 92!

A random variable is a (measurable) function $X : \Omega \to \mathbb{R}$.

Then we define the cumulative distribution function $G_{\mathsf{X}}(a) = P\{\mathsf{X} \le a\}$ for every $a \in \mathbb{R}$.

This is a nondecreasing function which is right-continuous.

$$\mathbf{E}[\mathsf{X}] \;=\; \int_{-\infty}^{\infty} x\, dG_{\mathsf{X}}(x)\,.$$

More generally,

$$\mathbf{E}[f(\mathsf{X})] \;=\; \int_{-\infty}^{\infty} f(x)\, dG_{\mathsf{X}}(x)\,.$$

Define the probability density function $g_{\mathsf{X}}(x) = \frac{d}{dx}(G_{\mathsf{X}}(x))$ if it exists.

Then if the p.d.f. exists,

$$\mathbf{E}[\mathsf{X}] \;=\; \int_{-\infty}^{\infty} x\, g_{\mathsf{X}}(x)\, dx\,, \quad \text{and} \quad \mathbf{E}[f(\mathsf{X})] \;=\; \int_{-\infty}^{\infty} f(x)\, g_{\mathsf{X}}(x)\, dx\,.$$

Mean and Variance

The mean of a random variable X is defined as

$$\mu_{\mathsf{X}} \;=\; \mathbf{E}[\mathsf{X}]\,.$$

It is sometimes called the “first moment”.

The variance is

$$\sigma_{\mathsf{X}}^2 = \mathrm{Var}(\mathsf{X}) = \mathbf{E}[(\mathsf{X} - \mu_{\mathsf{X}})^2] = \mathbf{E}[\mathsf{X}^2] - \mu_{\mathsf{X}}^2 .$$

(Neftci calls the variance the "second moment". But he is wrong in this. If you know the mean and variance, then you can calculate the second moment, which is $\mathbf{E}[\mathsf{X}]^2$. But it is not equal to the second moment.)

Some probability distributions.

Binomial distribution: Typically, in basic probability courses, we learn about binomial distributions with parameters $p \in [0,1]$ and $n \in \{1,2,3,\dots\}$:

X is a "discrete" random variable.

That is, it only takes values in a discrete set, rather than a continuous set.

A discrete set is a set which is either finite or countably infinite, like $\{0,1,2,\dots\}$.

The binomial random variable only takes values in the set $\{0,1,2,\dots,n\}$.

Moreover,

$$P\{\mathsf{X} = k\} = \binom{n}{k} p^k (1-p)^{n-k} .$$

Then we have

$$\mathbf{E}[\mathsf{X}] = np ,$$

and

$$\mathrm{Var}(\mathsf{X}) = np(1-p) .$$

For us, the binomial distribution is related to the binomial tree model.

But, unlike what I told several people in office hours, they are not the same!

The binomial tree model uses an exponentiated version of the binomial distribution, which we probably will not discuss.

Poisson distribution:

The Poisson random variable with parameter λ is another discrete random variable X, this time taking values in $\{0, 1, 2, \dots\}$:

$$\mathbf{P}\{\mathsf{X} = n\} \;=\; e^{-\lambda}\,\frac{\lambda^n}{n!}$$

Note that the power series for e^x is

$$e^x \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}\,.$$

So

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbf{P}\{\mathsf{X} = n\} \;&=\; \sum_{n=0}^{\infty} e^{-\lambda}\,\frac{\lambda^n}{n!} \\
&=\; e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \\
&=\; e^{-\lambda} \cdot e^{\lambda} \\
&=\; 1\,.
\end{aligned}$$

So this random variable does have total probability equal to 1, as is required.

For this random variable $\mathbf{E}[\mathsf{X}] = \lambda$ and $\mathrm{Var}(\mathsf{X}) = \lambda$, too.

This random variable models “rare events”.

Normal distribution:

The most ubiquitous distribution is the normal random variable.

This has parameters μ and σ^2.

This is a continuous random variable X, such that

$$g_{\mathsf{X}}(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}.$$

It has mean $\mu_{\mathsf{X}} = \mu$ and variance $\sigma^2_{\mathsf{X}} = \sigma^2$.

It comes up a lot because of the "Central Limit Theorem".

We will talk about these things more next time.

When we talk about a "standard" normal random variable, then we set $\mu = 0$ and $\sigma^2 = 1$.