Lecture Notes for Math 210 – 17 October 2007

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17 October 2007

Chapter 4: Pricing Derivatives

Risk-free portfolio for a call option

Let C_t be price of a call option.

We are looking for a function $f(x,t)$ so that we can write $C_t = f(S_t,t)$.

Risk-free portfolio value at time t (discounted):

$$e^{-rt}\,X_t \;=\; e^{-rt}\,f(S_t,t) + \int_0^t \Delta(S_\tau,\tau)\,d[e^{-rt}\,S_t]\,,$$

where $\Delta(x,t)$ comes out of Δ-hedging.

We need to find both $f(x,t)$ and $\Delta(x,t)$.

Since X_t is risk-free, we get $d[e^{-rt}X_t] = 0$.

Leads to

$$\boxed{-rf(S_t,t)\,dt + d[f(S_t,t)] - \Delta(S_t,t)\cdot(-rS_t\,dt + dS_t) \;=\; 0.}$$

Assuming S_t is differentiable lets us use the chain-rule for calculating $d[f(S_t,t)]$, and also rewrite $dS_t = \frac{d}{dt}(S_t)\,dt$.

Gives

$$\left[-rf(S_t,t)+\frac{\partial f}{\partial t}(S_t,t)-rS_t\Delta(S_t,t)\right]+\left[\frac{\partial f}{\partial x}(S_t,t)-\Delta(S_t,t)\right]\frac{d}{dt}(S_t) \;=\; 0$$

But this is wrong.

(1) Only 1 differential equation, but 2 unknown functions $f(x,t)$ and $\Delta(x,t)$.

(2) We *know* S_t is not differentiable just by looking at a newpaper.

If we revert to writing dS_t instead of $\frac{d}{dt}(S_t)\,dt$, but we keep using the chain-rule, then we get

$$\left[-rf(S_t,t)+\frac{\partial f}{\partial t}(S_t,t)-rS_t\Delta(S_t,t)\right]dt+\left[\frac{\partial f}{\partial x}(S_t,t)-\Delta(S_t,t)\right]dS_t \;=\; 0\,.$$

Both coefficients must be 0, so

$$-rf(S_t,t)+\frac{\partial f}{\partial t}(S_t,t)-rS_t\Delta(S_t,t) \;=\; 0\,,$$

and

$$\frac{\partial f}{\partial x}(S_t,t)-\Delta(S_t,t) \;=\; 0\,.$$

By the second equation, we know that

$$\boxed{\Delta(x,t) \;=\; \frac{\partial f}{\partial x}(x,t)\,.}$$

This is always true for a derivative that depends an asset in a differentiable way.

(There is an analogous formula for the lattice model which is a discrete model, not a continuous one.)

This also matches Hull's *definition* of Δ.

Plugging this in for $\Delta(x,t)$ in first equation gives

$$\begin{aligned} & -rf(x,t)+\frac{\partial f}{\partial t}(x,t)-rx\Delta(x,t) \;=\; 0 \\ \Rightarrow \qquad & -rf(x,t)+\frac{\partial f}{\partial t}(x,t)-rx\frac{\partial f}{\partial x}(x,t) \;=\; 0\,, \end{aligned}$$

This is a PDE for the function $f(x,t)$.

It is similar to one you solved in your homework.

Solving it, gives the formula for $C_t = f(S_t,t)$.

It is easy to solve this PDE:

$$f(x,t) \;=\; e^{-r(T-t)}\,f(e^{r(T-t)}x,T)\,. \qquad (!\#\$\%)$$

We have solved in terms of $f(*,T)$ because we know $f(S_T,T) = C_T = \max(S_T - K, 0)$.

Problem: Equation $(!\#\$\%)$ is <u>*not*</u> the Black-Scholes formula.

Something is wrong!

<u>*Correcting the correction. The Black-Scholes formula: Version II.*</u>

When we assumed that S_t was differentiable we made 2 mistakes.

One was in writing $dS_t = \frac{d}{dt}(S_t)\,dt$ to combine dt and dS_t.

We have now fixed that problem.

The second mistake was in using the chain-rule to rewrite $d[f(S_t,t)]$.

The formula,

$$d[f(S_t,t)] \;=\; \frac{\partial f}{\partial x}(S_t,t)\,dS_t+\frac{\partial f}{\partial t}(S_t,t)\,dt\,,$$

only works if $(dS_t)^2 = 0$.

If $(dS_t)^2 \neq 0$, we need a 2nd order partial derivative term, coming from Taylor's formula:

$$d[f(S_t,t)] \;=\; \frac{\partial f}{\partial x}(S_t,t)\, dS_t + \frac{\partial f}{\partial t}(S_t,t)\, dt + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}(S_t,t)\,(dS_t)^2\,.$$

It turns out that, for the *standard* model of a stock, dS_t is not differentiable, and $(dS_t)^2$ is not 0.

Instead $(dS_t)^2 = \sigma^2 (S_t)^2\, dt$,

where σ^2 is a measure of the volatility of the stock:

$\mathrm{Var}(S_{t+\Delta t} - S_t) = \sigma^2 (S_t)^2 \Delta t$.

The fact that $(dS_t)^2$ is not zero is very weird, and it can only be true because S_t is not differentiable.

I may call this "Brownian weirdness" (in analogy with the even stranger behaviors in physics related to "quantum weirdness"). Plugging this in does not affect the formula

$$\Delta(x,t) \;=\; \frac{\partial f}{\partial x}(x,t)\,.$$

That formula is always true, and it is why Hull is correct in *defining* Δ to be the partial derivative.

But it does affect the other equation, for $f(x,t)$.

Instead we get

$$\boxed{\frac{\partial f}{\partial t}(x,t) + rx\,\frac{\partial f}{\partial x}(x,t) + \frac{\sigma^2 x^2}{2}\cdot\frac{\partial^2 f}{\partial x^2}(x,t) - rf(x,t) \;=\; 0.}$$

This is the Black-Scholes equation.

It is also *easy* to solve this equation in terms of the heat-kernel that we investigated in the homework.

Actually the heat-kernel is the probability density function for a Normal/Gaussian random variable.

So we can solve this in terms of the expectation of a normal random variable, instead.

(The Black-Scholes formula uses the "error-function".)

But first we have to justify everything that we did today, using stochastic integrals, and Itô's formula.

Afterward:

Taking the zero-volatility limit, $\sigma = 0$, the Black-Scholes PDE reduces to the one we solved before.

The formula,

$$f(x,t) \;=\; e^{-r(T-t)}\, f(e^{r(T-t)}x, T)\,,$$

is similar to one of the problems in your homework.

It gives the price for another unusual derivative: a call option on a bond.

$U_T = \max(B_T - K, 0) = f(B_T, T)$.

That is because a bond is the zero-volatility limit of a stock.

Chapter 5: Tools in Probability Theory

Probability Sample space: Ω.

Outcome: ω.

New idea: Event space $\mathcal{F}$.

(The symbol $\mathcal{F}$ stands for "σ-field". We will not really define what that means.)

$\mathcal{F}$ consists of "measurable events", which are subsets of Ω.

Probability measure: a function $P : \mathcal{F} \to [0,1]$ satisfying the axioms of probability.

Neftci has some *nasty* notation.

If we ever integrate over P it will be something like

$$\int_{\Omega} dP(\omega) \;=\; 1\,,$$

NOT WHAT HE WROTE FOR EQUATION (2) ON PAGE 92!

A random variable is a (measurable) function $X : \Omega \to \mathbb{R}$.

Then we define the cumulative distribution function $G_{\mathsf{X}}(a) = P\{\mathsf{X} \le a\}$ for every $a \in \mathbb{R}$.

This is a nondecreasing function which is right-continuous.

$$\boldsymbol{E}[\mathsf{X}] \;=\; \int_{-\infty}^{\infty} x\, dG_{\mathsf{X}}(x)\,.$$

More generally,

$$\boldsymbol{E}[f(\mathsf{X})] \;=\; \int_{-\infty}^{\infty} f(x)\, dG_{\mathsf{X}}(x)\,.$$

Define the probability density function $g_{\mathsf{X}}(x) = \frac{d}{dx}(G_{\mathsf{X}}(x))$ if it exists.

Then if the p.d.f. exists,

$$\boldsymbol{E}[\mathsf{X}] \;=\; \int_{-\infty}^{\infty} x\, g_{\mathsf{X}}(x)\, dx\,, \quad \text{and} \quad \boldsymbol{E}[f(\mathsf{X})] \;=\; \int_{-\infty}^{\infty} f(x)\, g_{\mathsf{X}}(x)\, dx\,.$$