Lecture Notes for Math 210 – 15 October 2007

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Chapter 4: Pricing Derivatives

Risk-free portfolio for a call option

Let C_t be price of a call option.

Suppose $C_t = f(S_t, t)$ for a "nice" function $f(x, t)$.

Δ-hedging (mathematical idealization):

at each time $t \in [0, T)$, short-sell an amount $\Delta(S_t, t)$ of shares of stock,

to buy-back short time later, $t + dt$.

Discounted value of call option at time t:

$$e^{-rt} C_t \, = \, e^{-rt} f(S_t, t)\,.$$

Discounted value of short-sells at time t:

$$-\int_0^t \Delta(S_\tau, \tau)\, d[e^{-r\tau} S_\tau]\,.$$

Discounted value of risk-free portfolio:

$$e^{-rt} X_t \;=\; e^{-rt} f(S_t, t) - \int_0^t \Delta(S_\tau, \tau)\, d[e^{-r\tau} S_\tau]\,.$$

Risk-free condition: $d[e^{-rt} X_t] = 0$.

$$\boxed{d[e^{-rt} f(S_t, t)] - \Delta(S_t, t)\, d[e^{-rt} S_t] \;=\; 0.}$$

Product rule: $d[e^{-rt} f(S_t, t)] \;=\; e^{-rt}\left(-rf(S_t, t)\, dt + d[f(S_t, t)]\right)$

Product rule: $d[e^{-rt} S_t] \;=\; e^{-rt}\left(-rS_t\, dt + dS_t\right)$

$$\boxed{-rf(S_t, t)\, dt + d[f(S_t, t)] - \Delta(S_t, t) \cdot (-rS_t\, dt + dS_t) \;=\; 0.}$$

<u>*Naive interpretation: S_t is differentiable*</u>

Suppose S_t is differentiable.

Chain rule:

$$d[f(S_t, t)] \;=\; \frac{\partial f}{\partial x}(S_t, t)\, dS_t + \frac{\partial f}{\partial t}(S_t, t)\, dt\,.$$

So

$$-rf(S_t, t)\, dt + \frac{\partial f}{\partial x}(S_t, t)\, dS_t + \frac{\partial f}{\partial t}(S_t, t)\, dt - \Delta(S_t, t) \cdot (-rS_t\, dt + dS_t) \;=\; 0\,.$$

Collecting terms involving dt and dS_t together:

$$\left[-rf(S_t, t) + \frac{\partial f}{\partial t}(S_t, t) - rS_t \Delta(S_t, t)\right] dt + \left[\frac{\partial f}{\partial x}(S_t, t) - \Delta(S_t, t)\right] dS_t \;=\; 0\,.$$

Also $dS_t = \frac{d}{dt}(S_t)\, dt$. So

$$\left(\left[-rf(S_t, t) + \frac{\partial f}{\partial t}(S_t, t) - rS_t \Delta(S_t, t)\right] + \left[\frac{\partial f}{\partial x}(S_t, t) - \Delta(S_t, t)\right] \frac{d}{dt}(S_t)\right) dt \;=\; 0\,.$$

In order for this to be 0, we should make the coefficient of dt equal to 0:

$$\left[-rf(S_t,t) + \frac{\partial f}{\partial t}(S_t,t) - rS_t\Delta(S_t,t)\right] + \left[\frac{\partial f}{\partial x}(S_t,t) - \Delta(S_t,t)\right]\frac{d}{dt}(S_t) \;=\; 0$$

This is a PDE.

Two problems:

(1) There is only 1 differential equation, but 2 unknown functions: $f(x,t)$ and $\Delta(x,t)$.

(2) S_t is *not* differentiable.

Can use problem (2) to solve problem (1).

(This still leaves problem (2), however.)

A semi-stochastic correction: zero-volatility limit

Since S_t is not differentiable cannot write dS_t as $\frac{d}{dt}(S_t)\,dt$.

Cannot combine dS_t and dt terms.

Since their sum is 0, both coefficients must equal 0.

$$\left[-rf(S_t,t) + \frac{\partial f}{\partial t}(S_t,t) - rS_t\Delta(S_t,t)\right] dt + \left[\frac{\partial f}{\partial x}(S_t,t) - \Delta(S_t,t)\right] dS_t \;=\; 0\,.$$

So must have

$$-rf(S_t,t) + \frac{\partial f}{\partial t}(S_t,t) - rS_t\Delta(S_t,t) \;=\; 0\,,$$

and

$$\frac{\partial f}{\partial x}(S_t,t) - \Delta(S_t,t) \;=\; 0\,.$$

Replacing S_t by x, this leads to 2 coupled differential equations:

$$-rf(x,t) + \frac{\partial f}{\partial t}(x,t) - rx\Delta(x,t) \;=\; 0\,,$$

and

$$\frac{\partial f}{\partial x}(x,t) - \Delta(x,t) \;=\; 0\,.$$

This is actually *good*, because we also have 2 unknown functions: $\Delta(x,t)$ and $f(x,t)$.

We can hope to solve 2 differential equations in 2 unknown functions

Q: Consider 2 equations in 2 unknowns:

$$ax + by \;=\; e\,,$$
$$cx + dy \;=\; f\,,$$

where a, b, c, d, e, f are constants.

What is the condition to have a unique solution?

A: That $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \neq 0$.

By a standard approach, let us eliminate $\Delta(x,t)$ first.

Using second equation:

$$\frac{\partial f}{\partial x}(x,t) - \Delta(x,t) \;=\; 0 \qquad \Rightarrow \qquad \Delta(x,t) \;=\; \frac{\partial f}{\partial x}(x,t)\,.$$

The formula

$$\boxed{\Delta(x,t) \;=\; \frac{\partial f}{\partial x}(x,t)}$$

is always true.

This also matches Hull's *definition* of Δ.

Plugging this in for $\Delta(x,t)$ in first equation gives

$$\begin{aligned} & -rf(x,t) + \frac{\partial f}{\partial t}(x,t) - rx\Delta(x,t) \;=\; 0 \\ \Rightarrow \qquad & -rf(x,t) + \frac{\partial f}{\partial t}(x,t) - rx\frac{\partial f}{\partial x}(x,t) \;=\; 0\,, \end{aligned}$$

This is a PDE for the function $f(x,t)$.

It is similar to one you solved in your homework.

Solving it, gives the formula for $C_t = f(S_t, t)$.

It is easy to solve this PDE:

$$f(x,t) \;=\; e^{-r(T-t)}\, f(e^{r(T-t)}x, T)\,. \qquad (!\#\$\%)$$

We have solved in terms of $f(*, T)$ because we know $f(S_T, T) = C_T = \max(S_T - K, 0)$.

Problem: Equation (!#$%) is <u>*not*</u> the Black-Scholes formula.

Something is wrong!

<u>*Correcting the correction. The Black-Scholes formula: Version II.*</u>

When we assumed that S_t was differentiable we made 2 mistakes.

One was in writing $dS_t = \frac{d}{dt}(S_t)\,dt$ to combine dt and dS_t.

We have now fixed that problem.

The second mistake was in using the chain-rule to rewrite $d[f(S_t, t)]$.

The formula,

$$d[f(S_t,t)] \;=\; \frac{\partial f}{\partial x}(S_t,t)\,dS_t + \frac{\partial f}{\partial t}(S_t,t)\,dt\,,$$

only works if $(dS_t)^2 = 0$.

If $(dS_t)^2 \neq 0$, we need a 2nd order partial derivative term, coming from Taylor's formula:

$$d[f(S_t,t)] \;=\; \frac{\partial f}{\partial x}(S_t,t)\,dS_t + \frac{\partial f}{\partial t}(S_t,t)\,dt + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}(S_t,t)\,(dS_t)^2\,.$$

It turns out that, for the *standard* model of a stock, dS_t is not differentiable, and $(dS_t)^2$ is not 0.

Instead $(dS_t)^2 = \sigma^2 (S_t)^2\,dt$,

where σ^2 is a measure of the volatility of the stock:

$\mathrm{Var}(S_{t+\Delta t} - S_t) = \sigma^2 (S_t)^2 \Delta t$.

(This will also be related to the quadratic variation of S_t.)

Plugging this in does not affect the formula

$$\Delta(x,t) \;=\; \frac{\partial f}{\partial x}(x,t)\,.$$

That formula is always true, and it is why Hull is correct in *defining* Δ to be the partial derivative.

But it does affect the other equation, for $f(x,t)$.

Instead we get

$$\boxed{\frac{\partial f}{\partial t}(x,t)+rx\,\frac{\partial f}{\partial x}(x,t)+\frac{\sigma^2x^2}{2}\cdot\frac{\partial^2 f}{\partial x^2}(x,t)-rf(x,t)\;=\;0.}$$

This is the Black-Scholes equation.

It is also *easy* to solve this equation in terms of the heat-kernel that we investigated in the homework.

Actually the heat-kernel is the probability density function for a Normal/Gaussian random variable.

So we can solve this in terms of the expectation of a normal random variable, instead.

(The Black-Scholes formula uses the "error-function".)

But first we have to justify everything that we did today, using stochastic integrals, and Itô's formula.

Afterward:

Taking the zero-volatility limit, $\sigma = 0$, the Black-Scholes PDE reduces to the one we solved before.

The formula,

$$f(x,t)\;=\;e^{-r(T-t)}\,f(e^{r(T-t)}x,T)\,,$$

is similar to one of the problems in your homework.

It gives the price for another unusual derivative: a call option on a bond.

$U_T = \max(B_T - K, 0) = f(B_T, T)$.

That is because a bond is the zero-volatility limit of a stock.