

Lecture Notes for Math 210 – 12 October 2007

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Chapter 4: Pricing Derivatives

Risk-free portfolio for a call option

Real strategy:

Break $[0, T]$ into discrete time-steps:

$[t_i, t_i + \Delta t]$ for $0 \leq i \leq N - 1$

1. Buy 1 call option at $t = 0$.

Ignore fee.

2. For each i with $0 \leq i \leq N - 1$,

short sell $\Delta = \Delta(S_{t_i}, t_i)$ shares of stock
at time t_i .

3. Buy back $\Delta(S_{t_i}, t_i)$ shares of stock
at time $t_i + \Delta t$.

Repeat until time T .

Mathematical Idealization:

Use infinitesimal times steps,

$[\tau, \tau + d\tau]$ for $0 \leq \tau < T$.

1. Buy 1 call option at $t = 0$.

Ignore fee.

2. For each time τ with $0 \leq \tau < T$,

short sell $\Delta = \Delta(S_\tau, \tau)$ shares of stock
at time τ .

3. Buy back $\Delta(S_\tau, \tau)$ shares of stock
at time $\tau + d\tau$.

Continue for all $0 \leq \tau < T$.

Q: On average, do you expect the short-sells to make you money, or lose you money?

A: *Tricky question.*

Sometimes they make money, sometimes they lose money.

But, on average, we think that stock does better than bonds.

That is a tenet of “rational market theory” that is discussed in *Malkiel*.

Given this, the short-sells lose money on average.

But you can continue to make trades without actually losing money

because the portfolio finances itself.

It is *self-financing*.

Whenever the short-sell loses money, because stock rises in price,

the call option also rises in price, and this profit pays for loss in short-sell.

For this reason, such a portfolio called “self-financing portfolio” sometimes,

instead of a “risk-free portfolio”.

The Black-Scholes PDE: Version I.

In real-world situation, let X_{t_k} be value of portfolio

at time $t_k = k\Delta t$, for $1 \leq k \leq N$.

Try to choose $\Delta(x, t)$ so that X_{t_k} is *risk-free*.

$$X_{t_k} = C_{t_k} + \sum_{i=0}^{k-1} \Delta(S_{t_i}, t_i) [e^{r(t_k-t_i)} S_{t_i} - e^{r(t_k-t_i-\Delta t)} S_{t_i+\Delta t}].$$

Discounting,

$$\begin{aligned} e^{-rt_k} X_{t_k} &= e^{-rt_k} C_{t_k} + \sum_{i=0}^{k-1} \Delta(S_{t_i}, t_i) [e^{-rt_i} S_{t_i} - e^{-r(t_i+\Delta t)} S_{t_i+\Delta t}] \\ &= e^{-rt_k} f(S_{t_k}, t_k) - \sum_{i=0}^{k-1} \Delta(S_{t_i}, t_i) \Delta[e^{-rt_i} S_{t_i}], \end{aligned}$$

where

$$\Delta[e^{-rt_i} S_{t_i}] = e^{-r(t_i+\Delta t)} S_{t_i+\Delta t} - e^{-rt_i} S_{t_i}.$$

So

$$\Delta[e^{-rt_k} X_{t_k}] = \Delta[e^{-rt_k} f(S_{t_k}, t_k)] - \Delta(S_{t_k}, t_k) \Delta[e^{-rt_k} S_{t_k}].$$

For risk-free asset this should be 0:

$$\Delta[e^{-rt_k} f(S_{t_k}, t_k)] - \Delta(S_{t_k}, t_k) \Delta[e^{-rt_k} S_{t_k}] = 0.$$

Mathematical idealization:

For $0 \leq t \leq T$,

$$\begin{aligned} e^{-rt} X_t &= e^{-rt} C_t - \int_0^t \Delta(S_\tau, \tau) d[e^{-r\tau} S_\tau] \\ &= e^{-rt} f(S_t, t) - \int_0^t \Delta(S_\tau, \tau) d[e^{-r\tau} S_\tau]. \end{aligned}$$

So

$$d[e^{-rt} X_t] = d[e^{-rt} f(S_t, t)] - \Delta(S_t, t) d[e^{-rt} S_t].$$

If X_t is risk-free, then $d[e^{-rt} X_t] = 0$

(just like $d[e^{-rt} B_t] = d[B_0] = 0$).

So

$$d[e^{-rt} f(S_t, t)] - \Delta(S_t, t) d[e^{-rt} S_t] = 0.$$

This equation encodes all of the information of the Black-Scholes equation.

It is simply a very short version of the Black-Scholes equation.

But we have to know how to decode it.

Naive interpretation: Treat S_t as differentiable.

Remember the product rule:

$$\frac{d}{dx}[f(x)g(x)] = \frac{df}{dx}(x) g(x) + f(x) \frac{dg}{dx}(x)$$

This translates to

$$d[f(x, t)g(x, t)] = f(x, t) d[g(x, t)] + g(x, t) d[f(x, t)] ,$$

for total derivatives, where remember

$$d[f(x, t)] = \frac{\partial f}{\partial x}(x, t) dx + \frac{\partial f}{\partial t}(x, t) dt ,$$

and similarly for $d[g(x, t)]$.

So

$$\begin{aligned} d[e^{-rt}f(S_t, t)] &= f(S_t, t) d[e^{-rt}] + e^{-rt} d[f(S_t, t)] \\ &= -rf(S_t, t) e^{-rt} dt + e^{-rt} d[f(S_t, t)] \\ &= e^{-rt} (-rf(S_t, t) dt + d[f(S_t, t)]) . \end{aligned}$$

Similarly,

$$\begin{aligned} d[e^{-rt}S_t] &= S_t d[e^{-rt}] + e^{-rt} dS_t \\ &= -rS_t e^{-rt} dt + e^{-rt} dS_t \\ &= e^{-rt} (-rS_t dt + dS_t) . \end{aligned}$$

So putting these together, and using previous equation,

$$e^{-rt} (-rf(S_t, t) dt + d[f(S_t, t)] - \Delta(S_t, t) \cdot (-rS_t dt + dS_t)) = 0,$$

or just

$$\boxed{-rf(S_t, t) dt + d[f(S_t, t)] - \Delta(S_t, t) \cdot (-rS_t dt + dS_t) = 0.}$$

This equation is true, because so far we have avoided the assumption that S_t is differentiable.

Let us now use that assumption.

This has two consequences:

First, we can use the chain-rule/change-of-variables formula.

These two formulas are two sides of the same coin:

chain-rule works for derivatives, change-of-variables for integrals.

But it gives

$$d[f(S_t, t)] = \frac{\partial f}{\partial x}(S_t, t) dS_t + \frac{\partial f}{\partial t}(S_t, t) dt.$$

So

$$-rf(S_t, t) dt + \frac{\partial f}{\partial x}(S_t, t) dS_t + \frac{\partial f}{\partial t}(S_t, t) dt - \Delta(S_t, t) \cdot (-rS_t dt + dS_t) = 0.$$

Collecting terms involving dt and dS_t together:

$$\left[-rf(S_t, t) + \frac{\partial f}{\partial t}(S_t, t) - rS_t \Delta(S_t, t) \right] dt + \left[\frac{\partial f}{\partial x}(S_t, t) - \Delta(S_t, t) \right] dS_t = 0.$$

The second consequence of assuming that S_t is differentiable is that we can combine

dt and dS_t terms.

We can rewrite dS_t as $\frac{d}{dt}(S_t) dt$.

So

$$\left(\left[-rf(S_t, t) + \frac{\partial f}{\partial t}(S_t, t) - rS_t\Delta(S_t, t) \right] + \left[\frac{\partial f}{\partial x}(S_t, t) - \Delta(S_t, t) \right] \frac{d}{dt}(S_t) \right) dt = 0.$$

In order for this to be 0, we should make the coefficient of dt equal to 0:

$$\left[-rf(S_t, t) + \frac{\partial f}{\partial t}(S_t, t) - rS_t\Delta(S_t, t) \right] + \left[\frac{\partial f}{\partial x}(S_t, t) - \Delta(S_t, t) \right] \frac{d}{dt}(S_t) = 0$$

This is a PDE.

Two problems:

(1) There is only 1 differential equation, but 2 unknown functions: $f(x, t)$ and $\Delta(x, t)$.

(2) S_t is *not* differentiable.

Can use problem (2) to solve problem (1).

(This still leaves problem (2), however.)

A semi-stochastic correction: zero-volatility limit

Since S_t is not differentiable cannot write dS_t as $\frac{d}{dt}(S_t) dt$.

Cannot combine dS_t and dt terms.

Since their sum is 0, both coefficients must equal 0. So both coefficients must be 0.

$$\left[-rf(S_t, t) + \frac{\partial f}{\partial t}(S_t, t) - rS_t\Delta(S_t, t) \right] dt + \left[\frac{\partial f}{\partial x}(S_t, t) - \Delta(S_t, t) \right] dS_t = 0.$$

So must have

$$-rf(S_t, t) + \frac{\partial f}{\partial t}(S_t, t) - rS_t\Delta(S_t, t) = 0,$$

and

$$\frac{\partial f}{\partial x}(S_t, t) - \Delta(S_t, t) = 0.$$

Replacing S_t by x , this leads to 2 coupled differential equations:

$$-rf(x, t) + \frac{\partial f}{\partial t}(x, t) - rx\Delta(x, t) = 0,$$

and

$$\frac{\partial f}{\partial x}(x, t) - \Delta(x, t) = 0.$$

This is actually *good*, because we also have 2 unknown functions: $\Delta(x, t)$ and $f(x, t)$.

We can hope to solve 2 differential equations in 2 unknown functions (just like in matrix analysis when we can generally solve 2 equations in 2 unknown variables).

By standard approaches, let us eliminate $\Delta(x, t)$ first.

Using second equation:

$$\frac{\partial f}{\partial x}(x, t) - \Delta(x, t) = 0 \quad \Rightarrow \quad \Delta(x, t) = \frac{\partial f}{\partial x}(x, t).$$

Plugging this in for $\Delta(x, t)$ in first equation gives

$$\begin{aligned} -rf(x, t) + \frac{\partial f}{\partial t}(x, t) - rx\Delta(x, t) &= 0 \\ \Rightarrow -rf(x, t) + \frac{\partial f}{\partial t}(x, t) - rx\frac{\partial f}{\partial x}(x, t) &= 0, \end{aligned}$$

This is a PDE for the function $f(x, t)$.

It is similar to one you solved in your homework.

Solving it, gives the formula for $C_t = f(S_t, t)$.

It is easy to solve this PDE:

$$f(x, t) = e^{-r(T-t)} f(e^{r(T-t)}x, T). \quad (!\#\$\%)$$

We have solved in terms of $f(*, T)$ because we know $f(S_T, T) = C_T$:

it is the payoff at time T , which *defines* the derivative.

Problem: Equation (!\#\\$\%) is not the Black-Scholes formula.

Something is wrong!

Correcting the correction. The Black-Scholes formula: Version II. When we assumed

that S_t was differentiable we made 2 mistakes.

One was in writing $dS_t = \frac{d}{dt}(S_t) dt$ to combine dt and dS_t .

We have now fixed that problem.

The second mistake was in using the chain-rule to rewrite $d[f(S_t, t)]$ in terms of partial derivatives.

The formula,

$$d[f(S_t, t)] = \frac{\partial f}{\partial x}(S_t, t) dS_t + \frac{\partial f}{\partial t}(S_t, t) dt,$$

only works if $(dS_t)^2 = 0$.

If $(dS_t)^2 \neq 0$, we need a 2nd order partial derivative term, coming from Taylor's formula:

$$d[f(S_t, t)] = \frac{\partial f}{\partial x}(S_t, t) dS_t + \frac{\partial f}{\partial t}(S_t, t) dt + \frac{1}{2} \cdot \frac{\partial^2 f}{\partial x^2}(S_t, t) (dS_t)^2.$$

It turns out that, for the *standard* model of a stock, dS_t is not differentiable, and $(dS_t)^2$ is not 0.

Instead $(dS_t)^2 = \sigma^2(S_t)^2 dt$,

where σ^2 is a measure of the volatility of the stock:

$$\text{Var}(S_{t+\Delta t} - S_t) = \sigma^2(S_t)^2 \Delta t.$$

(This will also be related to the quadratic variation of S_t .)

Plugging this in does not affect the formula

$$\Delta(x, t) = \frac{\partial f}{\partial x}(x, t).$$

That formula is always true, and it is why Hull is correct in *defining* Δ to be the partial derivative.

But it does affect the other equation, for $f(x, t)$.

Instead we get

$$\frac{\partial f}{\partial t}(x, t) + rx \frac{\partial f}{\partial x}(x, t) + \frac{\sigma^2 x^2}{2} \cdot \frac{\partial^2 f}{\partial x^2}(x, t) - rf(x, t) = 0.$$

This is the Black-Scholes equation.

It is also *easy* to solve this equation in terms of the heat-kernel that we investigated in the homework.

Actually the heat-kernel is the probability density function for a Normal/Gaussian random variable.

So we can solve this in terms of the expectation of a normal random variable, instead.

(The Black-Scholes formula uses the “error-function”.)

But first we have to justify everything that we did today, using stochastic integrals, and Itô’s formula.

Afterward:

Taking the zero-volatility limit, $\sigma = 0$, the Black-Scholes PDE reduces to the one we solved before.

The formula,

$$f(x, t) = e^{-r(T-t)} f(e^{r(T-t)}x, T),$$

is similar to one of the problems in your homework.

It gives the price for another unusual derivative: a call option on a bond.

$$U_T = \max(B_T - K, 0) = f(B_T, T).$$

That is because a bond is the zero-volatility limit of a stock.

Note a bond must earn less return, on average, than a stock, in order to compensate the investor for risk.

But that is part of the description that we will learn from Malkiel.