Lecture Notes for Math 210 – 10 October 2007

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Chapter 4: Pricing Derivatives

Back-up and start again.

Call options: nonlinear derivatives.

Let r be the risk-free interest rate, for continuous compounding.

Let C_t be the price, at time t for a call option.

Assume $C_t = f(S_t, t)$ for some differentiable functions $f(x, t)$.

An example. Suppose at time $t = 0$ you buy a call option for $T = 1$ year, $K = \$100$.

Suppose $S_0 = \$90$.

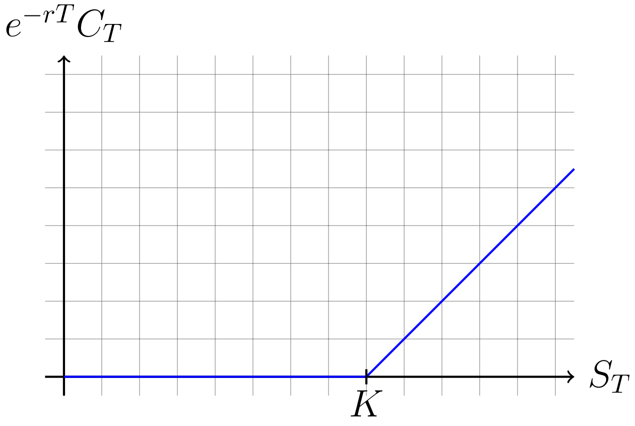
At time $t = 3$ months, $S_t = \$98$.

The higher S_T is, the greater your payoff at time T.

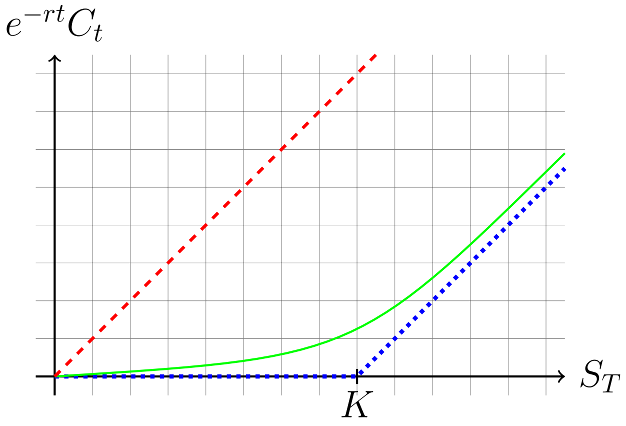
Q: What is the profit to you from the \$8 rise in the first 3 months?

It is more than \$8, equal to \$8, or less than \$8? Is it \$0?

A: The profit is bigger than \$0, but less than \$8.



Call option payoff, at time T, discounted to present day value, $t = 0$.



Call option price, at time t between 0 and T,

discounted to present day value, $t = 0$;

with payoff function overlaid in dotted blue line,

and the spot price of the stock overlaid in dashed red line.

The payoff of the call option is $C_T = \max(S_T - K, 0)$.

Therefore $C_T \leq S_T$.

So for all times t, we should have $C_t \leq S_t$.

Similarly, for $f(x,t)$ such that $C_t = f(S_t, t)$, we should have

$$0 \leq \frac{\partial f}{\partial x}(x,t) \leq 1 \,.$$

Consequence: Changes in the stock price lead to somewhat smaller changes in the call option price.

If we want to hedge our risk in owning a call option on 1 share,

we should short-sell a smaller fraction $0 \leq \Delta \leq 1$ of shares of the stock.

This is called "Δ-hedging".

- *Hull's definition.* **Hedge** A trade designed to reduce risk.
- *My definition.* **Hedging** Taking offsetting positions in two risky assets which are negatively correlated, in order to reduce the risk of each separately.
- *Hull's definition.* **Delta Hedging** A hedging scheme that is designed to make the price of a portfolio of derivatives insensitive to small changes in the underlying asset.
- *Hull's definition.* **Delta** The rate of change of the price of a derivative with the price of the underlying asset.

(Note: It is okay that Δ is a fraction or decimal, we can simply "scale-up" by some multiple of the portfolio to get approximate integers.)

As we will see shortly, Δ *is exactly the partial derivative of* f *in the* x *direction, which we also know is between* 0 *and* 1.

Risk-free portfolio:

1. Buy 1 call option (but we ignore the fee).

In other words, you are long 1 call option.

2. Fix N large. Break up the time interval $[0, T]$ into N equal subintervals:

$\Delta T = \frac{T}{N}$, $t_i = i\Delta t$ for $i = 0, 1, \ldots, N$.

- At each time t_i, for $0 \leq i \leq N-1$, call stockbroker,

  give the order to short-sell $\Delta = \Delta(S_{t_i}, t_i)$ shares of stock,

  at price S_{t_i}. Earns $\Delta \cdot S_{t_i}$ at time t_i.

- At time $t_i + \Delta t$, buy back $\Delta(S_{t_i}, t_i)$ shares

  at price $S_{t_i+\Delta t}$ to repay the short-sell.

  Costs $\Delta \cdot S_{t_i+\Delta t}$ at time $t_i + \Delta t$.

Do this continually for each time step $t_1, \ldots, t_N$.

Q: When you buy a call option, are you betting on the stock going up, or down?

What about when you short-sell a stock?

3. Try to choose the function $\Delta(x, t)$ so that portfolio value X_t is risk-free.

Let X_{t_k} be the value of the portfolio at time t_k for $1 \leq k \leq N$.

Then, discounting,

$$\begin{aligned} e^{-rt_k} X_{t_k} &= e^{-rt_k} C_{t_k} + \sum_{i=0}^{k-1} \Delta(S_{t_i}, t_i) \cdot [e^{-rt_i} S_{t_i} - e^{-r(t_i+\Delta t)} S_{t_i+\Delta t}] \\ &= e^{-rt_k} f(S_{t_k}, t_k) - \sum_{i=0}^{k-1} \Delta(S_{t_i}, t_i) \cdot \Delta[e^{-rt_i} S_{t_i}] \end{aligned}$$

TERRIBLE NOTATION!

If you take $N \to \infty$ limit, get

$$e^{-rt} X_t \;=\; e^{-rt} f(S_t, t) - \int_0^t \Delta(S_\tau, \tau)\, d[e^{-r\tau} S_\tau]\,,$$

for all $t \in [0, T]$.

So (by the Fundamental Theorem of Calculus)

$$\begin{aligned} d[e^{-rt} X_t] \;&=\; d[e^{-rt} f(S_t, t)] - d\left(\int_0^t \Delta(S_\tau, \tau)\, d[e^{-r\tau} S_\tau] \right) \\ &=\; d[e^{-rt} f(S_t, t)] - \Delta(S_t, t)\, d[e^{-rt} S_t]\,. \end{aligned}$$

This does not represent reality because the stockbroker cannot really make transactions continuously.

Cannot really take $\Delta t \to 0$, but presumably can get "close".

In the mathematical idealization, this is the limit.

Fact: Let r be the risk-free interest rate.

Let X_t be the price, at time t, for a risk-free asset.

Then $d[e^{-rt} X_t] = 0$.

For example, for a risk-free bond, $B_t = e^{rt} B_0$, so $e^{-rt} B_t = B_0$ which is constant.

So $d[e^{-rt} B_t] = d[B_0] = 0$.

So if the value of the portfolio is really risk-free,

then we must have $d[e^{-rt} X_t] = 0$.

$$\boxed{d[e^{-rt} f(S_t, t)] - \Delta(S_t, t)\, d[e^{-rt} S_t] \;=\; 0.}$$

Surprisingly, this equation, all by itself, tells us everything we want to know.

The stock price S_t is *not* differentiable.

But let us ignore that problem, and take derivatives anyway,

using the ordinary chain-rule:

$$\begin{aligned} d[e^{-rt}f(S_t,t)] &= f(S_t,t)\,d[e^{-rt}] + e^{-rt}\,d[f(S_t,t)] \\ &= f(S_t,t)\,(-re^{-rt}\,dt) + e^{-rt}\left(\frac{\partial f}{\partial x}(S_t,t)\,dS_t + \frac{\partial f}{\partial t}(S_t,t)\,dt\right) \\ &= e^{-rt}\left(\left[\frac{\partial f}{\partial t}(S_t,t) - rf(S_t,t)\right]\,dt + \frac{\partial f}{\partial x}(S_t,t)\,dS_t\right)\ , \end{aligned}$$

and,

$$\begin{aligned} d[e^{-rt}S_t] &= S_t\,d[e^{-rt}] + e^{-rt}\,dS_t \\ &= S_t\,(-re^{-rt}\,dt) + e^{-rt}\,dS_t \\ &= e^{-rt}\,(-rS_t dt + dS_t)\ . \end{aligned}$$

So

$$\begin{aligned} d[e^{-rt}f(S_t,t)] + \Delta(S_t,t)\,d[e^{-rt}S_t] &= e^{-rt}\Bigg(\left[\frac{\partial f}{\partial t}(S_t,t) - rf(S_t,t) + rS_t\,\Delta(S_t,t)\right]\,dt \\ &\qquad + \left[\frac{\partial f}{\partial x}(S_t,t) - \Delta(S_t,t)\right]\,dS_t\Bigg) \;=\; 0 \end{aligned}$$

Two problems:

(1) There is only 1 equation, but 2 unknown functions: $f(x,t)$ and $\Delta(x,t)$.

(2) S_t is *not* differentiable.

Can use problem (2) to solve problem (1).

(This still leaves problem (2), however.)

If S_t were differentiable, then we would have $dS_t = \frac{d}{dt}(S_t)dt\,dt$.

So we would combine the dt and dS_t terms.

If S_t is not differentiable cannot combine them.

So both coefficients must be 0.

$$e^{-rt}\left(\left[\frac{\partial f}{\partial t}(S_t,t) - rf(S_t,t) + rS_t\,\Delta(S_t,t)\right]\,dt + \left[\frac{\partial f}{\partial x}(S_t,t) - \Delta(S_t,t)\right]\,dS_t\right) \;=\; 0$$

$$\begin{aligned} \Longrightarrow \quad & \frac{\partial f}{\partial t}(S_t,t) - rf(S_t,t) + rS_t\,\Delta(S_t,t) \;=\; 0\,, \qquad \text{and} \\ & \frac{\partial f}{\partial x}(S_t,t) - \Delta(S_t,t) \;=\; 0\,. \end{aligned}$$

Substitute x for S_t:

$$\frac{\partial f}{\partial t}(x,t) - rf(x,t) + rx\,\Delta(x,t) \;=\; 0\,, \qquad \text{and}$$

$$\frac{\partial f}{\partial x}(x,t) - \Delta(x,t) \;=\; 0\,.$$

The second equation leads to

$$\Delta(x,t) \;=\; \frac{\partial f}{\partial x}(x,t)\,.$$

So, plugging this in for $\Delta(x,t)$ to the first equation, gives

$$\frac{\partial f}{\partial t}(x,t) + rx\,\frac{\partial f}{\partial x}(x,t) - rf(x,t) \;=\; 0\,.$$

This is a first-order PDE, which we will solve (easily) much later.

The solutions of this equation gives the price for an unusual asset: a call option written on a bond.

But, it is not the right equation for the actual Black-Scholes formula.

That is because of "Brownian weirdness".

Earlier, we wrote

$$d[f(S_t,t)] \;=\; \frac{\partial f}{\partial x}(S_t,t)\,dS_t + \frac{\partial f}{\partial t}(S_t,t)\,dt\,.$$

But that uses the change-of-variables formula, which is only valid when S_t is differentiable, and $(dS_t)^2 = 0$.

When we talk about Brownian motion, we are going to want to consider non-differentiable S_t, for which $(dS_t)^2 \neq 0$.

Then, we will need 2nd order terms in the change-of-variables formula.

By applying the type of chain-rule arguments we considered last chapter,

$$d[f(S_t,t)] \;=\; \frac{\partial f}{\partial x}(S_t,t)\,dS_t + \frac{\partial f}{\partial t}(S_t,t)\,dt + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}(S_t,t)\,(dS_t)^2\,.$$

It turns out that for the most common model of stock prices, $(dS_t)^2 \;=\; \sigma^2\,S_t^2\,dt$, where σ^2 is a measure of the "volatility" or variance of changes in the stock price in short times.

Therefore, we get

$$d[f(S_t,t)] \;=\; \frac{\partial f}{\partial x}(S_t,t)\,dS_t + \frac{\partial f}{\partial t}(S_t,t)\,dt + \frac{\sigma^2 S_t^2}{2}\cdot\frac{\partial^2 f}{\partial x^2}(S_t,t)\,dt\,.$$

This does not affect the formula for Δ:

$$\Delta(x,t) \;=\; \frac{\partial f}{\partial x}(x,t)\,.$$

But it does change the other PDE from

$$\frac{\partial f}{\partial t}(x,t) + rx\,\frac{\partial f}{\partial x}(x,t) - rf(x,t) \;=\; 0\,,$$

to

$$\boxed{\frac{\partial f}{\partial t}(x,t)+rx\,\frac{\partial f}{\partial x}(x,t)+\frac{\sigma^2x^2}{2}\cdot\frac{\partial^2 f}{\partial x^2}(x,t)-rf(x,t) \;=\; 0.}$$

We will also solve this PDE (easily) much later,

to get the Black-Scholes equation for $f(x,t)$.

Along the way, we will learn how to take stochastic Itô integrals

to check that the $\Delta t \to 0$ limit of the risk-free portfolio really makes sense.