Lecture Notes for Math 210 – 3 October 2007

Shannon Starr

3 October 2007

Chapter 4: Pricing Derivatives

In order to price derivatives, we use the no-arbitrage principle.

1. self-financing (risk-free) portfolios;

2. synthetic "equivalent" martingales.

Digression: (Discounting)

Let us denote present time as $t = 0$.

So t will now mean a time in future.

If you will have a risk-free income of N_t at time t in the future,

then its present value is $e^{-rt}N_t$, where

r = risk-free interest rate with continuous compounding.

Example: A Bond with initial investment B_0 is worth $B_t e^{rt}$ at time t, risk-free.

So it is worth B_0 today. That is exactly right.

Basic principle: Money which is not invested in any other asset is always invested in a bond or other risk-free asset. That is the base-line.

Another principle: All risk-free assets with the same expiration date are equivalent.

Total derivatives for bonds: Since $B_t = B_0 e^{rt}$,

$$dB_t = d[B_0 e^{rt}] = B_0\, r\, e^{rt}\, dt = r B_t\, dt$$

Also, if we discount for present value, to get $e^{-rt} B_t$ then we just get a constant B_0. So

$$d[e^{-rt} B_t] = 0\,.$$

Example: Short sell. Suppose you short-sell a stock today.

Q: (a) What is the value to you at time t in the future, keeping track both of the money earned and your liability?

(b) If you discount that money to present time by multiplying by e^{-rt} what do you get?

(c) What is the present value of the short-sell?

A: (a) You earn S_0 today from the short-sell. But at time t you are liable for the stock, so you owe S_t. So your value at time t is $S_0 e^{rt} - S_t$.

(b) If you discount the way it said, then you get $S_0 - e^{-rt} S_t$.

(c) Actually the present value is 0. Otherwise the short-sell would be a foolish thing to do. That is because the present value of S_t at time t is S_0, not $e^{-rt} S_t$. But it is still useful to consider the discounted amount of $S_0 - e^{-rt} S_t$, for various purposes.

Note:

$$e^{-rt}S_t - S_0 = \int_0^t d[e^{-r\tau}S_\tau]$$

by the fundamental theorem of calculus, where τ is Greek t is stand-in for t in integral.

§4.2.2 Forwards. Linear derivatives.

Suppose we enter a contract wherein we will receive 1 share of stock at time T in future.

Ignore fee/fair-forward-price, momentarily.

Present time is $t = 0$.

The payoff at time T is S_T, spot price of stock at time T.

Q: What is the value of this contract to us, today?

A: It is worth S_0 to us today.

Risk-free portfolio:

- Long position in the contract. You will get S_T at time T.
- Short-sell 1 share of stock at present, to be repaid at time T. Earn S_0 today, risk-free.

Time T: You get 1 share of stock from contract, and use that to pay-back investor you borrowed from for short-sell. Whole portfolio is risk-free.

Present value is S_0: so value at time T is $e^{rT}S_0$.

Fair forward price (to be paid at time T) is $e^{rT}S_0$.

Q: Same contract: At time T you will get S_T. What is the value at a time t for $0 < t \leq T$?

A: It is worth S_t at time t.

Portfolio is risk-free.

Portfolio is worth S_0 today.

So portfolio is worth S_0e^{rt} at time t.

But short-sell by itself is worth $S_0e^{rt} - S_t$ at time t.

So contract by itself must be worth S_t at time t.

Basic premise: Suppose we have some derivative, whose payoff at time T is $g(S_T)$ for some function $g(x)$.

Then at all times t with $0 \leq t \leq T$, we should be able to calculate the value of the derivative at time t as $f(S_t, t)$ for some function $f(x, t)$.

All of the useful information about the stock and our expectations of its future value S_T should be encoded in S_t, at time t.

So we should be able to then find the value of the derivative from S_t at time t, as well.

Of course $f(x, T) = g(x)$.

The payoff for the contract we just considered was S_T.

If we write this as $g(S_T)$ for some function $g(x)$ then we just have $g(x) = x$.

We then found that the value at time t is given by $f(S_t, t)$, where $f(x, t) = x$ for all times $0 \leq t \leq T$.

So $f(x, t)$ is always linear in x.

Call options: nonlinear derivatives.

The payoff for a call option is not linear.

It is $g(S_T)$ for $f(x) = \max(x - K, 0)$.

Main question: What is $f(x,t)$ for all $0 \le t \le T$? Particularly, what is $f(x,0)$?

($f(S_0, 0)$ = correct price for the call option today.)

Idea: (1) Linearize by using calculus.

(2) Make a risk-free portfolio.

(3) The risk-free portfolio will be very simple. It just has 1 call option (long position), and is short some amount Δ of shares of the stock.

(4) The amount, Δ, varies during the lifetime of the portfolio $\Delta = \Delta(t)$. (5) But $\Delta(t_2)$ at some time $0 \le t_2 \le T$ only depends on S_{t_1} at some earlier time $0 \le t_1 \le t_2$.

That is reasonable. We do not have access to any look-ahead models.

We will work backwards to try to figure out Δ_t.

Also, we will completely ignore the "Brownian weirdness" which I kept alluding to in the last chapter.

(I) Working completely backwards: start with last step. We know $f(x,t)$.

(II) Next, also suppose we know Δ at all times.

(III) Third, suppose we have constructed the risk-free portfolio.

The portfolio changes over time.

It is alway long 1 call option.

At time t it is also short $\Delta(t)$ shares of stock.

Suppose $\Delta(t)$ is constant on a short interval $[t, t + \Delta t]$.

Sorry for the confusing notation! There is no getting around it.

Δ-hedging is Δ-hedging!

(IV) Evaluate the value of our portfolio, and make sure that it is risk-free.

At time t the portfolio is worth

$$X_t \;=\; f(S_t, t) - \Delta(t) \cdot S_t \,.$$

At time $t + \Delta t$, the portfolio is worth

$$X_{t+\Delta t} \;=\; f(S_{t+\Delta t}, t + \Delta t) - \Delta(t) \cdot S_{t+\Delta t} \,.$$

We want to compare these two moneys. So we discount both to today's value by multiplying by e^{-rt} and $e^{-r(t+\Delta t)}$.

$$\begin{aligned}
\Delta[e^{-rt} X_t] \;&=\; e^{-r(t+\Delta t)} X_{t+\Delta t} - e^{-rt} X_t \\
&=\; e^{-r(t+\Delta t)} [f(S_{t+\Delta t}, t + \Delta t) - \Delta(t) \cdot S_{t+\Delta t}] - e^{-rt}[f(S_t, t) - \Delta(t) \cdot S_t] \\
&=\; \int_t^{t+\Delta t} d[e^{-r\tau} f(S_\tau, \tau)] - \Delta(t)\, d[e^{-r\tau} S_\tau] \,.
\end{aligned}$$

Now, this should be equal to 0 if the portfolio is risk-free.

That is because we have adjusted all incomes to present day, and we should have $d[e^{-rt}X_t] = 0$ so that $\Delta[e^{-rt}X_t] = \int_t^{t+\Delta t} d[e^{-r\tau}X_\tau] = 0$.

But that requires

$$d[e^{-r\tau} f(S_\tau, \tau)] - \Delta(t)\, d[e^{-r\tau} S_\tau] \;=\; 0 \,,$$

for all $\tau \in (t, t + \Delta t)$.

Later, taking $\Delta t \to 0$, we can let $\Delta(t)$ just be τ, and then we can replace τ by t.

We get

$$d[e^{-rt}\, f(S_t, t)] \;-\; \Delta(t)\, d[e^{-rt}\, S_t] \;=\; 0\,. \qquad (*)$$

Surprisingly, this equation, all by itself, tells us everything we want to know.

Even though we do not know $\Delta(t)$, we can use this equation to figure out $f(x, t)$, and then we can use that to figure out $\Delta(t)$!

This differential equation is equivalent to the Black-Scholes PDE.

But, we have to first figure out what this equation is really telling us.

(V) Naive interpretation of $(*)$.

Let us treat S_t as if it were differentiable.

Then using the rules of calculus,

$$\begin{aligned} d[e^{-rt}\, f(S_t, t)] \;&=\; f(S_t, t)\, d[e^{-rt}] + e^{-rt}\, d[f(S_t, t)] \\ &=\; f(S_t, t)\, [-re^{-rt}\, dt] + e^{-rt}\, \left[\frac{\partial f}{\partial x}(S_t, t)\, dS_t + \frac{\partial f}{\partial t}(S_t, t)\, dt\right]\,. \end{aligned}$$

Also,

$$\begin{aligned} d[e^{-rt}\, S_t] \;&=\; S_t\, d[e^{-rt}] + e^{-rt}\, dS_t \\ &=\; S_t\, [-re^{-rt}\, dt] + e^{-rt}\, dS_t \end{aligned}$$

So putting this all together, and collecting terms, we get

$$e^{-rt}\, \left[-rf(S_t, t) + \frac{\partial f}{\partial t}(S_t, t) + r\Delta(t) \cdot S(t)\right]\, dt + e^{-rt}\, \left[\frac{\partial f}{\partial x}(S_t, t) - \Delta(t)\right]\, dS_t \;=\; 0\,.$$

Problem: (1) S_t is not differentiable so this cannot be right.

(2) This is only 1 equation in 2 unknowns. We do not know $f(x,t)$ or $\Delta(t)$.

Solution of (2): Use Problem (1) to solve Problem (2).

If S_t were differentiable this would only be 1 equation because we could rewrite it as

$$e^{-rt}\left[-rf(S_t,t)+\frac{\partial f}{\partial t}(S_t,t)+r\Delta(t)S(t)+\left(\frac{\partial f}{\partial x}(S_t,t)-\Delta(t)\right)\frac{dS_t}{dt}\right]dt \;=\; 0\,.$$

But, if dS_t is not differentiable, we cannot do that.

Then the only way for the whole equation to be true is if both coefficients of dt and dS_t are 0:

$$-rf(S_t,t)+\frac{\partial f}{\partial t}(S_t,t)+r\Delta(t)\cdot S(t) \;=\; 0\,,$$
$$\frac{\partial f}{\partial x}(S_t,t)-\Delta(t) \;=\; 0\,.$$

(One can think that dS_t and dt take place on different scales.)

This is 2 equations in 2 unknowns, which we can solve.

The second equation leads to

$$\Delta(t) \;=\; \frac{\partial f}{\partial x}(S_t,t)\,.$$

So, plugging this in for $\Delta(t)$ to the first equation, gives

$$-rf(S_t,t)+\frac{\partial f}{\partial t}(S_t,t)+r\frac{\partial f}{\partial x}(S_t,t)\cdot S_t \;=\; 0\,.$$

Rewriting S_t as x, we just get the PDE:

$$-rf(x,t) + \frac{\partial f}{\partial t}(x,t) + r\frac{\partial f}{\partial x}(x,t) \cdot x \;=\; 0\,.$$

This is a first-order PDE, closely related to the 1d advection or wave equation.

It is easy to solve this equation, and this would give the zero-volatility limit of the true Black-Scholes equation[1].

But, it is not the right equation for the actual Black-Scholes formula, and that is because of "Brownian weirdness".

Earlier, we wrote

$$d[f(S_t,t)] \;=\; \frac{\partial f}{\partial x}(S_t,t)\,dS_t + \frac{\partial f}{\partial t}(S_t,t)\,dt\,.$$

But that uses the change-of-variables formula, which is valid when S_t is differentiable, and $(dS_t)^2 = 0$.

When we talk about Brownian motion, we are going to want to consider non-differentiable S_t, for which $(dS_t)^2 \neq 0$.

Then, we will need 2nd order terms in the change-of-variables formula, coming from the 2nd-order contributions from Taylor's formula.

This will lead to a 2nd-order PDE which is more closely related to the heat equation than the wave equation.

Also, along the way, we will learn how to take stochastic Itô integrals, and using those we can check that our risk-free portfolio really makes sense.

[1] Actually this is the value for a call option written on a bond, since the bond is the zero-volatility limit of a stock.