Lecture Notes for Math 210 – 1 October 2007

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Chapter 3: Deterministic and Stochastic Calculus

§5. *Partial Derivatives*

Consider $f(x,y) =$ function of two variables.

$\frac{\partial f}{\partial x}(x,y) =$ the derivative with respect to x holding y fixed.

Formally,

$$\frac{\partial f}{\partial x}(x,y) \;=\; \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}\,.$$

In other words, for each fixed value of y, define a new function $g_y(x) = f(x,y)$ for that fixed value of y.

Then $\frac{\partial f}{\partial x}(x,y) = \frac{d}{dx} g_y(x) = g_y'(x)$.

E.g., for

$$f(x,y) = x\,\sin(xy^2)\,,$$

we have

$$\frac{\partial f}{\partial x}(x,y) \;=\; 1\cdot\sin(xy^2) + x\cdot\cos(xy^2)\cdot y^2 \;=\; \sin(xy^2) + xy^2\,\cos(xy^2)\,.$$

Similarly,

$$\frac{\partial f}{\partial y}(x,y) \;=\; x\,\cos(xy^2)\cdot 2xy \;=\; 2x^2y\,\cos(xy^2)\,.$$

Note: Unfortunately, a common notation is to write f_x for $\frac{\partial f}{\partial x}$, and f_y for $\frac{\partial f}{\partial y}$.

But this does not work well with other notations such as B_t, S_t and C_t for asset prices at time t.

In class, I will try to use the longer notation $\frac{\partial f}{\partial x}$.

<u>Chain rule</u> for partial derivatives.

Suppose $x = x(t)$ and $y = y(t)$ are differentiable curves.

Then $f(x(t), y(t))$ is a differentiable function of t, and

$$\frac{d}{dt}\,f(x(t),y(t)) \;=\; \frac{\partial f}{\partial x}(x(t),y(t))\,\frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t),y(t))\,\frac{dy}{dt}\,.$$

This is particularly important for a special case.

Suppose X_t is an asset price which is differentiable in time t.

Then, thinking of $f(X_t, t)$ as a function $f(x,t)$ evaluated at $x = X_t$, we have

$$\frac{d}{dt}f(X_t,t) \;=\; \frac{\partial f}{\partial x}(X_t,t)\,\frac{dX_t}{dt} + \frac{\partial f}{\partial t}(X_t,t)\,.$$

We often write this as

$$df(X_t,t) \;=\; \frac{\partial f}{\partial x}(X_t,t)\,dX_t + \frac{\partial f}{\partial t}(X_t,t)\,dt\,.$$

This is called a “total derivative”.

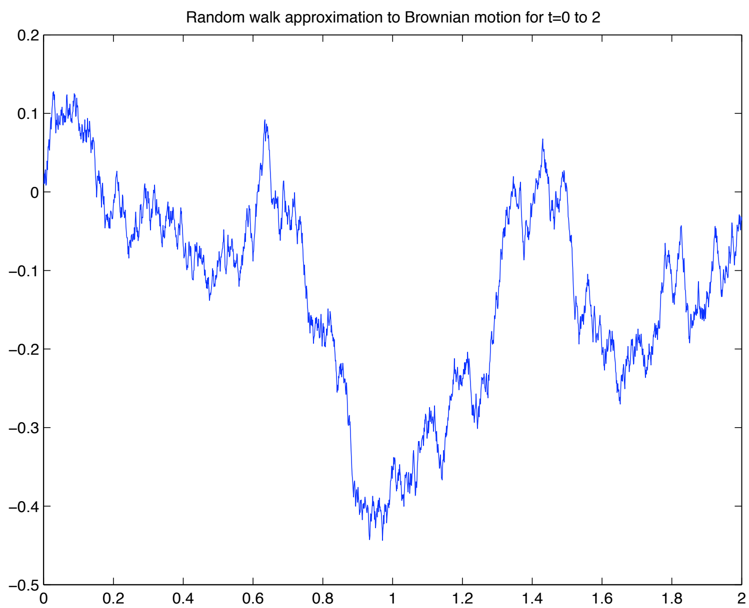
One useful thing you can do with this is integrate it:

$$f(X_T, T) - f(X_0, 0) = \int_0^T df(X_t, t) = \int_0^T \frac{\partial f}{\partial x}(X_t, t)\, dX_t + \int_0^T \frac{\partial f}{\partial t}(X_t, t)\, dt\,,$$

where the first integral is treated as a Stieltjes integral.

Beware: If the asset X_t is not differentiable in t, this simple formula does not work. In that case we need more terms of the Taylor series of f.

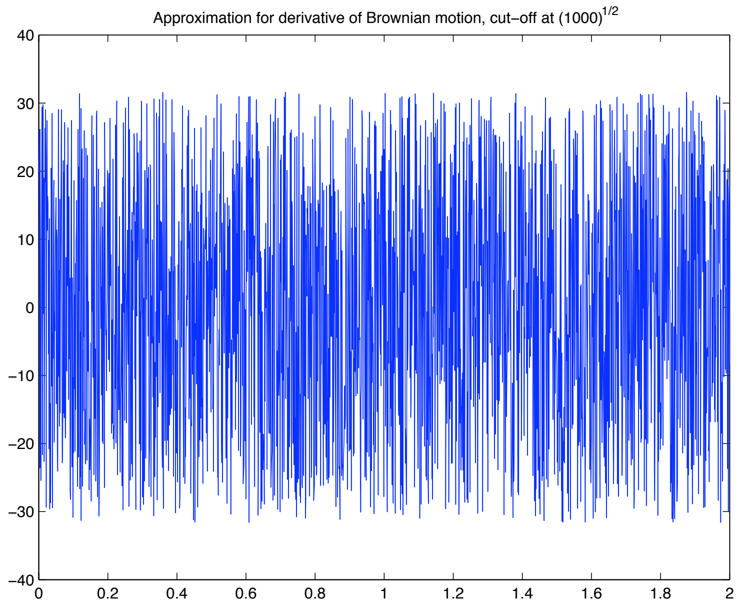
An example of an asset that is differentiable in t is the integral of a stock price over some window.



We have plotted a numerical approximation to Brownian motion from time $t = 0$ to $t = 2$. This has some of the same qualitative features of a stock prices S_t, except that Brownian motion, itself, can be negative (which no stock price can ever be). Although it is not 100% obvious from the picture, the graph of Brownian motion is continuous, but not differentiable. (Remember that a loose definition of continuity says that you can draw it without lifting your pen/pencil.)

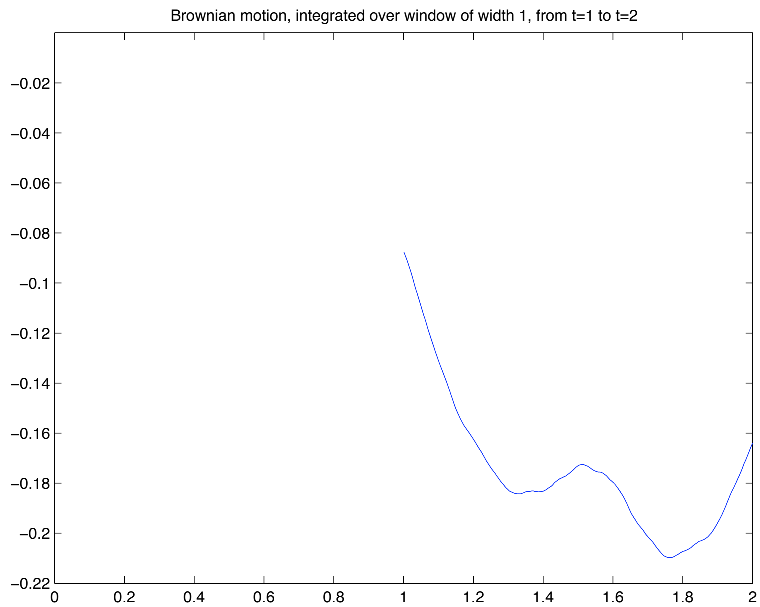
Next we show a plot of the finite-difference approximation to the derivative of

Brownian motion.



We cut this off at $\Delta t = 1000$, which results in a cut-off of the derivative at $\approx \sqrt{1000}$. (We made the approximation for Brownian motion out of a random walk, so that in our approximation, the absolute max and min for this approximate derivative are at $\pm\sqrt{1000} \approx \pm 31.62$.) The feature you see is that this graph fills out space up to the cut-off. The derivative of Brownian motion is white noise, and one of its features is that it has fluctuations on all scales. It is not a function. It is a generalized function or distribution, like the Dirac-delta function that you learn about in physics. One of the problems with White noise is that, just like the Dirac-δ function, you cannot square it, or really take any nonlinear power of it.

Finally, we show a plot for an asset $X_t = \int_{t-1}^{t} S_t \, dt$, where we are modelling S_t right now by Brownian motion. (This is wrong because Brownian motion can be negative, whereas S_t cannot be. But qualitative features other than that are similar.)



This graph is much different than the others. Not only is it continuous, it is also differentiable. Therefore, even though this is also a random function, we can take the derivative of X_t. (We would get $\frac{d}{dt}X_t = S_t - S_{t-1}$ by the Fundamental Theorem of Calculus.) So if we had some function $f(X_t, t)$ for this asset, the total derivative would be

$$df(X_t, t) \;=\; \frac{\partial f}{\partial x}(X_t, t)\, dX_t + \frac{\partial f}{\partial t}(X_t, t)\, dt\,.$$

However, since S_t is not differentiable the same type of formula would *not* work for $df(S_t, t)$. In fact, as we will see much later, we will need to add an extra term to the right-hand-side

$$df(S_t, t) \;=\; \frac{\partial f}{\partial x}(S_t, t)\, dS_t + \frac{\partial f}{\partial t}(S_t, t)\, dt + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}(S_t, t)\,(dS_t)^2\,.$$

Chapter 4: Pricing Derivatives

In order to price derivatives, we use the no-arbitrage principle.

But there are two ways to do this: (1) self-financing (risk-free) portfolios; (2) synthetic "equivalent" martingales.

Digression: (Discounting) One tool that Neftci does not use much is discounting to the present value.

Suppose some asset has value X_t at time t.

Q: How much is that worth to you today, at time $t = 0$?

A: $e^{-rt}X_t$. If you have N dollars today, it is worth $e^{rt}N$ at time t because you can invest it at the risk-free rate r, with continuous compounding. If that equals X_t, then we must have $e^{rt}N = X_t$ so $N = e^{-rt}X_t$.

Many authors (for example John Hull) like to adjust all future earnings to the present value because it gives a common standard for all future moneys. It also simplifies the math.

Q: Suppose that X_t is the value of a risk-free investment. What is

$$d[e^{-rt}X_t]\,?$$

A: For a bond, we know that $B_t = e^{rt}B_0$, where B_0 was the initial investment. So $d[e^{-rt}B_t] = d[B_0] = 0$ because B_0, the initial investment, is a constant. By the no-arbitrage principle, all risk-free investments are equivalent. In other words, "my money is as green as yours". So if X_t is any risk-free investment, then $d[e^{-rt}X_t] = 0$.

Neftci prefers to say that if X_t is the value for a risk-free investment then $d[X_t] = rX_t\,dt$. This is also useful.

§2.2 *Forwards.*

Consider a (European) call option, with strike price K, written today at time $t = 0$, with expiration date T in the future. Let C_t be the price of the call option at all times $0 \leq t \leq T$. We are going to look for a function $F(x, t)$ such that $C_t = F(S_t, t)$. There are many philosophical reasons that such a function should exist. The most basic is that, at time t, everything useful that the entire market knows about the stock *should* be represented in the stock price S_t. This is a deep statement that requires a lot of explanation. We will delay the explanation until Chapter 8 of Neftci, and when we read selections from Burton Malkiel's book, *A Random Walk Down Wall Street.*

For now, let us consider some general implications of the existence of such a function:

- $C_T = F(S_T, T)$. But $C_T = \max(S_T - K, 0)$. So $F(x, T) = \max(x - K, 0)$. This means we know $F(x, t)$ completely at time $t = T$. But what we really want is C_0, the value of the call option today. So we want to determine $F(x, t)$ at time $t = 0$.

- The value at time T is $F(x, T) = \max(x - K, 0)$ is a nonlinear function. It is not linear. The reason this is important is as follows.

- Suppose that we had a different asset whose value at time t was given by $X_t = g(S_t, t)$ and that $g(x, T) = ax + b$. Then $X_T = aS_T + b$. That means that the present value of X_T is $X_0 = aS_0 + be^{-rT}$. That is simply because the present value of the stock is S_0, while the present value of b in cash, risk-free, is be^{-rT}. Therefore, $g(x,0) = ax + be^{-rT}$. In fact, $g(x,t) = ax + be^{-r(T-t)}$. This exactly reproduces a forward contract. Therefore, in a very real sense, all of the difficulties in calculating $C_t = F(S_t, t)$ comes from the fact that $F(x,T)$ is not linear.

- The method of the self-financing portfolio will attempt to "linearize" $F(x,t)$ in x.

The *linearization* of $F(x,t)$ about the point x_0 is

$$F(x,t) \approx F(x_0,t) + \frac{\partial F}{\partial x}(x_0,t) \cdot (x - x_0)\,.$$

We will consider this more next time.