Lecture Notes for Math 210 – 5 September 2007

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1 Class Info

There is a webpage for the class

- http://www.math.rochester.edu/courses/210/home

Look at that webpage. There is important information there. There is also a class handout available on that webpage with a syllabus for the course. We will probably deviate a little bit from the syllabus but it gives you an idea *what you should* read for class.

Here is some important information:

- 1 midterm: November 20 in class.
- HW: 1 assignment per week starting next week. Assigned Monday, due the following Wednesday in class. Must turn in at the beginning of class (within first 5 minutes).
- Final: December 20
- Essay: On a topic related to Burton Malkiel's book. Due the last day of classes.

2 Idea of the course

- Learning how to determine the correct prices for *financial derivatives*.

Definition: A *financial derivative* is a financial instrument whose underlying value is derived from the value of another more basic financial instrument.

- Example: A stock is *not* a financial derivative.
- A stock is a basic financial instrument. Its value is not derived from any more basic asset.

A stock's value is determined by the value of the company and on the size of the dividends that the company is paying to its stockholders.

So a stock is not really an example of a financial derivative, but an example of something that is not a financial derivative. But an example of something that is a financial derivative is the following.

- Example: A forward contract is an example of a financial derivative.

- A *forward* is a contract to buy an asset such as a stock, at a time T in the future, for a price F_t. The price F_t is also paid at time T in the future.

Notation:

- We will often use the letter t for the time, today.

We will sometimes let t refer to a time between today and T. But for now, let us suppose t just means today. (This is not a "hard and fast rule" about letting t stand for the time today. In the solutions to Neftci's problems, I use t_0 to stand for today, if there is some reason to want to use t to stand for a time between t_0 and T, later in the example.)

- F_t = price we agree to pay (at time T) for the asset (to be delivered at time T) if we make the agreement today.

Not surprisingly F_t does rely on the the asset's underlying price today. So the forward is a *derivative*. Its value is *derived* from the value of the underlying asset. Whereas, a stock itself is not a derivative because its value is not derived from anything more basic.

It turns out to be very easy to calculate the correct price F_t which is the fair forward price in the sense that neither party in the contract has a 100% probability of making money from the other party. The price just depends on the price of the stock today.

- S_t = price of the stock today.

- There is a simple formula for F_t in terms of S_t.

We will calculate the formula for F_t at the end of the lecture.

But there is another financial derivative, called a "European call option," which is not so simple to calculate.

- A *European call option* is a contract which gives the holder the *right* to buy a particular asset (such as a stock) at a time T in the future, for a pre-determined price K. But it does not obligate the holder to buy if they do not wish to.

This is the central object of study for most of the course. Let us consider how this works.

Let us suppose the call option is on a particular stock. Let us define

- S_T = price of the stock at time T.

Of course, today, we do not know what S_T is. But let us denote this unknown variable by S_T.

Suppose, at time T, that S_T is greater than K.

- If S_T is greater than K, then the holder of the option will choose to exercise: they will buy at time T for price K.
  - Since the market price is S_T and $S_T > K$ the holder could buy the stock for price K, and turn around and sell it on the open market for S_T.
  - Profit = $S_T - K$.

- If S_T is less than K, then the holder of the option will not choose to exercise.
  - If they did buy for K, the value of the stock would only be S_T on the open market. (Could have bought for S_T instead.)

  - So would make $S_T - K$, but that is negative: so it is a loss.
  - Better not to exercise: make no profit, but more importantly make no loss.

From this we see that, at time T, the value of the option is

$$\begin{cases} S_T - K & \text{if } S_T > K; \\ 0 & \text{if } S_T < K. \end{cases}$$

We can also write this as

- $\max(S_T - K, 0)$

  Today, we only know

- Today we only know S_t = the price of the stock today.

We do not know S_T.

- So how can we determine the value of the European call option today, just from S_t?
- The answer is know. It is the Black-Scholes formula.

But the Black-Scholes formula is not easy. It will require us to learn a lot about probability theory. In particular we will learn something about stochastic integration and Ito's formula, which is considered to be a non-trivial topic in math. We will spend the entire semester deriving the Black-Scholes formula, developing all the mathematics we need along the way.

3 A problem

- A certain stock has price today $S_t = \$100$.
- A forward contract is created which is an agreement to buy 1 share of the stock at time $T = 1$year in the future, for a price F_t. F_t = the fair forward price. Assume the following:
  (a) The stock has no dividends between now and T.
  (b) You can borrow money from the bank at an interest rate of $r = 5\%$. So, if you borrow N today, in 1 year you must repay $N(1 + r) = 1.05 \times N$.
  (c) You can lend money to the bank at the same interest rate of $r = 5\%$. If you invest N today, then in 1 year you will get $N(1 + r)$ from the bank.
- What is the fair forward price F_t, such that neither the holder nor the writer of the contract can make a risk-free profit from the other side?

Note that the assumptions imply that you can borrow and lend money at the same rate. For most students this is not true. The bank charges more interest from you if you borrow than it gives you for investing in a savings account. But, for large financial institutions, such as other banks, this is approximately true, as long as the large institution has good credit so that the risk of default is low.

- Solution: $F_t = (1 + r)S_t$

We will prove that $F_t = (1 + r)S_t$ by a two-step argument.

3.1 Proof that $F_t \leq (1+r)S_t$

The basic underlying assumption is that it should not be possible to make a risk-free profit for either party of the contract. If that was possible, then the other party would have to make the corresponding risk-free loss, and that party would never willingly agree.

> **Definition:** An *arbitrage* opportunity is an opportunity to make a risk-free profit. The basic assumption of this course is that no arbitrage opportunity can exist in the free open market.

We want to prove that $F_t \leq (1+r)S_t$.

- Note that the statement, "it is true that $F_t \leq (1+r)S_t$" is logically equivalent to the statement "it is false that $F_t > (1+r)S_t$".

- To prove that $F_t \leq (1+r)S_t$, we start by assuming that $F_t > (1+r)S_t$ and showing that this leads to a contradiction (so must be false).

- The contradiction will be that, assuming $F_t > (1+r)S_t$, there will be an arbitrage opportunity. This contradicts our basic "no arbitrage" assumption, which we make for the entire course.

- Suppose $F_t > (1+r)S_t$ to reach a contradiction.

- Take the short position in the forward contract. I.e., you enter the contract with somebody else, where you agree to deliver the asset at time T, and in exchange they pay you F_t at that time.

- Now, at time t, borrow S_t from the bank.

- Buy 1 share of the stock, now, at time t.

- At time T, you will owe the bank $(1+r)S_t$.

- At time T, you will exchange your 1 stock for F_t, as per the forward agreement.

- Since $F_t > (1+r)S_t$, your net profit is $F_t - (1+r)S_t$.

- This means you made a risk-free profit, which contradicts the no-arbitrage assumption.

- Therefore, either the no-arbitrage assumption is wrong, or else it is false that $F_t > (1+r)S_t$.

- Since we refuse to give up the no-arbitrage assumption, this means "it is false that $F_t > (1+r)S_t$. In other words, "it is true that $F_t \leq (1+r)S_t$."

- That is what we wanted to prove.

3.2 Proof that $F_t \geq (1+r)S_t$

Now we want to prove that $F_t \geq (1+r)S_t$. The proof will be of a similar form to that just above. Namely it will be a *contradiction proof*. But first we need another definition.

Definition: To *short sell* a stock today, means that you sell a stock for its current price S_t, today, by first borrowing from some other investor who owns the stock. You must repay the stock to the investor at a later time T (when the stock's value will be S_T).

It is generally assumed that it is possible for you to short-sell a stock, unless otherwise specified. The way this works is that your stockbroker finds another investor who owns the stock, usually who also is a client of the same stockbroker, and borrows the stock from them, for some amount of time.

Now we can complete the proof.

- Suppose $F_t < (1+r)S_t$, to reach a contradiction.
- Take the long position in the forward contract. I.e., you enter the contract with somebody else, where you agree to pay F_t, at time T, in exchange for the asset (delivered also at time T).
- Now, at time t, short-sell one share of the stock, which generates income of S_t today.
- Invest your S_t in the bank.
- At time T, you will collect $(1+r)S_t$ from the bank.
- At time T, you will pay F_t to 1 share of the stock, as per the forward agreement.
- You take that 1 share and repay the investor whom you borrowed from, in order to make the short-sell.
- At time T you got $(1+r)S_t$ from the bank and you paid F_t for the share. So your net profit is $(1+r)S_t - F_t$.
- This means you made a risk-free profit, which contradicts the no-arbitrage assumption.
- Therefore, either the no-arbitrage assumption is wrong, or else it is false that $F_t < (1+r)S_t$.
- Since we refuse to give up the no-arbitrage assumption, this means "it is false that $F_t < (1+r)S_t$. In other words, "it is true that $F_t \geq (1+r)S_t$."
- That is what we wanted to prove.

3.3 Proof that $F_t \geq (1+r)S_t$

Since we now know both that $F_t \leq (1+r)S_t$ and $F_t \geq (1+r)S_t$, this means that $F_t = (1+r)S_t$, as we claimed.

- This type of argument is fool-proof. It does not matter how the price of the stock S_t will change between now and S_T. As long as we accept that there are no arbitrage opportunities in the market, we conclude that the fair forward price must be $F_t = (1+r)S_t$ for this problem.