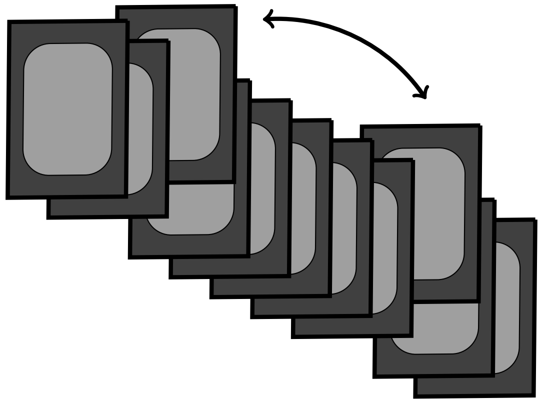
Math 391 – Mixing Times for Markov Chains
Summary of Computer Lab #4
Monday, Feb. 25

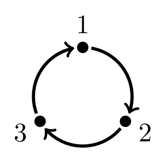
Goal: Generate a simulation for the "random transposition" shuffle.



Fact 1: The symmetric group, S_n, is the group of all permutations of n objects. A permutation is a rearrangement or a choice of ordering. Permutations are closely related to the problem of shuffling cards. A permutation is often denoted by $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ where π_k is the place that the kth card gets sent. So it is sometimes also written as an $n \times 2$ array:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ \pi_1 & \pi_2 & \pi_3 & \ldots & \pi_n \end{pmatrix}.$$

For example the permutation $(\pi_1, \pi_2, \pi_3) = (1, 2, 3)$ is the identity permutation: $\pi_i = i$ for all i. On the other hand, if $(\pi_1, \pi_2, \pi_3) = (2, 3, 1)$ then this represents a cycle of length 3, as shown below:



Another way we can represent a permutation is by a permutation matrix, A. We define A to be the $n \times n$ matrix such that $A_{ij} = 1$ if and only if $\pi_i = j$. Otherwise $A_{ij} = 0$. (Note that this is slightly different than I defined in class. With this definition, we do not have a representation, but we have an anti-homomorphism, so sending π to A^T would be a representation. Of course you do **not** have to know what representations are to understand this lab.)

Q1: Suppose that π is the identity permutation on n elements, so that $\pi_i = i$ for $i = 1, \ldots, n$. Then what is the permutation matrix associated to π?

A1: It is the identity matrix ,

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

For Matlab, start by typing the following:

```
>> N = 11
N =
     11
>> E = eye(N)
E =
     1 0 0 0 0 0 0 0 0 0 0
     0 1 0 0 0 0 0 0 0 0 0
     0 0 1 0 0 0 0 0 0 0 0
     0 0 0 1 0 0 0 0 0 0 0
     0 0 0 0 1 0 0 0 0 0 0
     0 0 0 0 0 1 0 0 0 0 0
     0 0 0 0 0 0 1 0 0 0 0
     0 0 0 0 0 0 0 1 0 0 0
     0 0 0 0 0 0 0 0 1 0 0
     0 0 0 0 0 0 0 0 0 1 0
     0 0 0 0 0 0 0 0 0 0 1
```

This is the permutation matrix for the identity permutation: $\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & 11 \\ 1 & 2 & 3 & \dots & 11 \end{pmatrix}$. We now want to generate a random transposition. The identity permutation fixes every number from 1 to 11. A transposition is a permutation which only switches two numbers (by swapping them with each other).

Q2: What does the following Matlab command do: `ceil(N*rand)`? Try it several times and see.

```
>> >> ceil(N*rand)
ans =
     9
>> ceil(N*rand)
ans =
     10
>> ceil(N*rand)
ans =
     2
>> ceil(N*rand)
ans =
     11
>> ceil(N*rand)
ans =
     7
```

A2: It generates a random number between 1 and N, which is $N = 11$ in this case. Note that the ceiling function is $\text{ceil}(x) = \lceil x \rceil$, which is the smallest integer k such that $x \leq k$. So $\text{ceil}(1) = 1$ but $\text{ceil}(1.1) = 2$. The ceiling function is related to the floor function.

Now try the following:

```
>> I = ceil(N*rand)
I =
     2
>> J = ceil(N*rand)
J =
     4
>> B = E;
>> B(I,:) = E(J,:);
>> B(J,:) = E(I,:);
```

```
>> B
B =
     1 0 0 0 0 0 0 0 0 0 0
     0 0 0 1 0 0 0 0 0 0 0
     0 0 1 0 0 0 0 0 0 0 0
     0 1 0 0 0 0 0 0 0 0 0
     0 0 0 0 1 0 0 0 0 0 0
     0 0 0 0 0 1 0 0 0 0 0
     0 0 0 0 0 0 1 0 0 0 0
     0 0 0 0 0 0 0 1 0 0 0
     0 0 0 0 0 0 0 0 1 0 0
     0 0 0 0 0 0 0 0 0 1 0
     0 0 0 0 0 0 0 0 0 0 1
```

Note that, since I and J are random, there is some possibility that they will be equal. In fact, in this case the probability is $1/N = 1/11$. If that happens you would probably want to re-run the experiment to see what happens when $I \neq J$.

In the case that I had, $I = 2$ and $J = 4$. The B matrix is like the E matrix, except that it has rows 2 and 4 interchanged. In general, if A is a matrix in Matlab, then $A(I,:)$ gives the row vector which is row I of A. Alternatively, if we want to *redefine* the entire Ith row of A we can say $A(I,:)\ =$ and then put whatever row vector we want. So the command "`B(I,:) = E(J,:)`" redefines the Ith row of B to be equal to the Jth row of E. Similarly, the command "`B(J,:) = E(I,:)`" redefines the Jth row of B to be equal to the Ith row of E. So the effect of these two operations together is to make B look just like E (because we started with "`B = E`") except that rows I and J have been switched.

Now we want to put this all in a for-loop to take many steps of the "random-transposition walk on the symmetric group".

```
>> A = eye(N);
>> for n=1:200,
B=A;
I=ceil(N*rand);
J=ceil(N*rand);
B(I,:)=A(J,:);
B(J,:)=A(I,:);
A=B;
end
```

```
>> A
A =
     0 0 0 0 0 0 0 0 0 1 0
     0 0 0 0 0 0 1 0 0 0 0
     1 0 0 0 0 0 0 0 0 0 0
     0 1 0 0 0 0 0 0 0 0 0
     0 0 0 1 0 0 0 0 0 0 0
     0 0 0 0 0 0 0 0 0 0 1
     0 0 0 0 1 0 0 0 0 0 0
     0 0 0 0 0 0 0 1 0 0 0
     0 0 1 0 0 0 0 0 0 0 0
     0 0 0 0 0 1 0 0 0 0 0
     0 0 0 0 0 0 0 0 1 0 0
>> (1:11)*A
ans =
     3 4 9 5 7 10 2 8 11 1 6
```

We have taken 200 steps of the "random transposition walk" on the symmetric group. At the end we obtain a matrix A which is a permutation matrix for the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 4 & 9 & 5 & 7 & 10 & 2 & 8 & 11 & 1 & 6 \end{pmatrix}$$

Actually we have not generated the "random transposition walk" on the symmetric group, but rather, the "lazy random transposition walk" on the symmetric group. Because of this, the transition matrix is aperiodic, as well as irreducible. (To prove that it is irreducible is not completely trivial. It is an exercise in abstract algebra to prove that all permutations can be written as products of finitely many transpositions. But, actually, it is an easy exercise, once you understand what all the terms means.)

Q3: The reason that this is the "lazy" random walk is that we can have $I = J$, with some probability (probability $1/N$). If we change I and J, to condition on the event that $I \neq J$, then is the resulting "random transposition walk" aperiodic?

A3: No. It is periodic with period 2. If we look at the "signature" of the permutation, then it flips between the two values 1 and -1. At even times it equals 1 and at odd times it equals -1. One way to calculate the signature is to take the determinant of the permutation matrix. Consider, as an example, $N = 3$. Then we start with

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that $\det(A_0) = \det(I) = 1$. If we condition on $I \neq J$ then there are three possibilities for the unordered set $\{I, J\}$ with the resulting matrices:

$$\{I,J\} = \{1,2\} : A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \{I,J\} = \{1,3\} : A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad \{I,J\} = \{2,3\} : A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easy to check directly that all three of these matrices have determinant equal to -1. More generally whenever you switch two rows the determinant changes sign. (This is because you can switch two rows like follows: $A(J,:) =$

$A(J,:) - A(I,:)$, then $A(I,:) = A(I,:) + A(J,:)$ then $A(J,:) = A(J,:) - A(I,:)$ then $A(J,:) = -A(J,:)$ [thanks to Lam Tran for pointing this out].) So all permutation matrices for a transposition have determinant -1.

But the step of the random transposition walk on the symmetric group is to multiply by a permutation matrix for a randomly chosen transposition at each step, and $\det(AB) = \det(A)\det(B)$. So classifying permutations as "even" and "odd" depending on whether the determinant of their permutation matrix is 1 or -1, we see that at even times you could not get any odd permutation and at odd times you could not get any even permutation. But, as we mentioned before, since $I = J$ with probability $1/N$, we are actually performing a "lazy" walk. So the transition matrix is aperiodic as well as irreducible.

Task 1: Calculate the number of fixed points for the random permutation.

Finally, we want to look at some statistics to try to see if this Markov chain has mixed up the permutation well. One thing to look at is fixed points. Consider the permutation $(1, 4, 2, 5, 3)$ whose matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

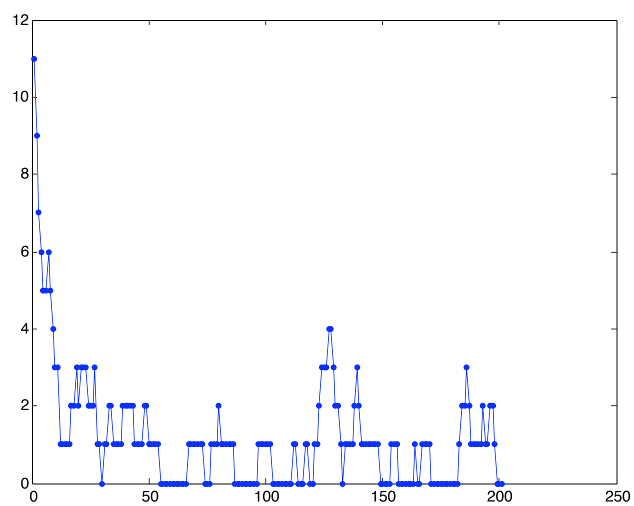
We want to calculate the number of fixed points. That equals the number of 1's on the diagonal. So one way to calculate it is as $\mathrm{tr}(A)$. So, now, type the following:

```
>> A = eye(N);
>> T = N;
>> for n=1:200,
B=A;
I=ceil(N*rand);
J=ceil(N*rand);
B(I,:)=A(J,:);
B(J,:)=A(I,:);
A=B;
T = [T,trace(A)];
end
```

This creates a vector T which has recorded the number of fixed points at each step. You can see the effect by typing:

```
>> plot(T), hold on, plot(T,'.')
```

$\longrightarrow$



Assignment: Email me your Matlab figures that you get when you plot T.

Q4: What is the expected number of fixed points in a uniform random permutation?

A4: This is a common type of question from introductory probability. The probability that $\pi_i = i$ is $1/N$ for each $i = 1, \ldots, N$. Note that these events are highly dependent. But since the expectation is additive (or linear) this still gives that the expected number of fixed points is $N(1/N) = 1$. In fact, the exact probabilities for the number of fixed points is calculated in the "Matching Problem". As $N \to \infty$ the random variable equal to the number of fixed points of a uniform random permutation of size N converges in-distribution to a Poisson random variable with parameter $\lambda = 1$. If N is large enough, and if n is large enough, we would expect something similar for the output of the random transposition random walk.