

Math 391 – Mixing Times for Markov Chains

Homework Set #3

Assigned: Thursday, Mar. 20

Due: Tuesday, Apr. 7 (by 5pm in my office 1017 Hylan)

Do the following Exercises from Levin, Peres and Wilmer:

Chapter 4: #4.1 on pg. 31, #4.2 on pg. 32, and #4.3 and #4.4 on pg. 34.

In addition, do the following problem(s).

A1. Look back at Example 4.1, on page 29 of Levin, Peres and Wilmer's text, which is the (blocked) random walk on $\{1, \dots, n\}$. Let

$$\mathbf{P}_x(X_t = z) = \text{Probability that } \{X_t = z\} \text{ conditional on the fact that } \{X_0 = x\} = P^t(x, z),$$

which is the same notation introduced on pages 13 and 14.

(a) Suppose m is any number in $\{1, \dots, n\}$. Modify the proof from Example 4.1 to show that

$$\mathbf{P}_x(X_t \in \{m, m+1, \dots, n\}) \leq \mathbf{P}_y(X_t \in \{m, m+1, \dots, n\}),$$

if $x \leq y$.

(b) Let \mathcal{M} be the set of all functions on $\{1, \dots, n\}$ that satisfy $f(j) \leq f(k)$ if $j \leq k$. This is the set of monotone functions. Show that

$$\mathbf{E}_x(f(X_t)) \leq \mathbf{E}_y(f(X_t)),$$

for any $f \in \mathcal{M}$, if $x \leq y$. (When this property is satisfied one says that the Markov chain is “monotone”.)

A2. Let Ω_N be the set of all N -tuples of numbers (x_1, x_2, \dots, x_N) such that x_1, x_2, \dots, x_N are all drawn from the set $\{-1, 0, 1\}$, and satisfying the further condition that the 1's and -1 's must alternate. So two allowable configurations, when $N = 6$, are

$$(+1, -1, 0, +1, 0, 0, -1) \quad \text{and} \quad (0, -1, 0, +1, 0, 0, -1).$$

(a) Prove that $|\Omega_N| = 2^{N+1} - 1$. (Hint: Given $(x_1, \dots, x_N) \in \Omega_N$, consider $(|x_1|, \dots, |x_N|)$ which consists of just 1's and 0's. Prove that the number of such absolute-value configurations is 2^N , and then argue from there.)

(b) Let $\mathbb{Z}_3 = \mathbb{Z}/(3\mathbb{Z})$ be the integers modulo 3. We say that 0, 3, -3 , 6, -6 , \dots are all equivalent modulo 3 and write them all using the symbol [0]. Similarly, [1] contains all the numbers 1, 4, -2 , 7, -5 , \dots , and [2] contains all the numbers 2, 5, -1 , 8, -4 , \dots . These are the only equivalence classes: [0], [1] and [2]. Given $(x_1, x_2, \dots, x_N) \in \Omega_N$, let (y_1, \dots, y_N) be defined as elements of \mathbb{Z}_3 through the formula

$$y_n = x_1 + \dots + x_n \text{ mod } 3 = [x_1 + \dots + x_n].$$

Let $\tilde{\Omega}_N$ denote the set of all configurations (y_1, \dots, y_N) , obtained in this way. Imagine defining a random walk on the triangle with vertices [0], [1] and [2]. For the purpose of this part of the problem, define $y_0 = [0]$, as well, so that the walk is (y_0, y_1, \dots, y_N) . What geometric or topological condition on the walk does the alternating-signs condition translate to?

(c) Let $\hat{\Omega}_N = \{(y_1, \dots, y_N) \in \tilde{\Omega}_N : y_1 \neq [0]\}$. Let (Y_1, \dots, Y_N) be a uniformly randomly chosen element of $\hat{\Omega}_N$. Prove that this forms a Markov chain with some transition matrix P and some initial probability μ for Y_1 . More specifically, find P and μ . What state do you never hit?

(d) Suppose $N = 3k + 1$, what is $|\{(y_1, \dots, y_N) \in \hat{\Omega}_N : \#\{n : y_{n+1} = y_n\} = k\}|$?