Math 391 : Markov Chains and Mixing Times
Homework #1
Assigned Tuesday, May 9. Due Tuesday, May 16.

This problem set reviews the Geometric random variable, using Calculus tools.

1. Suppose that $x \neq 1$. Prove that

$$\sum_{k=0}^{n-1} x^k \;=\; \frac{1-x^n}{1-x}\,,$$

and

$$\sum_{k=1}^{n-1} kx^{k-1} \;=\; \frac{d}{dx}\left(\frac{1-x^n}{1-x}\right).$$

2. Suppose that $0 \leq x < 1$. Prove that

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} x^k \;=\; \frac{1}{1-x}\,,$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} kx^{k-1} \;=\; \frac{1}{(1-x)^2}\,.$$

(*Hint: You can use L'Hospital's rule to prove that* $\lim_{n\to\infty} nx^n = 0$ *for* $0 < x < 1$.)

3. For $0 < p \leq 1$, we define the Geometric random variable to take values $1, 2, \ldots$ with probabilities

$$p(n) \;=\; p(1-p)^{n-1}\,.$$

Using Problem 3, check that

$$\sum_{n=1}^{\infty} p(n) \;=\; 1\,,$$

by relating this to the Geometric series in that problem, and also

$$\mathbb{E}X \;:=\; \sum_{n=1}^{\infty} np(n) \;=\; \frac{1}{p}\,.$$