CALCULATIONS IN THE STEENROD ALGEBRA

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ABSTRACT. For the mod 2 and mod 3 Steenrod algebras, we calculate the admissible bases, the Adem relations, the anti-automorphism, and the Milnor Bockstein operations.

CONTENTS

1. INTRODUCTION

The mod 2 Steenrod algebra $A_2 = \mathrm{H}\mathbb{F}_2^*(\mathrm{H}\mathbb{F}_2)$ is large and has many relations among its elements. Calculations in the Steenrod algebra A_2 done by hand are lengthy and time consuming, but the `Steenrod` package for Maple written by Ken Monks can make calculations easy. The `Steenrod` package is available for download at
`http://math.scranton.edu/monks/software/Steenrod/steen.html`
and the source code used for many of the calculations is in the last section of this paper.

Only the mod 2 Steenrod algebra is supported in Ken Monks's `Steenrod` package for Maple, so I have written a few unrefined procedures for making some calculations in $\mathrm{H}\mathbb{F}_p^*(\mathrm{H}\mathbb{F}_p)$ for $p > 2$.

Notation: We write $\mathrm{Sq}^{i,j}$ for $\mathrm{Sq}^i \circ \mathrm{Sq}^j = \mathrm{Sq}^i\mathrm{Sq}^j$, $P^{i,j}$ for $P^i \circ P^j = P^iP^j$, and write elements of the Steenrod algebra in admissible form.

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2. THE MOD 2 STEENROD ALGEBRA A_2

2.1. **The Adem relations in A_2.** An Adem relation has the form

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_{j=0}^{\lfloor a/2\rfloor} \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j$$

where $a < 2b$, $\lfloor a/2 \rfloor$ denotes the greatest integer $\leq a/2$, and the binomial coefficient is taken mod 2. We use the convention that $\binom{x}{y} = 0$ if $x < y$ or if $y < 0$. The Adem relations are discussed in detail in chapter 3 of [MT68]. In each degree, we give the admissible basis for the Steenrod algebra as a vector space over $\mathbb{F}_2$, and common Adem relations among elements.

An element $\mathrm{Sq}^{i_1,\ldots,i_r}$ in the mod 2 Steenrod algebra is admissible if $i_j \geq 2i_{j+1}$ for every $j < r$ (and it is vacuously satisfied if $r \leq 1$).

Degree	Basis elements (in braces) followed by Adem relations
0	$\{\mathbf{1}\}$
1	$\{\mathbf{Sq^1}\}$
2	$\{\mathbf{Sq^2}\}$ $Sq^{1,1} = 0$
3	$\{Sq^3,\ Sq^{2,1}\}$ $Sq^{1,2} = Sq^3$
4	$\{\mathbf{Sq^4},\ Sq^{3,1}\}$ $Sq^{1,3} = 0,\ Sq^{2,2} = Sq^{3,1}$
5	$\{Sq^5,\ Sq^{4,1}\}$ $Sq^{1,4} = Sq^5,\ Sq^{2,3} = Sq^5 + Sq^{4,1},\ Sq^{3,2} = 0$
6	$\{Sq^6,\ Sq^{5,1},\ Sq^{4,2}\}$ $Sq^{1,5} = 0,\ Sq^{2,4} = Sq^6 + Sq^{5,1},\ Sq^{3,3} = Sq^{5,1}$
7	$\{Sq^7,\ Sq^{6,1},\ Sq^{5,2},\ Sq^{4,2,1}\}$ $Sq^{1,6} = Sq^7,\ Sq^{2,5} = Sq^{6,1},\ Sq^{3,4} = Sq^7,\ Sq^{4,3} = Sq^{5,2}$
8	$\{\mathbf{Sq^8},\ Sq^{7,1},\ Sq^{6,2},\ Sq^{5,2,1}\}$ $Sq^{1,7} = 0,\ Sq^{2,6} = Sq^{7,1},\ Sq^{3,5} = Sq^{7,1},\ Sq^{4,4} = Sq^{7,1} + Sq^{6,2},\ Sq^{5,3} = 0$
9	$\{Sq^9,\ Sq^{8,1},\ Sq^{7,2},\ Sq^{6,3},\ Sq^{6,2,1}\}$ $Sq^{1,8} = Sq^9,\ Sq^{2,7} = Sq^9 + Sq^{8,1},\ Sq^{3,6} = 0,\ Sq^{4,5} = Sq^9 + Sq^{8,1} + Sq^{7,2},\ Sq^{5,4} = Sq^{7,2}$
10	$\{Sq^{10},\ Sq^{9,1},\ Sq^{8,2},\ Sq^{7,3},\ Sq^{7,2,1},\ Sq^{6,3,1}\}$ $Sq^{1,9} = 0,\ Sq^{2,8} = Sq^{10} + Sq^{9,1},\ Sq^{3,7} = Sq^{9,1},\ Sq^{4,6} = Sq^{10} + Sq^{8,2},$ $Sq^{5,5} = Sq^{9,1},\ Sq^{6,4} = Sq^{7,3}$
11	$\{Sq^{11},\ Sq^{10,1},\ Sq^{9,2},\ Sq^{8,3},\ Sq^{8,2,1},\ Sq^{7,3,1}\}$ $Sq^{1,10} = Sq^{11},\ Sq^{2,9} = Sq^{10,1},\ Sq^{3,8} = Sq^{11},\ Sq^{4,7} = Sq^{11} + Sq^{9,2},$ $Sq^{5,6} = Sq^{11} + Sq^{9,2},\ Sq^{6,5} = Sq^{9,2} + Sq^{8,3},\ Sq^{7,4} = 0$
12	$\{Sq^{12},\ Sq^{11,1},\ Sq^{10,2},\ Sq^{9,3},\ Sq^{8,4},\ Sq^{9,2,1},\ Sq^{8,3,1}\}$ $Sq^{1,11} = 0,\ Sq^{2,10} = Sq^{11,1},\ Sq^{3,9} = Sq^{11,1},\ Sq^{4,8} = Sq^{12} + Sq^{11,1} + Sq^{10,2},\ Sq^{5,7} = 0,$ $Sq^{6,6} = Sq^{11,1} + Sq^{10,2} + Sq^{9,3},\ Sq^{7,5} = Sq^{9,3}$
13	$\{Sq^{13},\ Sq^{12,1},\ Sq^{11,2},\ Sq^{10,3},\ Sq^{9,4}, Sq^{10,2,1},\ Sq^{9,3,1},\ Sq^{8,4,1}\}$ $Sq^{1,12} = Sq^{13},\ Sq^{2,11} = Sq^{13} + Sq^{12,1},\ Sq^{3,10} = 0,\ Sq^{4,9} = Sq^{12,1} + Sq^{11,2},$ $Sq^{5,8} = Sq^{13} + Sq^{11,2},\ Sq^{6,7} = Sq^{13} + Sq^{12,1} + Sq^{10,3},\ Sq^{7,6} = Sq^{11,2},\ Sq^{8,5} = Sq^{9,4}$
14	$\{Sq^{14},\ Sq^{13,1},\ Sq^{12,2},\ Sq^{11,3},\ Sq^{10,4},\ Sq^{11,2,1},\ Sq^{10,3,1},\ Sq^{9,4,1},\ Sq^{8,4,2}\}$ $Sq^{1,13} = 0,\ Sq^{2,12} = Sq^{14} + Sq^{13,1},\ Sq^{3,11} = Sq^{13,1},\ Sq^{4,10} = Sq^{12,2},\ Sq^{5,9} = Sq^{13,1},$ $Sq^{6,8} = Sq^{14} + Sq^{13,1} + Sq^{11,3},\ Sq^{7,7} = Sq^{13,1} + Sq^{11,3},\ Sq^{8,6} = Sq^{11,3} + Sq^{10,4},\ Sq^{9,5} = 0$
15	$\{Sq^{15},\ Sq^{14,1},\ Sq^{13,2},\ Sq^{12,3},\ Sq^{11,4},\ Sq^{10,5},\ Sq^{12,2,1},\ Sq^{11,3,1},\ Sq^{10,4,1},\ Sq^{9,4,2},\ Sq^{8,4,2,1}\}$ $Sq^{1,14} = Sq^{15},\ Sq^{2,13} = Sq^{14,1},\ Sq^{3,12} = Sq^{15},\ Sq^{4,11} = Sq^{13,2},\ Sq^{5,10} = Sq^{13,2},$ $Sq^{6,9} = Sq^{14,1} + Sq^{13,2} + Sq^{12,3},\ Sq^{7,8} = Sq^{15},\ Sq^{8,7} = Sq^{13,2} + Sq^{12,3} + Sq^{11,4},\ Sq^{9,6} = Sq^{11,4}$

2.2. **The anti-automorphism in A_2.** Let $\chi : A_2 \to A_2$ be the anti-automorphism of the Steenrod algebra. Since χ is an anti-automorphism of algebras, it is a $\mathbb{F}_2$-module homomorphism such that $\chi(ab) = \chi(b)\chi(a)$ for all $a, b \in A_3$. For $n \geq k$, the values of χ are determined recursively by the formulas

$$\chi(\mathrm{Sq}^{2^n-k}) = \mathrm{Sq}^{2^{n-1}}\mathrm{Sq}^{2^{n-2}}\cdots\mathrm{Sq}^{2^{k-1}}(\chi(\mathrm{Sq}^{2^{k-1}-k}))$$

and

$$\chi(\mathrm{Sq}^{2^{k-1}-k}) = \mathrm{Sq}^{2^{k-2}}\chi(\mathrm{Sq}^{2^{k-2}-k}) + \mathrm{Sq}^{2^{k-2}-1}\mathrm{Sq}^{2^{k-3}-1}\cdots\mathrm{Sq}^3\mathrm{Sq}^1$$

by [Dav74].

$$\begin{aligned}
\chi(\mathrm{Sq}^0) &= \mathrm{Sq}^0 = \textit{identity} \\
\chi(\mathrm{Sq}^1) &= \mathrm{Sq}^1 \\
\chi(\mathrm{Sq}^2) &= \mathrm{Sq}^2 \\
\chi(\mathrm{Sq}^3) &= \mathrm{Sq}^{2,1} \\
\chi(\mathrm{Sq}^4) &= \mathrm{Sq}^4 + \mathrm{Sq}^{3,1} \\
\chi(\mathrm{Sq}^5) &= \mathrm{Sq}^{4,1} \\
\chi(\mathrm{Sq}^6) &= \mathrm{Sq}^{4,2} \\
\chi(\mathrm{Sq}^7) &= \mathrm{Sq}^{4,2,1} \\
\chi(\mathrm{Sq}^8) &= \mathrm{Sq}^8 + \mathrm{Sq}^{7,1} + \mathrm{Sq}^{6,2} + \mathrm{Sq}^{5,2,1} \\
\chi(\mathrm{Sq}^9) &= \mathrm{Sq}^{8,1} + \mathrm{Sq}^{6,2,1} \\
\chi(\mathrm{Sq}^{10}) &= \mathrm{Sq}^{8,2} + \mathrm{Sq}^{7,3} + \mathrm{Sq}^{6,3,1} \\
\chi(\mathrm{Sq}^{11}) &= \mathrm{Sq}^{8,2,1} + \mathrm{Sq}^{7,3,1} \\
\chi(\mathrm{Sq}^{12}) &= \mathrm{Sq}^{8,4} + \mathrm{Sq}^{8,3,1} \\
\chi(\mathrm{Sq}^{13}) &= \mathrm{Sq}^{8,4,1} \\
\chi(\mathrm{Sq}^{14}) &= \mathrm{Sq}^{8,4,2} \\
\chi(\mathrm{Sq}^{15}) &= \mathrm{Sq}^{8,4,2,1} \\
\chi(\mathrm{Sq}^{16}) &= \mathrm{Sq}^{16} + \mathrm{Sq}^{12,4} + \mathrm{Sq}^{10,4,2} + \mathrm{Sq}^{15,1} + \mathrm{Sq}^{9,4,2,1} + \mathrm{Sq}^{11,4,1} + \mathrm{Sq}^{13,2,1} + \mathrm{Sq}^{12,3,1} + \mathrm{Sq}^{14,2}
\end{aligned}$$

2.3. **The Milnor Bockstein operations Q_i in A_2.** At the prime $p = 2$, the Milnor Bockstein operation $Q_0 = \mathrm{Sq}^1$, and Q_i is defined inductively by $Q_{i+1} = \mathrm{Sq}^{2^{i+1}}Q_i + Q_i\mathrm{Sq}^{2^{i+1}}$. The Milnor Bockstein operations are discussed in detail in [Mil58] and [Mil60].

$$\begin{aligned}
Q_0 &= \mathrm{Sq}^1 \\
Q_1 &= \mathrm{Sq}^3 + \mathrm{Sq}^{2,1} \\
Q_2 &= \mathrm{Sq}^7 + \mathrm{Sq}^{6,1} + \mathrm{Sq}^{5,2} + \mathrm{Sq}^{4,2,1} \\
Q_3 &= \mathrm{Sq}^{15} + \mathrm{Sq}^{14,1} + \mathrm{Sq}^{13,2} + \mathrm{Sq}^{11,4} + \mathrm{Sq}^{12,2,1} + \mathrm{Sq}^{11,3,1} + \mathrm{Sq}^{10,4,1} + \mathrm{Sq}^{9,4,2} + \mathrm{Sq}^{8,4,2,1}
\end{aligned}$$

2.4. **Maple source code for calculations in A_2.**

```
> with(Steenrod);
>
> # to list the basis elements in degree 7
> AdmisAnBasis(7,7);
> # to generate a list of basis elements from degree 1 to 20
> for i from 1 to 20 do AdmisAnBasis(i,20); od;
>
> # to calculate some Adem relations
> convert(Sq(1) &* Sq(2), Admissible);
> convert(&*(Sq(1),Sq(2),Sq(1)), Admissible);
> # to make a list of Adem relations
> for i from 1 to 10 do convert(Sq(i) &* Sq(10-i), Admissible); od;
>
> # to calculate \chi(Sqˆ4)
> convert( chi(Sq(4)), Admissible);
> # to generate a list
> for i from 1 to 10 do convert(chi(Sq(i)),Admissible); od;
```

3. THE MOD 3 STEENROD ALGEBRA A_3

The mod p Steenrod algebra A_p is the free $\mathbb{Z}/p$-algebra on the generators β, P^j, $j \geq 1$, modulo the Adem relations. Here β is the Bockstein operation of degree 1, and P^j is the reduced power operation of degree $2j(p-1)$. For more details, see [Koc96, p.51-52].

3.1. **The Adem relations in A_3.**

The Adem relations in the mod p Steenrod algebra A_p are

$$P^a P^b = \sum_{j=0}^{\lfloor a/p \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j$$

and

$$\begin{aligned} P^a \beta P^b \;=\; & \sum_{j=0}^{\lfloor a/p \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta P^{a+b-j} P^j \\ & + \sum_{j=0}^{\lfloor (a-1)/p \rfloor} (-1)^{a+j-1} \binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j} \beta P^j \end{aligned}$$

whenever $0 < a < pb$.

An element $\beta^{\epsilon_1} P^{i_1} \beta^{\epsilon_2} P^{i_2} \cdots$ in the mod p Steenrod algebra A_p is admissible if $i_j \geq \epsilon_{j+1} + p i_{j+1}$ for every j, where ϵ_j takes either the value 0 or 1.

In A_3 the basis elements (over $\mathbb{F}_3$) and Adem relations are

Degree	Basis elements (in braces) followed by Adem relations
0	$\{1\}$
1	$\{\beta\}$
2	$\beta^2 = 0$
3	
4	$\{P^1\}$
5	$\{\beta P^1, P^1\beta\}$
6	$\{\beta P^1\beta\}$
7	
8	$\{P^2\}$
	$P^{1,1} = 2P^2$
9	$\{\beta P^2,\ P^2\beta\}$
	$P^1\beta P^1 = \beta P^2 + P^2\beta$
10	$\{\beta P^2\beta\}$
11	
12	$\{P^3\}$
	$P^{1,2} = 0,\ P^{2,1} = 0$
13	$\{\beta P^3,\ P^3\beta\}$
	$P^1\beta P^2 = 2\beta P^3 + P^3\beta,\ P^2\beta P^1 = \beta P^3 + 2P^3\beta$
14	$\{\beta P^3\beta\}$
15	
16	$\{P^4,\ P^{3,1}\}$
	$P^{1,3} = P^4,\ P^{2,2} = 0$
17	$\{\beta P^4,\ P^4\beta,\ \beta P^{3,1},\ P^{3,1}\beta\}$
	$P^1\beta P^3 = P^4\beta,\ P^2\beta P^2 = 0,\ \textit{derived} : P^3\beta P^1 = 2\beta P^{3,1}$
18	$\{\beta P^4\beta,\ \beta P^{3,1}\beta\}$
19	
20	$\{P^5,\ P^{4,1}\}$
	$P^{1,4} = 2P^5, P^{2,3} = P^5,\ P^{3,2} = 2P^5 + P^{4,1}$
21	$\{\beta P^5,\ P^5\beta,\ \beta P^{4,1},\ P^{4,1}\beta\}$
	$P^1\beta P^4 = \beta P^5 + P^5\beta,\ P^2\beta P^3 = P^5\beta,\ P^3\beta P^2 = 2\beta P^5 + \beta P^{4,1}$
22	$\{\beta P^5\beta,\ \beta P^{4,1}\beta\}$
23	

Degree	Basis elements (in braces) followed by Adem relations
24	$\{P^6,\ P^{5,1}\}$ $P^{1,5} = 0,\ P^{2,4} = 0,\ P^{3,3} = 2P^6 + P^{5,1},\ P^{4,2} = 2P^{5,1}$
25	$\{\beta P^6,\ P^6\beta,\ \beta P^{5,1},\ P^{5,1}\beta\}$ $P^1\beta P^5 = 2\beta P^6 + P^6\beta,\ P^2\beta P^4 = \beta P^6 + 2P^6\beta,\ P^3\beta P^3 = \beta P^6 + \beta P^{5,1} + P^6\beta,$ $P^4\beta P^2 = \beta P^6 + \beta P^{5,1} + 2P^6\beta + P^5\beta P^1$
26	
27	
28	$\{P^7,\ P^{6,1}\}$ $P^{1,6} = P^7,\ P^{2,5} = 0,\ P^{3,4} = P^7 + P^{6,1},\ P^{4,3} = 2P^7,\ P^{5,2} = 0$
29	$\{\beta P^7,\ P^7\beta,\ \beta P^{6,1},\ P^{6,1}\beta\}$ $P^1\beta P^6 = P^7\beta,\ P^2\beta P^5 = 0,\ P^3\beta P^4 = \beta P^7 + \beta P^{6,1},$ $P^4\beta P^3 = 2\beta P^{6,1} + 2P^7\beta + P^6\beta P^1,\ P^5\beta P^2 = \beta P^{6,1} + 2P^6\beta P^1$
30	$\{\beta P^7\beta,\ \beta P^{6,1}\beta\}$
31	
32	$\{P^8,\ P^{7,1},\ P^{6,2}\}$ $P^{1,7} = 2P^8,\ P^{2,6} = P^8,\ P^{3,5} = P^{7,1},\ P^{4,4} = 2P^8 + P^{7,1},\ P^{5,3} = 2P^8$
33	$\{\beta P^8,\ P^8\beta,\ \beta P^{7,1},\ P^{7,1}\beta,\ \beta P^{6,2},\ P^{6,2}\beta\}$ $P^1\beta P^7 = \beta P^8 + P^8\beta,\ P^2\beta P^6 = P^8\beta,\ P^3\beta P^5 = \beta P^{7,1},$ $P^4\beta P^4 = \beta P^8 + P^8\beta + P^7\beta P^1,\ P^5\beta P^3 = 2P^8\beta$
34	$\{\beta P^8\beta,\ \beta P^{7,1}\beta,\ \beta P^{6,2}\beta\}$
35	
36	$\{P^9,\ P^{8,1},\ P^{7,2}\}$ $P^{1,8} = 0,\ P^{2,7} = 0,\ P^{3,6} = P^{8,1},\ P^{4,5} = 2P^{8,1},\ P^{5,4} = P^{8,1},\ P^{6,3} = 2P^{8,1} + P^{7,2}$
37	$\{\beta P^9,\ P^9\beta,\ \beta P^{8,1},\ P^{8,1}\beta,\ \beta P^{7,1},\ P^{7,1}\beta\}$ $P^1\beta P^8 = 2\beta P^9 + P^9\beta,\ P^2\beta P^7 = \beta P^9 + 2P^9\beta,\ P^3\beta P^6 = 2\beta P^9 + \beta P^{8,1} + P^9\beta,$ $P^4\beta P^5 = \beta P^{8,1} + P^8\beta P^1,\ P^5\beta P^4 = \beta P^9 + 2P^9\beta + P^8\beta P^1,$ $P^6\beta P^3 = \beta P^9 + 2\beta P^{8,1} + \beta P^{7,2} + 2P^9\beta$
38	$\{\beta P^9\beta,\ \beta P^{8,1}\beta,\ \beta P^{7,2}\beta\}$
39	
40	$\{P^{10},\ P^{9,1},\ P^{8,2}\}$ $P^{1,9} = P^{10},\ P^{2,8} = 0,\ P^{3,7} = 2P^{10} + P^{9,1},\ P^{4,6} = 0,$ $P^{5,5} = 0,\ P^{6,4} = P^{10} + 2P^{9,1} + P^{8,2},\ P^{7,3} = 2P^{8,2}$

3.2. **The anti-automorphism in A_3.** Let $\chi : A_3 \to A_3$ be the anti-automorphism of the Steenrod algebra. Since χ is an anti-automorphism of algebras, it is a $\mathbb{F}_3$-module homomorphism such that $\chi(ab) = \chi(b)\chi(a)$ for all $a, b \in A_3$.

We use the methods of [BM82] to calculate the anti-automorphism χ at odd primes. Let $\alpha_p(L)$ denote the sum of the coefficients in the p-adic expansion of L, and let $\nu_p(L)$ be the p-adic valuation of L which takes the value ℓ when $L = p^\ell m$ and $\gcd(p, m) = 1$. By [CCH96, p.2276], α_p and ν_p are related by $(p-1)\nu_p(L!) = L - \alpha_p(L)$. By Theorem 3.1 in Barratt and Miller's paper,

$$\chi(P^N) = (-1)^{L+1}\left(\sum_{L<j\le N}\binom{j-1}{L}P^j\chi(P^{N-j})\right)$$

whenever $(p-1)N > pL - \alpha_p(L)$. Using this theorem, we calculate

$$\begin{aligned}
\chi(P^0) &= P^0 = \mathit{identity}\\
\chi(P^1) &= 2P^1\\
\chi(P^2) &= P^2\\
\chi(P^3) &= 2P^3\\
\chi(P^4) &= 2P^{3,1}\\
\chi(P^5) &= P^5 + 2P^{4,1}\\
\chi(P^6) &= P^6 + P^{5,1}\\
\chi(P^7) &= 2P^{6,1}\\
\chi(P^8) &= P^{6,2}\\
\chi(P^9) &= 2P^9 + P^{8,1} + 2P^{7,2}\\
\chi(P^{10}) &= P^{9,1} + P^{8,2}
\end{aligned}$$

Remark: Every primitive element θ of the Steenrod algebra satisfies $\chi(\theta) = -\theta$ by [Mil58, p.167]. Since the elements P^1 and the Milnor Bockstein operations Q_i are primitive, we have we have $\chi(P^1) = -P^1$, and $\chi(Q_i) = -Q_i$. In particular, for $Q_0 = \beta$ we have $\chi(\beta) = -\beta$.

3.3. **The Milnor Bockstein operations Q_i in A_3.** At an odd prime p, the Milnor Bockstein operation $Q_0 = \beta$, and Q_i is defined inductively by $Q_{i+1} = P^{p^i}Q_i - Q_iP^{p^i}$. The Milnor Bockstein operations are discussed in detail in [Mil58] and [Mil60].

$$\begin{aligned}
Q_0 &= \beta\\
Q_1 &= P^1\beta + 2\beta P^1\\
Q_2 &= P^{3,1}\beta + 2\beta P^{3,1} + 2P^4\beta + \beta P^4
\end{aligned}$$

3.4. **Maple source code for calculations in A_p at odd primes p.**

```
> restart:
>
> # A mod p binomial coefficient
> nCr:=(p,x,y)->binomial(x,y) mod p;
>
> # one of the mod p Adem relations
> Adem:=(p,a,b)->
  if 0<a and a<p*b then RETURN(
  add( (((-1)^(a+j))*nCr(p,(p-1)*(b-j)-1,a-p*j) mod p)
  * P^(a+b-j) @ P^j,j=0..floor(a/p)) );
  else RETURN(NULL);
  fi:
```

```
> # p is the prime (not to be confused with capital P)
> # a and b are exponents on the composition P^a P^b
> # P is the reduced power operation at the prime p
> # @ is composition
> # Adem(p,a,b) = P^a \circ P^b = \sum ... (the Adem relation)
>
> ListAdem:=proc(p,x,y)
  local i,k;
  # p is the prime,
  # x is the lowest (starting) degree
  # y is the highest (ending) degree
  for i from x to y do
   for k from 1 to (i-1) do
    if k<p*(i-k) then print(P^k , P^(i-k) = Adem(p,k,i-k)) fi;
   od;
   print(`\n`);
  od;
  end:
> ListAdem(3,2,10);
>
> AdemWithBeta:=(p,a,b)->
  if 0<a and a<p*b then RETURN(
  add( (((-1)^(a+j))*nCr(p,(p-1)*(b-j),a-p*j) mod p) * beta @
  P^(a+b-j) @ P^j,j=0..floor(a/p))
  + add( (((-1)^(a+j-1))*nCr(p,(p-1)*(b-j)-1,a-p*j-1) mod p)
  * P^(a+b-j) @ beta @ P^j,j=0..floor((a-1)/p))
  )
  else RETURN(NULL); fi:
>
> ListAdemWithBeta:=proc(p,x,y)
  local i,k;
  for i from x to y do
   for k from 1 to i do
     if k<p*(i-k) then print(P^k,beta,P^(i-k)
     = AdemWithBeta(p,k,i-k)); fi;
   od;
   print(`\n`);
  od;
  end:
>
> ListAdemWithBeta(3,2,10);
>
> #--------------------------------------------
> restart:
> alpha:=(p,L)->L-((p-1)*padic[ordp](L!,p));
> seq(alpha(3,L),L=0..30);
> nCr:=(p,x,y)->binomial(x,y) mod p;
> # Theorem 3.1 says that chi(P^N) = Q(p,N,L) under certain conditions.
```

```
  # This Q is not the Milnor Bockstein operation!
> Q:=(p,N,L)->if (p-1)*N > p*L-alpha(p,L)
  then RETURN((-1)*(-1)^L*(add(nCr(p,j-1,L)
  *P^j @ chi(P^(N-j)),j=L+1..N)) mod p);
  fi;
> Q(3,0,0);
>
> Q(3,1,0);
> Q(3,1,1);
>
> Q(3,2,0);
> Q(3,2,1);
> Q(3,2,2);
>
> Q(3,3,0);
> Q(3,3,1);
> Q(3,3,2);
> Q(3,3,3);
>
> # etc.


> restart:
> isadmis:=proc(p::prime,L)
  # p is the prime you're working at
  # L is a list of exponents,
  # e.g. P^3 \beta P^1 \beta = \beta^0 P^3 \beta^1 P^1 \beta^1 P^0
  # is the list [0,3,1,1,1,0]
  # always start with the exponent of \beta (even if that exponent is zero),
  # alternate between exponents of \beta and P^k,
  # and end with the zero exponent of P^k
  # if this procedure returns one or more Not Admissible!!!!!!,
  # then the element is not admissible
  local epsilon,i,k;
  epsilon:=[seq(L[1+2*j],j=0..nops(L)/2-1)]; # print(epsilon);
  i:=[seq(L[2+2*j],j=0..nops(L)/2-1)]; # print(i);
  for k from 1 to (nops(L)/2-1)
   do
    if (i[k] >= epsilon[k+1] + p*i[k+1]) then print(`Admissible`);
    else print(`Not Admissible!!!!!!!!`); fi;
   od;
  end proc:
> isadmis(3,[1,3,0,1,0,0]); # \beta P^{3,1}
> isadmis(3,[0,3,1,1,0,0]); # P^3 \beta P^1
> isadmis(3,[0,3,0,1,1,0]); # P^{3,1} \beta
```

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