THE MOD p COHOMOLOGY OF BROWN-GITLER SPECTRA

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Abstract. We calculate the mod 2 cohomology and mod 3 cohomology of some Brown-Gitler spectra, and describe the mod p cohomology of the Brown-Gitler spectra. The role these cohomology calculations play in determining the mod p Dieudonné ring of a cohomology theory E with cup products is briefly discussed in the introduction. Some calculations of the anti-automorphism of the mod p Steenrod algebra are given as a reference.

Contents

1. Introduction

Brown-Gitler spectra have had many important applications in homotopy theory, notably their role in the construction of the η_j family in the stable homotopy groups of spheres in [Mah77], and their role in the immersions of manifolds [Coh85]. We are interested in them because for a ring spectrum E, the E homology of Brown-Gitler spectra not only surjects to the mod p Dieudonné ring of E, but is an isomorphism much of the time (see [Goe99, Prop. 11.1] for details).

Date: November 21, 2006, 9:00am.

The method of calculating the Dieudonné ring $D_*\mathrm{H}\mathbb{F}_{p*}\underline{E}_*$ we often use involves a change of rings theorem

$$\mathrm{Ext}_A^{s,t}(\mathrm{H}\mathbb{F}_p^*(B(n) \wedge E), \Sigma^t\mathbb{F}_p) \cong \mathrm{Ext}_C^{s,t}(\mathrm{H}\mathbb{F}_p^*(B(n)), \Sigma^t\mathbb{F}_p),$$

for some subalgebra C of the Steenrod algebra A depending on E, the Adams spectral sequence

$$\mathrm{Ext}_C^s(\mathrm{H}\mathbb{F}_p^*(B(n)), \Sigma^t\mathbb{F}_p) \implies \pi_{t-s}(B(n) \wedge E)$$

where $\pi_{t-s}(B(n) \wedge E) = B(n)_{t-s}(E) \cong E_{t-s}(B(n))$, and the surjection

$$B(n)_nE \to D_n\mathrm{H}\mathbb{F}_{p*}\Omega^\infty E$$

of [Goe99, Prop. 11.1]. Here $\underline{E}_*$ denotes the collection of spaces in the Ω-spectrum associated to the spectrum E, and $\underline{E}_k = \Omega^\infty\Sigma^k E$.

In this short note, we aim to calculate the cohomology of a few Brown-Gitler spectra for input to the Adams spectral sequence.

Remarks on Brown-Gitler spectra: At the prime 2, we index the Brown-Gitler spectra by the integers rather than the half-integers, following the notational conventions of [Goe99], so the spectrum originally called $B([n/2])$ by Brown and Gitler in [BG73] is called $B(n)$ here. At odd primes p, the indexing we use for Brown Gitler spectra is different from that in [Coh81]. The indexing we use is compatible with the Dieudonné ring calculations we have in mind.

2. THE MOD 2 COHOMOLOGY OF BROWN-GITLER SPECTRA

Let $A = \mathrm{H}\mathbb{F}_2^*(\mathrm{H}\mathbb{F}_2)$ denote the mod 2 Steenrod algebra. Let $B(n)$ denote the nth Brown-Gitler spectrum over $\mathrm{H}\mathbb{F}_2$. For each integer $n \geq 0$, $B(n)$ is a $\mathbb{Z}/2$-complete spectrum. Set $B(0) = S^0$ completed at 2, and $B(2n) \simeq B(2n+1)$. Then for $n \geq 2$, there is an isomorphism of left A-modules

$$\mathrm{H}\mathbb{F}_2^*(B(n)) \cong A/A\{\chi(\mathrm{Sq}^i) : 2i > n\}$$

by [Goe99, equation 11.1], [GLM93, Theorem 3.1], where $A\{\cdot\}$ deontes the left ideal. For each integer $n \geq 1$ there is a cofiber sequence

$$(B(2n-1) \simeq)\ B(2n-2) \to B(2n) \to \Sigma^n B(n)$$

that induces a short exact sequence of left A-modules

$$0 \leftarrow \mathrm{H}\mathbb{F}_2^*(B(2n-2)) \leftarrow \mathrm{H}\mathbb{F}_2^*(B(2n)) \leftarrow \Sigma^n\mathrm{H}\mathbb{F}_2^*(B(n)) \leftarrow 0$$

in which $\Sigma^n 1 \in \Sigma^n\mathrm{H}\mathbb{F}_2^*(B(n))$ maps to $\chi(\mathrm{Sq}^n) \in \mathrm{H}\mathbb{F}_2^*(B(2n))$. Using this short exact sequence and induction on n, it is possible to show that the set

$$S = \{\, \chi(\mathrm{Sq}^{i_1,i_2,\ldots,i_j}) \mid i_1 \leq n, i_2 \neq 0, \text{ and } i_1, i_2, \ldots, i_j \text{ is admissible} \,\}$$

is a basis for $\mathrm{H}\mathbb{F}_2^*(B(2n))$ as an $\mathbb{F}_2$-module. We use the compact notation $\mathrm{Sq}^{i_1,i_2,\ldots,i_j}$ for the composite $\mathrm{Sq}^{i_1} \circ \mathrm{Sq}^{i_2} \circ \cdots \circ \mathrm{Sq}^{i_j}$.

In all of the calculations and diagrams in this section, we will use the basis S. Sometimes the choice of S as a basis will cause us to modify notation. Such notation changes will be explained as they arise.

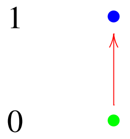
2.1. **Calculating** $\mathrm{H}\mathbb{F}_2^*(B(0)) = \mathrm{H}\mathbb{F}_2^*(B(1))$**.** This is straightforward since $B(0) \simeq B(1) \simeq S^0$ (completed at 2).

$\mathrm{H}\mathbb{F}_2^*(B(0))$

2.2. **Calculating** $\mathrm{H}\mathbb{F}_2^*(B(2)) = \mathrm{H}\mathbb{F}_2^*(B(3))$**.** We use the short exact sequence of left A-modules

$$0 \leftarrow \mathrm{H}\mathbb{F}_2^*(B(0)) \leftarrow \mathrm{H}\mathbb{F}_2^*(B(2)) \leftarrow \Sigma^1\mathrm{H}\mathbb{F}_2^*(B(1)) \leftarrow 0$$

to help us calculate $\mathrm{H}\mathbb{F}_2^*(B(2))$. We highlight in green the part of $\mathrm{H}\mathbb{F}_2^*(B(2))$ coming from $\mathrm{H}\mathbb{F}_2^*(B(0))$, and in blue the part coming from $\Sigma^1\mathrm{H}\mathbb{F}_2^*(B(1))$. We color in red the Steenrod square operations that connect the parts of $\mathrm{H}\mathbb{F}_2^*(B(2))$ coming from $\mathrm{H}\mathbb{F}_2^*(B(0))$ and $\Sigma^1\mathrm{H}\mathbb{F}_2^*(B(1))$.

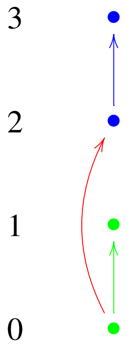


$\mathrm{H}\mathbb{F}_2^*(B(2))$

2.3. **Calculating** $\mathrm{H}\mathbb{F}_2^*(B(4)) = \mathrm{H}\mathbb{F}_2^*(B(5))$**.** We use the short exact sequence of left A-modules

$$0 \leftarrow \mathrm{H}\mathbb{F}_2^*(B(2)) \leftarrow \mathrm{H}\mathbb{F}_2^*(B(4)) \leftarrow \Sigma^2\mathrm{H}\mathbb{F}_2^*(B(2)) \leftarrow 0$$

to help us calculate $\mathrm{H}\mathbb{F}_2^*(B(4))$. We use the same color conventions as before.

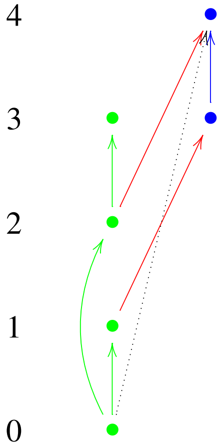


$\mathrm{H}\mathbb{F}_2^*(B(4))$

2.4. Calculating $\mathrm{H}\mathbb{F}_2^*(B(6)) = \mathrm{H}\mathbb{F}_2^*(B(7))$. We use the short exact sequence of left A-modules

$$0 \leftarrow \mathrm{H}\mathbb{F}_2^*(B(4)) \leftarrow \mathrm{H}\mathbb{F}_2^*(B(6)) \leftarrow \Sigma^3\mathrm{H}\mathbb{F}_2^*(B(3)) \leftarrow 0$$

to help us calculate $\mathrm{H}\mathbb{F}_2^*(B(6))$. We use the same color conventions as before. The dotted black arrow is a Sq^4 that is decomposable because of the relation $\chi(\mathrm{Sq}^4) = \mathrm{Sq}^4 + \mathrm{Sq}^{1,2,1} \sim 0$. We will often denote decomposable squaring operations by dotted black arrows, and leave them out of subsequent diagrams.

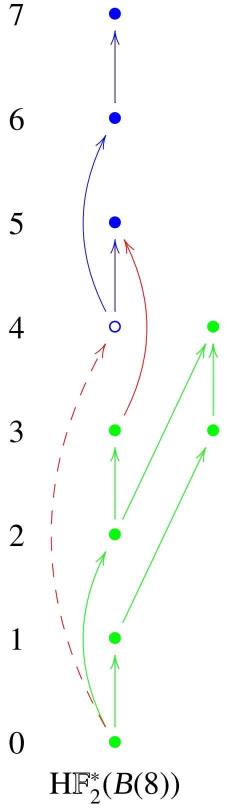


$\mathrm{H}\mathbb{F}_2^*(B(6))$

2.5. **Calculating** $\mathrm{H}\mathbb{F}_2^*(B(8)) = \mathrm{H}\mathbb{F}_2^*(B(9))$**.** We use the short exact sequence of left A-modules

$$0 \leftarrow \mathrm{H}\mathbb{F}_2^*(B(6)) \leftarrow \mathrm{H}\mathbb{F}_2^*(B(8)) \leftarrow \Sigma^4\mathrm{H}\mathbb{F}_2^*(B(4)) \leftarrow 0$$

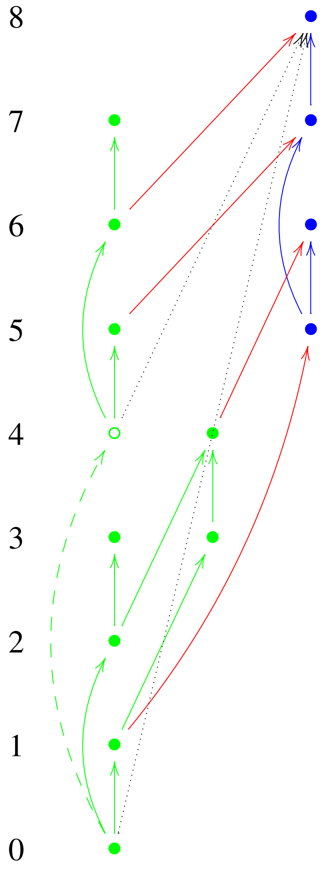
to help us calculate $\mathrm{H}\mathbb{F}_2^*(B(8))$. We use the same color conventions as before. One of the cohomology classes in degree 4 is depicted by a blue circle instead of a blue dot because it is represented by $\chi(\mathrm{Sq}^4) = \mathrm{Sq}^4 + \mathrm{Sq}^{3,1}$, not Sq^4. Similarly, the red arrow connecting the cohomology classes 1 and $\chi(\mathrm{Sq}^4)$ is dashed to indicate that the arrow is $\chi(\mathrm{Sq}^4)$, not Sq^4. We could perform a change of basis and replace ○ $= \chi(\mathrm{Sq}^4)$ by ● $= \mathrm{Sq}^4$, but this would take us away from the basis for $\mathrm{H}\mathbb{F}_2^*(B(8))$ given naturally by the short exact sequence above.



2.6. **Calculating** $\mathrm{H}\mathbb{F}_2^*(B(10)) = \mathrm{H}\mathbb{F}_2^*(B(11))$**.** We use the short exact sequence of left A-modules

$$0 \leftarrow \mathrm{H}\mathbb{F}_2^*(B(8)) \leftarrow \mathrm{H}\mathbb{F}_2^*(B(10)) \leftarrow \Sigma^5\mathrm{H}\mathbb{F}_2^*(B(5)) \leftarrow 0$$

to help us calculate $\mathrm{H}\mathbb{F}_2^*(B(10))$. We use the same color conventions as before. As with the cohomology of $B(8)$, both the green circle and dashed green arrow indicate that the cohomology class and arrow are $\chi(\mathrm{Sq}^4)$, not Sq^4. The dotted black arrows are a decomposable Sq^4 and a decomposable Sq^8 resulting from the relations $\chi(\mathrm{Sq}^{4,2}) \sim 0$ and $\chi(\mathrm{Sq}^8) \sim 0$.

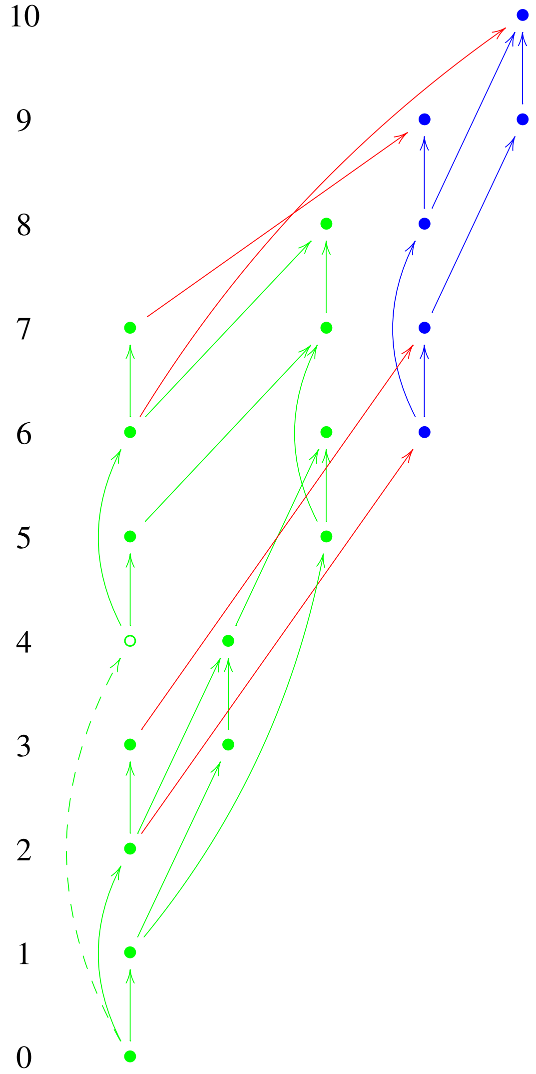


$\mathrm{H}\mathbb{F}_2^*(B(10))$

2.7. **Calculating** $H\mathbb{F}_2^*(B(12)) = H\mathbb{F}_2^*(B(13))$**.** We use the short exact sequence of left A-modules

$$0 \leftarrow H\mathbb{F}_2^*(B(10)) \leftarrow H\mathbb{F}_2^*(B(12)) \leftarrow \Sigma^6 H\mathbb{F}_2^*(B(6)) \leftarrow 0$$

to help us calculate $H\mathbb{F}_2^*(B(12))$. We use the same color conventions as before. We remark that in $H\mathbb{F}_2^*(B(12))$, $Sq^4\chi(Sq^4) \sim Sq^{1,2,4,1} + Sq^{2,4,2}$ because of the relation $\chi(Sq^7) = 0$, and $Sq^8 \sim Sq^{7,1} + Sq^{6,2} + Sq^{5,2,1} \sim Sq^{1,2,4,1} + Sq^{2,4,2}$ because of the relation $\chi(Sq^8) = 0$, so the black dotted arrows Sq^4 and Sq^8 from $H\mathbb{F}_2^*(B(10))$ are not in the diagram for $H\mathbb{F}_2^*(B(12))$.

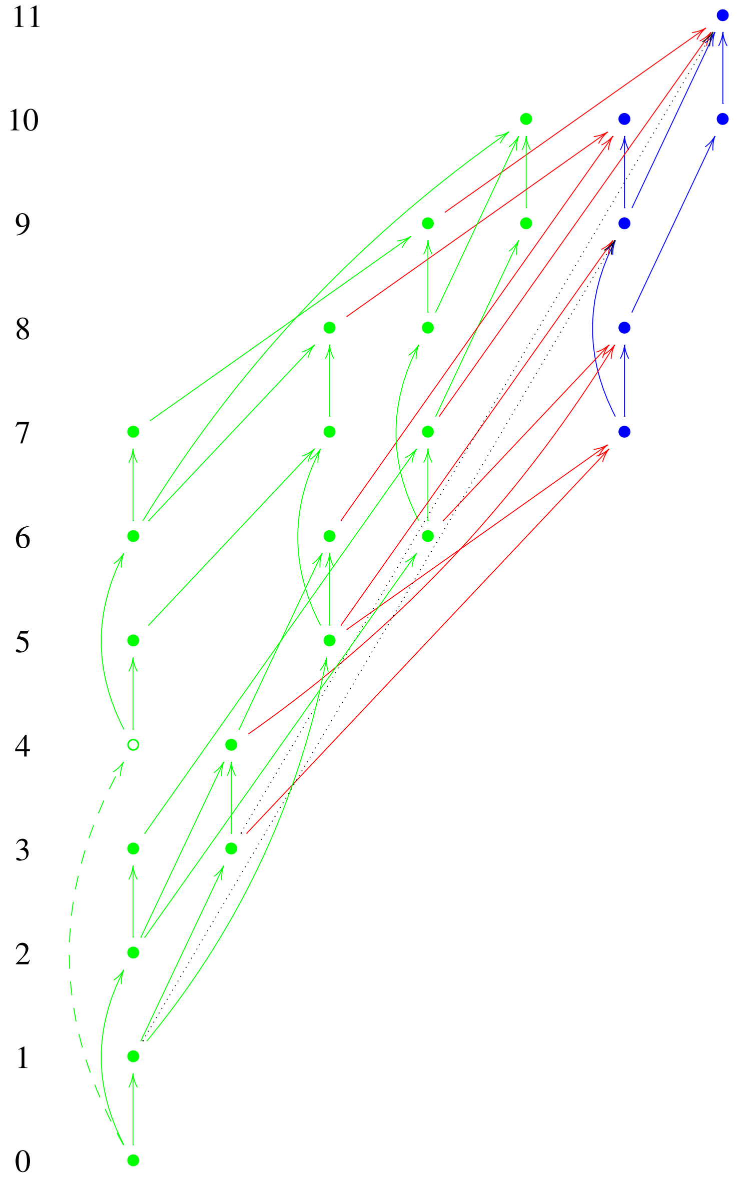


$H\mathbb{F}_2^*(B(12))$

2.8. **Calculating** $H\mathbb{F}_2^*(B(14)) = H\mathbb{F}_2^*(B(15))$**.** We use the short exact sequence of left A-modules

$$0 \leftarrow H\mathbb{F}_2^*(B(12)) \leftarrow H\mathbb{F}_2^*(B(14)) \leftarrow \Sigma^7 H\mathbb{F}_2^*(B(7)) \leftarrow 0$$

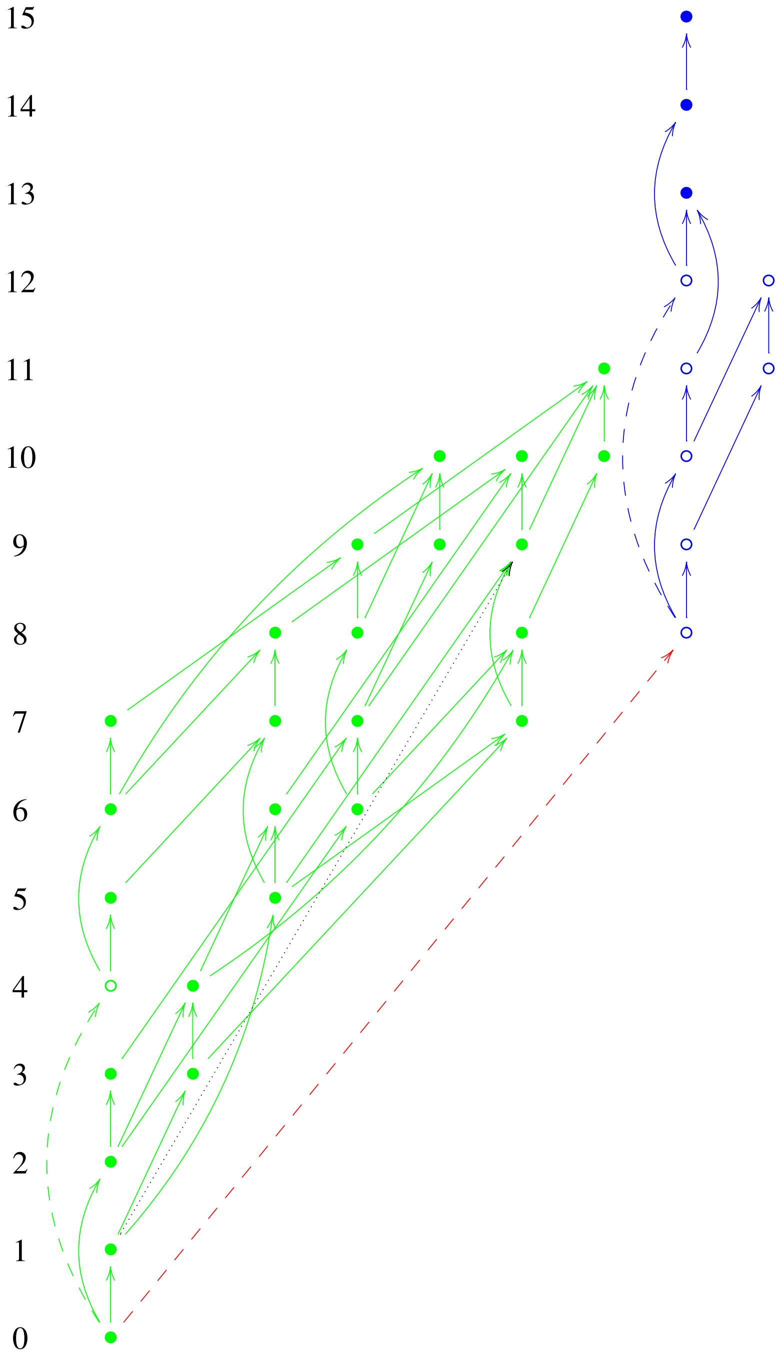
to help us calculate $H\mathbb{F}_2^*(B(14))$. We use the same conventions as before.



$H\mathbb{F}_2^*(B(14))$

2.9. **Calculating** $\mathrm{H}\mathbb{F}_2^*(B(16)) = \mathrm{H}\mathbb{F}_2^*(B(17))$**.** We use

$$0 \leftarrow \mathrm{H}\mathbb{F}_2^*(B(14)) \leftarrow \mathrm{H}\mathbb{F}_2^*(B(16)) \leftarrow \Sigma^8\mathrm{H}\mathbb{F}_2^*(B(8)) \leftarrow 0.$$



$\mathrm{H}\mathbb{F}_2^*(B(16))$

We have intentionally left out many red arrows in $\mathrm{H}\mathbb{F}_2^*(B(16))$. The reader is left to fill them in. In particular, there are not very many Sq^{2^i} from the green dots to the blue dots, but there are $\chi(\mathrm{Sq}^{2^i})$ that could be in their place. Thus, we would have to make choices about which kind of arrows are worth showing. It seems more profitable to leave the diagram unadorned with red arrows so that the reader may fill it in with arrows that suit his or her purpose.

3. A basis for $\mathrm{H}\mathbb{F}_*(B(2n))$

Recall that a basis for $\mathrm{H}\mathbb{F}^*(B(2n))$ over $\mathbb{F}$ is $S = \{\chi(\mathrm{Sq}^I) : i_1 \leq n\}$. The $\mathbb{F}$-linear dual of $\mathrm{H}\mathbb{F}^*(B(2n))$ is easily described in as a subobject of $\mathrm{Hom}_{\mathbb{F}-mod}(\mathrm{H}\mathbb{F}^*(\mathrm{H}\mathbb{F}), \mathbb{F}) = \mathrm{H}\mathbb{F}_*(\mathrm{H}\mathbb{F})$, the dual of the Steenrod algebra, using a weight function. Let $\zeta_i = \chi(\xi_i)$ where $\xi_i = (\mathrm{Sq}^{2^{i-1}}\mathrm{Sq}^{2^{i-2}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1)^*$. Define a weight function on monomials by $w(\zeta_1^{\alpha_1} \cdots \zeta_k^{\alpha_k}) = \sum_{i=1}^k \alpha_i 2^{i-1}$.

1**. Theorem.** *As a module over* $\mathbb{F}$.

$$\mathrm{Hom}_{\mathbb{F}-mod}(\mathrm{H}\mathbb{F}^*(B(2n)), \mathbb{F}) = \mathrm{H}\mathbb{F}_*(B(2n)) = \mathbb{F}\{\zeta_1^{\alpha_1} \cdots \zeta_k^{\alpha_k} : w(\zeta_1^{\alpha_1} \cdots \zeta_k^{\alpha_k}) \leq n\}.$$

Further, these are isomorphisms of canonical right A modules, where the canonical right A module structure on $\mathrm{Hom}_{\mathbb{F}-mod}(\mathrm{H}\mathbb{F}^*(B(2n)), \mathbb{F}) = \mathrm{H}\mathbb{F}_*(B(2n))$ *is induced by the canonical left A module structure on* $\mathrm{H}\mathbb{F}^*(B(2n)) = A/A\{\chi(\mathrm{Sq}^i) : i > n\}$.

4. THE ANTI-AUTOMORPHISM OF THE MOD 2 STEENROD ALGEBRA

Let $\chi : A \to A$ be the anti-automorphism of the Steenrod algebra. Since χ is an anti-automorphism of algebras, it is an $\mathbb{F}_2$-module homomorphism such that $\chi(ab) = \chi(b)\chi(a)$ for all $a, b \in A$. In this section, we denote $\mathrm{Sq}^i \circ \mathrm{Sq}^j$ by $\mathrm{Sq}^{i,j}$ for economy. For $n \geq k$, the values of χ are determined recursively by the formulas

$$\chi(\mathrm{Sq}^{2^n-k}) = \mathrm{Sq}^{2^{n-1},2^{n-2},\ldots,2^k,2^{k-1}} \circ (\chi(\mathrm{Sq}^{2^{k-1}-k}))$$

and

$$\chi(\mathrm{Sq}^{2^{k-1}-k}) = \mathrm{Sq}^{2^{k-2}} \circ (\chi(\mathrm{Sq}^{2^{k-2}-k})) + \mathrm{Sq}^{2^{k-2}-1,2^{k-3}-1,\ldots,3,1}$$

from [Dav74]. We list some values of χ below.

$$\begin{aligned}
\chi(1) &= 1 \\
\chi(\mathrm{Sq}^1) &= \mathrm{Sq}^1 \\
\chi(\mathrm{Sq}^2) &= \mathrm{Sq}^2 \\
\chi(\mathrm{Sq}^3) &= \mathrm{Sq}^{2,1} \\
\chi(\mathrm{Sq}^4) &= \mathrm{Sq}^4 + \mathrm{Sq}^{3,1} \\
\chi(\mathrm{Sq}^5) &= \mathrm{Sq}^{4,1} \\
\chi(\mathrm{Sq}^6) &= \mathrm{Sq}^{4,2} \\
\chi(\mathrm{Sq}^7) &= \mathrm{Sq}^{4,2,1} \\
\chi(\mathrm{Sq}^8) &= \mathrm{Sq}^8 + \mathrm{Sq}^{7,1} + \mathrm{Sq}^{6,2} + \mathrm{Sq}^{5,2,1} \\
\chi(\mathrm{Sq}^9) &= \mathrm{Sq}^{8,1} + \mathrm{Sq}^{6,2,1} \\
\chi(\mathrm{Sq}^{10}) &= \mathrm{Sq}^{8,2} + \mathrm{Sq}^{7,3} + \mathrm{Sq}^{6,3,1} \\
\chi(\mathrm{Sq}^{11}) &= \mathrm{Sq}^{8,2,1} + \mathrm{Sq}^{7,3,1} \\
\chi(\mathrm{Sq}^{12}) &= \mathrm{Sq}^{8,4} + \mathrm{Sq}^{8,3,1} \\
\chi(\mathrm{Sq}^{13}) &= \mathrm{Sq}^{8,4,1} \\
\chi(\mathrm{Sq}^{14}) &= \mathrm{Sq}^{8,4,2} \\
\chi(\mathrm{Sq}^{15}) &= \mathrm{Sq}^{8,4,2,1} \\
\chi(\mathrm{Sq}^{16}) &= \mathrm{Sq}^{16} + \mathrm{Sq}^{15,1} + \mathrm{Sq}^{14,2} + \mathrm{Sq}^{12,4} + \mathrm{Sq}^{13,2,1} \\
&\quad + \mathrm{Sq}^{12,3,1} + \mathrm{Sq}^{11,4,1} + \mathrm{Sq}^{10,4,2} + \mathrm{Sq}^{9,4,2,1}
\end{aligned}$$

5. THE MOD p COHOMOLOGY OF BROWN-GITLER SPECTRA

Let $A = A_p = \mathrm{H}\mathbb{F}_p^*(\mathrm{H}\mathbb{F}_p)$ denote the mod p Steenrod algebra for an odd prime p. Let $B(n)$ denote the Brown-Gitler spectrum at the integer n, following the indexing conventions of [Goe99]. For each integer $n \geq 0$, $B(n)$ is a p-complete spectrum. Set $B(0) = S^0$ (completed at p), and if $n \equiv \pm 1 \bmod 2p$, set $B(n) = B(n-1)$. Then, for $n \geq 2$ there is an isomorphism of left A-modules

$$\mathrm{H}\mathbb{F}_p^*(B(n)) \cong A/A\{\, \chi(\beta^\epsilon P^i) : 2pi + 2\epsilon > n, \epsilon = 0, 1 \,\}$$

by [Goe99, equation 11.2], [GLM93, Theorem 3.1], where $A\{\cdot\}$ deontes the left ideal. For each integer $n \geq 0$ there is a cofiber sequence

$$(B(2pn-1) \simeq)\ B(2pn-2) \to B(2pn) \to \Sigma^{2pn-2n} B(2n)$$

that induces a short exact sequence of left A-modules

$$0 \leftarrow \mathrm{H}\mathbb{F}_p^*(B(2pn-2)) \leftarrow \mathrm{H}\mathbb{F}_p^*(B(2pn)) \leftarrow \Sigma^{2pn-2n}\mathrm{H}\mathbb{F}_p^*(B(2n)) \leftarrow 0$$

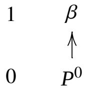
in which $\Sigma^{2pn-2n} 1 \in \Sigma^{2pn-2n}\mathrm{H}\mathbb{F}_p^*(B(2n))$ maps to $\chi(P^n) \in \mathrm{H}\mathbb{F}_p^*(B(2n))$.

6. THE MOD 3 COHOMOLOGY OF BROWN-GITLER SPECTRA

6.1. **Calculating** $\mathrm{H}\mathbb{F}_3^*(B(0)) = \mathrm{H}\mathbb{F}_3^*(B(1))$**.** This is straightforward since $B(0) \simeq B(1) \simeq S^0$.

$\mathrm{H}\mathbb{F}_3^*(B(0))$

6.2. **Calculating** $\mathrm{H}\mathbb{F}_3^*(B(2)) = \cdots = \mathrm{H}\mathbb{F}_3^*(B(5))$**.** Remarks: $\chi(P^1) = -P^1 = 0$, and all higher degree operations vanish.

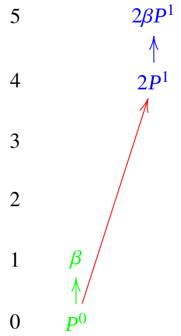


$\mathrm{H}\mathbb{F}_3^*(B(2)) = \cdots = \mathrm{H}\mathbb{F}_3^*(B(5))$

6.3. **Calculating** $H\mathbb{F}_3^*(B(6)) = H\mathbb{F}_3^*(B(7))$**.** Remarks: $\chi(\beta^1 P^1) = P^1\beta^1 = 0$, and all higher degree operations vanish. The cofiber sequence $B(4) \to B(6) \to \Sigma^4 B(2)$ induces a short exact sequence of left A-modules

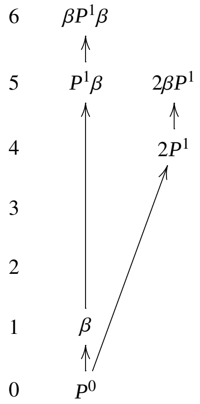
$$0 \leftarrow H\mathbb{F}_3^*(B(4)) \leftarrow H\mathbb{F}_3^*(B(6)) \leftarrow \Sigma^4 H\mathbb{F}_3^*(B(2)) \leftarrow 0$$

which we may use to calculate $H\mathbb{F}_3^*(B(6))$.



$H\mathbb{F}_3^*(B(6)) = H\mathbb{F}_3^*(B(7))$

6.4. **Calculating** $H\mathbb{F}_3^*(B(8)) = \cdots = H\mathbb{F}_3^*(B(11))$**.** Remarks: $\chi(P^2) = P^2 = 0$, and all higher degree operations vanish.

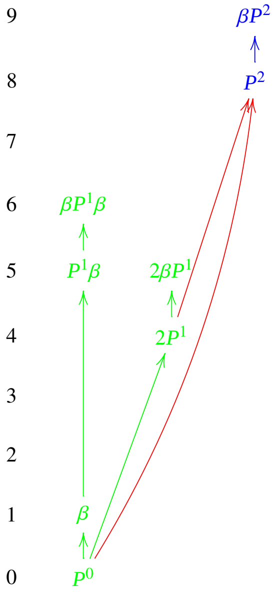


$H\mathbb{F}_3^*(B(8)) = \cdots = H\mathbb{F}_3^*(B(11))$

6.5. **Calculating $\mathrm{H}\mathbb{F}_3^*(B(12)) = \mathrm{H}\mathbb{F}_3^*(B(13))$.** Remarks: $\chi(\beta P^2) = -P^2\beta = 0$, and all higher degree operations vanish. The cofiber sequence $B(10) \to B(12) \to \Sigma^8 B(4)$ induces a short exact sequence of left A-modules

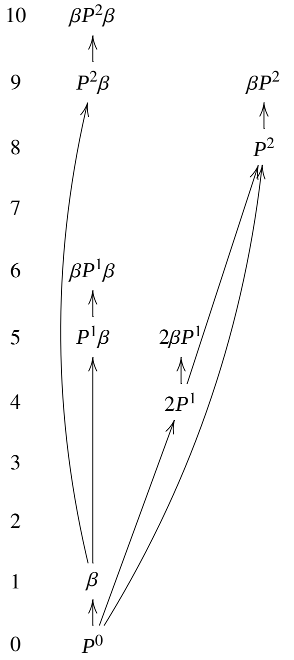
$$0 \leftarrow \mathrm{H}\mathbb{F}_3^*(B(10)) \leftarrow \mathrm{H}\mathbb{F}_3^*(B(12)) \leftarrow \Sigma^8\mathrm{H}\mathbb{F}_3^*(B(4)) \leftarrow 0$$

which we may use to calculate $\mathrm{H}\mathbb{F}_3^*(B(12))$.



$\mathrm{H}\mathbb{F}_3^*(B(12)) = \mathrm{H}\mathbb{F}_3^*(B(13))$

6.6. **Calculating** $\mathrm{H}\mathbb{F}_3^*(B(14)) = \cdots = \mathrm{H}\mathbb{F}_3^*(B(17))$**.** Remarks: $\chi(P^3) = -P^3 = 0$, and all higher degree operations vanish.

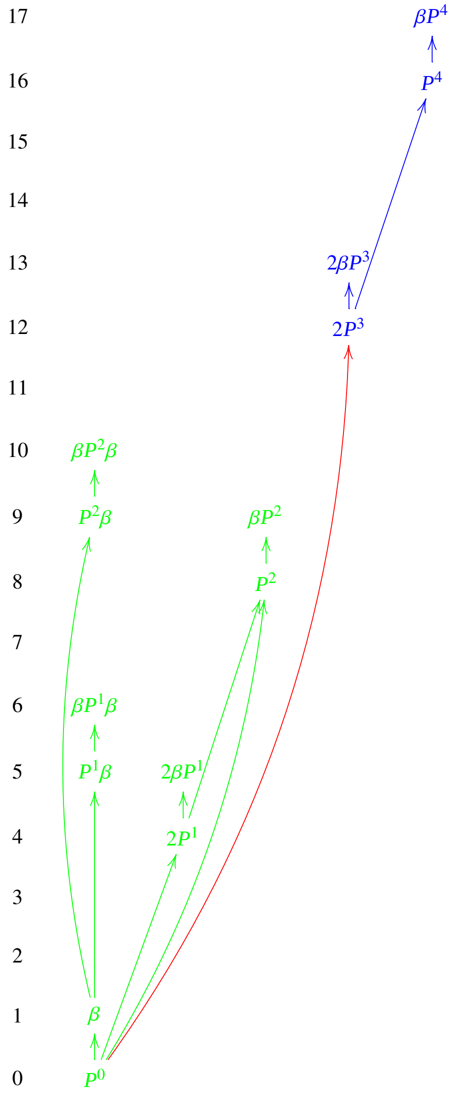


$\mathrm{H}\mathbb{F}_3^*(B(14)) = \cdots = \mathrm{H}\mathbb{F}_3^*(B(17))$

6.7. **Calculating** $\mathrm{H}\mathbb{F}_3^*(B(18)) = \mathrm{H}\mathbb{F}_3^*(B(19))$**.** Remarks: $\chi(\beta P^3) = (-P^3)(-\beta) = P^3\beta = 0, \chi(P^4) = -P^{3,1}, \chi(\beta P^4) = P^{3,1}\beta$, and all operations in degrees ≥ 18 vanish. The cofiber sequence $B(16) \to B(18) \to \Sigma^{12}B(6)$ induces a short exact sequence of left A-modules

$$0 \leftarrow \mathrm{H}\mathbb{F}_3^*(B(16)) \leftarrow \mathrm{H}\mathbb{F}_3^*(B(18)) \leftarrow \Sigma^{12}\mathrm{H}\mathbb{F}_3^*(B(6)) \leftarrow 0$$

which we may use to calculate $\mathrm{H}\mathbb{F}_3^*(B(18))$.



$\mathrm{H}\mathbb{F}_3^*(B(18)) = \mathrm{H}\mathbb{F}_3^*(B(19))$

7. THE ANTI-AUTOMORPHISM IN A_3

Let $\chi : A_3 \to A_3$ be the anti-automorphism of the Steenrod algebra. Since χ is an anti-automorphism of algebras, it is a $\mathbb{F}_3$-module homomorphism such that $\chi(ab) = \chi(b)\chi(a)$ for all $a, b \in A_3$.

We use the methods of [BM82] to calculate the anti-automorphism χ at odd primes. Let $\alpha_p(L)$ denote the sum of the coefficients in the p-adic expansion of L, and let $\nu_p(L)$ be the p-adic valuation of L which takes the value ℓ when $L = p^\ell m$ and $\gcd(p, m) = 1$. By [CCH96, p.2276], α_p and ν_p are related by $(p-1)\nu_p(L!) = L - \alpha_p(L)$. By Theorem 3.1 in Barratt and Miller's paper,

$$\chi(P^N) = (-1)^{L+1}\left(\sum_{L<j\le N}\binom{j-1}{L}P^j\chi(P^{N-j})\right)$$

whenever $(p-1)N > pL - \alpha_p(L)$. Using this theorem, we calculate

$$\begin{array}{rcl}
\chi(P^0) & = & P^0 = identity \\
\chi(P^1) & = & 2P^1 \\
\chi(P^2) & = & P^2 \\
\chi(P^3) & = & 2P^3 \\
\chi(P^4) & = & 2P^{3,1} \\
\chi(P^5) & = & P^5 + 2P^{4,1} \\
\chi(P^6) & = & P^6 + P^{5,1} \\
\chi(P^7) & = & 2P^{6,1} \\
\chi(P^8) & = & P^{6,2} \\
\chi(P^9) & = & 2P^9 + P^{8,1} + 2P^{7,2} \\
\chi(P^{10}) & = & P^{9,1} + P^{8,2}
\end{array}$$

Remark: Every primitive element θ of the Steenrod algebra satisfies $\chi(\theta) = -\theta$ by [Mil58, p.167]. Since the elements P^1 and the Milnor Bockstein operations Q_i are primitive, we have we have $\chi(P^1) = -P^1$, and $\chi(Q_i) = -Q_i$. In particular, for $Q_0 = \beta$ we have $\chi(\beta) = -\beta$.

7.1. **Maple source code for calculations in A_p at odd primes p.**

```
> restart:
>
> # A mod p binomial coefficient
> nCr:=(p,x,y)->binomial(x,y) mod p;
>
> # one of the mod p Adem relations
> Adem:=(p,a,b)->
  if 0<a and a<p*b then RETURN(
```

```
  add( (((-1)^(a+j))*nCr(p,(p-1)*(b-j)-1,a-p*j) mod p)
  * P^(a+b-j) @ P^j,j=0..floor(a/p)) );
  else RETURN(NULL);
  fi:
> # p is the prime (not to be confused with capital P)
> # a and b are exponents on the composition P^a P^b
> # P is the reduced power operation at the prime p
> # @ is composition
> # Adem(p,a,b) = P^a \circ P^b = \sum ... (the Adem relation)
>
> ListAdem:=proc(p,x,y)
  local i,k;
  # p is the prime,
  # x is the lowest (starting) degree
  # y is the highest (ending) degree
  for i from x to y do
   for k from 1 to (i-1) do
    if k<p*(i-k) then print(P^k , P^(i-k) = Adem(p,k,i-k)) fi;
   od;
   print(`\n`);
  od;
  end:
> ListAdem(3,2,10);
>
> AdemWithBeta:=(p,a,b)->
  if 0<a and a<p*b then RETURN(
  add( (((-1)^(a+j))*nCr(p,(p-1)*(b-j),a-p*j) mod p) * beta @
  P^(a+b-j) @ P^j,j=0..floor(a/p))
  + add( (((-1)^(a+j-1))*nCr(p,(p-1)*(b-j)-1,a-p*j-1) mod p)
  * P^(a+b-j) @ beta @ P^j,j=0..floor((a-1)/p))
  )
  else RETURN(NULL); fi:
>
> ListAdemWithBeta:=proc(p,x,y)
  local i,k;
  for i from x to y do
   for k from 1 to i do
     if k<p*(i-k) then print(P^k,beta,P^(i-k)
     = AdemWithBeta(p,k,i-k)); fi;
   od;
   print(`\n`);
  od;
```

```
  end:
>
> ListAdemWithBeta(3,2,10);
>
> #------------------------------------------
> restart:
> alpha:=(p,L)->L-((p-1)*padic[ordp](L!,p));
> seq(alpha(3,L),L=0..30);
> nCr:=(p,x,y)->binomial(x,y) mod p;
> # Theorem 3.1 says that chi(P^N) = Q(p,N,L) under certain conditions.
  # This Q is not the Milnor Bockstein operation!
> Q:=(p,N,L)->if (p-1)*N > p*L-alpha(p,L)
  then RETURN((-1)*(-1)^L*(add(nCr(p,j-1,L)
  *P^j @ chi(P^(N-j)),j=L+1..N)) mod p);
  fi;
> Q(3,0,0);
>
> Q(3,1,0); Q(3,1,1);
>
> Q(3,2,0); Q(3,2,1); Q(3,2,2);
>
> Q(3,3,0); Q(3,3,1); Q(3,3,2); Q(3,3,3); # etc.
```

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