Chapter 7
Sheaf theory

The theory of sheaves has come to play a central rôle in the theories of several complex variables and holomorphic differential geometry. The theory is also essential to real analytic geometry. The theory of sheaves provides a framework for solving "local to global" problems of the sort that are normally solved using partitions of unity in the smooth case. In this chapter we provide a fairly comprehensive overview of sheaf theory. Since we are interested in fairly concrete applications of the theory, our presentation is correspondingly concrete. When one delves deeply into sheaf theory, a categorical approach is significantly more efficient than the direct approach we undertake here. However, for many first-timers to the world of sheaves—particularly those coming to sheaves from the differential geometric rather than the algebraic world—the categorical setting for sheaf theory is an impediment to understanding the point of the theory. This being said, the reader looking for more than the superficial understanding of sheaves we will provide here will benefit from a more sophisticated approach, and for this we refer to [Kashiwara and Schapira 1990] for a geometric treatment, or to [Godement 1958] or [Bredon 1997] for a treatment with the focus on algebraic topological applications of sheaf theory. A concise overview can be found in [Warner 1983, Chapter 5], along with some differential geometric applications.

7.1 Elementary sheaf theory

In this section we review those parts of the theory that will be useful for us. Our interest in sheaves arises primarily in the context of holomorphic and real analytic functions and sections of real analytic vector bundles. However, in order to provide some colour for the particular setting in which we are interested, we give a treatment with greater generality. The treatment, however, is far from comprehensive, and we refer to the references at the beginning of the chapter for more details.

One of the places we do engage in some degree of generality is the class of functions and sections for which we consider sheaves. While our applications of sheaf theory will focus on the holomorphic and real analytic cases, we will also treat the cases of general differentiability. Specifically, we consider sheaves of functions and sections of class C^r for $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$. The manifolds on which we consider a certain class of differentiability will, of course, vary with the degree of differentiability. To encode this, we shall use the language, “ let $r' \in \{\infty, \omega, \mathrm{hol}\}$ be as required.” By this we mean that $r' = \infty$ if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, that $r' = \omega$ if $r = \omega$, and $r' = \mathrm{hol}$ if $r = \mathrm{hol}$. Also, we shall implicitly or explicitly let $\mathbb{F} = \mathbb{R}$ if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$.

7.1.1 Presheaves

The basic ingredient in the theory of sheaves is a presheaf. We shall need various sorts of presheaves, and will define these separately. This is admittedly a little laboured, and is certainly a place where a categorical presentation of the subject is more efficient. But we elect not to follow this abstract approach.

Since nothing is made more complicated by doing so at this point, we give our general definition of presheaf in terms of topological spaces.

7.1.1 Definition (Presheaf of sets) Let $(\mathcal{S}, \mathscr{O})$ be a topological space. A ***presheaf of sets*** over $\mathcal{S}$ is an assignment to each $\mathcal{U} \in \mathscr{O}$ a set $F(\mathcal{U})$ and to each $\mathcal{V}, \mathcal{U} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$ a mapping $r_{\mathcal{U},\mathcal{V}} \colon F(\mathcal{U}) \to F(\mathcal{V})$ called the ***restriction map***, with these assignments having the following properties:

(i) $r_{\mathcal{U},\mathcal{U}}$ is the identity map;

(ii) if $\mathcal{W}, \mathcal{V}, \mathcal{U} \in \mathscr{O}$ with $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$, then $r_{\mathcal{U},\mathcal{W}} = r_{\mathcal{V},\mathcal{W}} \circ r_{\mathcal{U},\mathcal{V}}$.

We shall frequently use a single symbol, like $\mathscr{F}$, to refer to a presheaf, with the understanding that $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$, and that the restriction maps are understood. •

7.1.2 Definition (Presheaf of rings) Let $(\mathcal{S}, \mathscr{O})$ be a topological space. A ***presheaf of rings*** over $\mathcal{S}$ is an assignment to each $\mathcal{U} \in \mathscr{O}$ a set $R(\mathcal{U})$ and to each $\mathcal{V}, \mathcal{U} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$ a ring homomorphism $r_{\mathcal{U},\mathcal{V}} \colon R(\mathcal{U}) \to R(\mathcal{V})$ called the ***restriction map***, with these assignments having the following properties:

(i) $r_{\mathcal{U},\mathcal{U}}$ is the identity map;

(ii) if $\mathcal{W}, \mathcal{V}, \mathcal{U} \in \mathscr{O}$ with $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$, then $r_{\mathcal{U},\mathcal{W}} = r_{\mathcal{V},\mathcal{W}} \circ r_{\mathcal{U},\mathcal{V}}$.

We shall frequently use a single symbol, like $\mathscr{R}$, to refer to a presheaf, with the understanding that $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$, and that the restriction maps are understood. •

7.1.3 Definition (Presheaf of modules) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$ be a presheaf of rings over $\mathcal{S}$ with restriction maps denote by $r^{\mathscr{R}}_{\mathcal{U},\mathcal{V}}$. A ***presheaf of $\mathscr{R}$-modules*** over $\mathcal{S}$ is an assignment to each $\mathcal{U} \in \mathscr{O}$ a set $E(\mathcal{U})$ and to each $\mathcal{V}, \mathcal{U} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$ a mapping $r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}} \colon E(\mathcal{U}) \to E(\mathcal{V})$ called the ***restriction map***, with these assignments having the following properties:

(i) $r^{\mathscr{E}}_{\mathcal{U},\mathcal{U}}$ is the identity map;

(ii) if $\mathcal{W}, \mathcal{V}, \mathcal{U} \in \mathscr{O}$ with $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$, then $r^{\mathscr{E}}_{\mathcal{U},\mathcal{W}} = r^{\mathscr{E}}_{\mathcal{V},\mathcal{W}} \circ r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}}$;

(iii) the relations

$$r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}}(s + t) = r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}}(s) + r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}}(t), \qquad s, t \in E(\mathcal{U}),$$
$$r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}}(f s) = r^{\mathscr{R}}_{\mathcal{U},\mathcal{V}}(f) r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}}(s), \qquad f \in R(\mathcal{U}),\ s \in E(\mathcal{U})$$
•

hold.

We shall frequently use a single symbol, like $\mathscr{E}$, to refer to a presheaf of $\mathscr{R}$-modules, with the understanding that $\mathscr{E} = (E(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$, and that the restriction maps are understood. •

Said more compactly, a presheaf of rings is a presheaf of sets where all sets are rings and where the restriction maps are homomorphisms. One can easily imagine doing this for algebraic structures of all sorts, but we shall stick to what we need here. Also, if we wish to talk about a presheaf and wish to include all of the flavours of presheaves, we shall simply write the presheaf as $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$. An element $s \in F(\mathcal{U})$ is called a ***section of $\mathscr{F}$ over $\mathcal{U}$*** and an element of $F(\mathcal{S})$ is called a ***global section***. If $\mathcal{U} \in \mathscr{O}$ then we denote by $\mathscr{F}|\mathcal{U}$ the ***restriction*** of $\mathscr{F}$ to $\mathcal{U}$, which is the presheaf over $\mathcal{U}$ whose sections over $\mathcal{V} \subseteq \mathcal{U}$ are simply $F(\mathcal{V})$.

Sometimes, presheaves of rings with certain properties a prescribed.

7.1.4 Definition (Ringed space) A ***ringed space*** is a pair $((\mathcal{S}, \mathscr{O}), \mathscr{R})$ where $(\mathcal{S}, \mathscr{O})$ is a topological space and where $\mathscr{R}$ is a presheaf of rings such that

(i) $R(\mathcal{U}) \subseteq \mathrm{C}^0(\mathcal{U})$ for each $\mathcal{U} \in \mathscr{O}$ and

(ii) $r_{\mathcal{U},\mathcal{V}}(f)$ is the restriction of the function $f \in R(\mathcal{U})$ to $\mathcal{V}$ for every $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ for which $\mathcal{V} \subseteq \mathcal{U}$. •

Let us look at the principal examples we shall use in this book.

7.1.5 Example (Presheaves)

1. If X is a set, a ***constant presheaf*** of sets $\mathscr{F}_X = (F_X(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ on a topological space $(\mathcal{S}, \mathscr{O})$ is defined by $F_X(\mathcal{U}) = X$ for every $\mathcal{U} \in \mathscr{O}$. The restriction maps are taken to be $r_{\mathcal{U},\mathcal{V}} = \mathrm{id}_X$ for every $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$. If the set X has a ring structure, then we have a constant presheaf of rings.
2. Let us denote by $\mathbb{Z}_{\mathcal{S}}$ the constant presheaf over a topological space $(\mathcal{S}, \mathscr{O})$ assigning the ring $\mathbb{Z}$ to every open set. Then an $\mathbb{Z}_{\mathcal{S}}$-module is a sheaf of Abelian groups, in the sense that to every $\mathcal{U} \in \mathscr{O}$ we assign an $\mathbb{Z}$-module, i.e., an Abelian group.

We now let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$, let $r' \in \{\infty, \omega, \mathrm{hol}\}$ be as required, and let $\mathbb{F} = \mathbb{R}$ if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. We let M be a manifold of class C^r and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r.

3. The presheaf of functions on M of class C^r assigns to each open $\mathcal{U} \subseteq \mathsf{M}$ the ring $\mathrm{C}^r(\mathcal{U})$. The restriction map $r_{\mathcal{U},\mathcal{V}}$ for open sets $\mathcal{V}, \mathcal{U} \subseteq \mathsf{M}$ with $\mathcal{V} \subseteq \mathcal{U}$ is simply the restriction of functions on $\mathcal{U}$ to $\mathcal{V}$. These maps clearly satisfy the conditions for a presheaf of rings. This presheaf we denote by $\mathscr{C}^r_{\mathsf{M}}$.
4. In rather similar manner, the presheaf of sections of E of class C^r assigns to each open $\mathcal{U} \subseteq \mathsf{M}$ the $\mathrm{C}^r(\mathcal{U})$-module $\Gamma^r(\mathsf{E}|\mathcal{U})$. The restriction map $r_{\mathcal{U},\mathcal{V}}$ for open sets $\mathcal{V}, \mathcal{U} \subseteq \mathsf{M}$ with $\mathcal{V} \subseteq \mathcal{U}$ is again just the restriction of sections on $\mathcal{U}$ to $\mathcal{V}$. These maps satisfy the conditions for a presheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. This presheaf we denote by $\mathscr{G}^r_{\mathsf{E}}$.

5. Generalising the preceding example a little, a ***presheaf of $\mathscr{C}^{\mathrm{r}}_{\mathsf{M}}$-modules*** is a presheaf $\mathscr{E} = (E(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ such that $E(\mathcal{U})$ is a $\mathrm{C}^r(\mathcal{U})$-module and such that the restriction maps satisfy the natural algebraic conditions

$$r_{\mathcal{U},\mathcal{V}}(s+t) = r_{\mathcal{U},\mathcal{V}}(s) + r_{\mathcal{U},\mathcal{V}}(t), \qquad s,t \in E(\mathcal{U}),$$
$$r_{\mathcal{U},\mathcal{V}}(fs) = r_{\mathcal{U},\mathcal{V}}(f) r_{\mathcal{U},\mathcal{V}}(s), \qquad f \in \mathrm{C}^r(\mathcal{U}),\ s \in E(\mathcal{U}). \qquad \bullet$$

The value of a presheaf is that it allows us to systematically deal with objects that are not globally defined, but are only locally defined. We have seen in various places, most explicitly at the end of Section 4.1.3, that there is value in doing this, especially in the holomorphic and real analytic cases.

7.1.2 Sheaves

The notion of a sheaf, which we are about to define, allows us to patch locally defined objects together to produce an object defined on a union of open sets.

7.1.6 Definition (Sheaf (of sets, rings, or modules)) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and suppose that we have a presheaf $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ (of sets, rings, or modules) with restriction maps $r_{\mathcal{U},\mathcal{V}}$ for $\mathcal{U},\mathcal{V} \in \mathscr{O}$ satisfying $\mathcal{V} \subseteq \mathcal{U}$.

(i) The presheaf $\mathscr{F}$ is ***separated*** when, if $\mathcal{U} \in \mathscr{O}$, if $(\mathcal{U}_a)_{a\in A}$ is an open covering of $\mathcal{U}$, and if $s,t \in F(\mathcal{U})$ satisfy $r_{\mathcal{U},\mathcal{U}_a}(s) = r_{\mathcal{U},\mathcal{U}_a}(t)$ for every $a \in A$, then $s = t$;

(ii) The presheaf $\mathscr{F}$ has the ***gluing property*** when, if $\mathcal{U} \in \mathscr{O}$, if $(\mathcal{U}_a)_{a\in A}$ is an open covering of $\mathcal{U}$, and if, for each $a \in A$, there exists $s_a \in F(\mathcal{U}_a)$ with the family $(s_a)_{a\in A}$ satisfying

$$r_{\mathcal{U}_{a_1},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{a_1}) = r_{\mathcal{U}_{a_2},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{a_2})$$

for each $a_1, a_2 \in A$, then there exists $s \in F(\mathcal{U})$ such that $s_a = r_{\mathcal{U},\mathcal{U}_a}(s)$ for each $a \in A$.

(iii) The presheaf (of sets, rings, or modules) $\mathscr{F}$ is a ***sheaf*** (of sets, rings, or modules) if it is separated and has the gluing property. •

It is fairly easy to show that the presheaves $\mathscr{C}^r_{\mathsf{M}}$ and $\mathscr{G}^r_{\mathsf{E}}$ are sheaves, and let us record this here.

7.1.7 Proposition (Presheaves of functions and sections are sheaves) *Let* $\mathrm{r} \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$, *let* $\mathrm{r}' \in \{\infty, \omega, \mathrm{hol}\}$ *be as required, and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* M *be a manifold of class* C^{r} *and let* $\pi\colon \mathsf{E} \to \mathsf{M}$ *be a vector bundle of class* C^{r}. *Then the presheaves* $\mathscr{C}^{\mathrm{r}}_{\mathsf{M}}$ *and* $\mathscr{G}^{\mathrm{r}}_{\mathsf{E}}$ *are sheaves.*

Proof We shall prove this for functions, the proof for sections following *mutatis mutandis*. Let $\mathcal{U} \subseteq \mathsf{M}$ be open and let $(\mathcal{U}_a)_{a\in A}$ be an open cover for $\mathcal{U}$. To prove condition (i), if $f, g \in \mathrm{C}^r(\mathcal{U})$ agree on each neighbourhood $\mathcal{U}_a$, $a \in A$, then it follows that $f(x) = g(x)$ for every $x \in \mathcal{U}$ since $(\mathcal{U}_a)_{a\in A}$ covers $\mathcal{U}$. To prove condition (ii) let $f_a \in \mathrm{C}^r(\mathcal{U}_a)$ satisfy

$$r_{\mathcal{U}_{a_1},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(f_{a_1}) = r_{\mathcal{U}_{a_2},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(f_{a_2})$$

for each $a_1, a_2 \in A$. Define $f\colon \mathcal{U} \to \mathbb{F}$ by $f(x) = f_a(x)$ if $x \in \mathcal{U}_a$. This gives f as being well-defined by our hypotheses on the family $(f_a)_{a\in A}$. It remains to show that f is of class

C^r. This, however, follows since f as defined agrees with f_a on $\mathcal{U}_a$, and f_a is of class C^r for each $a \in A$. ■

Let us also give some examples of presheaves that are not sheaves.

7.1.8 Example (Presheaves that are not sheaves)

1. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and take $\mathsf{M} = \mathbb{R}$. Let us define a presheaf $\mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R})$ over $\mathbb{R}$ by

$$\mathscr{C}^r_{\mathrm{bdd}}(\mathcal{U}) = \{f \in C^r(\mathcal{U}) \mid f \text{ is bounded}\}.$$

The restriction maps are, of course, just restriction of functions, and one readily verifies that this defines a presheaf of rings. It is not a sheaf. Indeed, let $(\mathcal{U}_a)_{a \in A}$ be a covering of $\mathbb{R}$ by bounded open sets and define $f_a \in \mathscr{C}^r_{\mathrm{bdd}}(\mathcal{U})$ by $f_a(x) = x$. Then we certainly have $f_a(x) = f_b(x)$ for $x \in \mathcal{U}_a \cap \mathcal{U}_b$. However, it does not hold that there exists $f \in \mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R})$ such that $f(x) = f_a(x)$ for every $x \in \mathcal{U}_a$ and for every $a \in A$, since any such function would necessarily be unbounded. The difficulty in this case is that presheaves are designed to carry local information, and so they do not react well to cases where local information does not carry over to global information, in this case boundedness. Note that the defect in this example comes in the form of the violation of gluing condition (ii) in Definition 7.1.6; condition (i) still holds.

2. Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let X be a set. As in Example 7.1.5–1, $\mathscr{F}_X = (F_X(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$ denotes the constant presheaf defined by $F_X(\mathcal{U}) = X$. It is clear that $\mathscr{F}_X$ satisfies the separation condition. We claim that $\mathscr{F}_X$ does not generally satisfy the gluing condition. Indeed, let $\mathcal{U}_1, \mathcal{U}_2 \in \mathscr{O}$ be disjoint and take $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$. Let $s_1 \in F_X(\mathcal{U}_1)$ and $s_2 \in F(\mathcal{U}_2)$. If $s_1 \neq s_2$ then there is no $s \in F_X(\mathcal{U})$ for which $r_{\mathcal{U},\mathcal{U}_1}(s) = s_1$ and $r_{\mathcal{U},\mathcal{U}_2}(s) = x_2$.

3. An example of a presheaf that is not separated is a little less relevant, but we give it for the sake of completeness. Let $\mathcal{S} = \{0, 1\}$ have the discrete topology and define a presheaf $\mathscr{F}$ by requiring that $F(\emptyset) = \emptyset$ and that $F(\mathcal{U}) = \mathbb{R}^{\mathcal{U}}$ (i.e., the set of maps from $\mathcal{U}$ into $\mathbb{R}$). The restriction maps are defined by asking that $r_{\mathcal{U},\mathcal{V}}(s) = \zeta_{\mathcal{V}}$ whenever $\mathcal{V}$ is a proper subset of $\mathcal{U}$, where $\zeta_{\mathcal{V}} \colon \mathcal{V} \to \mathbb{R}$ is defined by $\zeta_{\mathcal{V}}(x) = 0$. Now let $s, t \in F(\{0, 1\})$ be defined by

$$s(0) = s(1) = 1, \quad t(0) = t(1) = -1.$$

Note that $(\{0\}, \{1\})$ is an open cover for $\{0, 1\}$ and

$$r_{\{0,1\},\{0\}}(s) = r_{\{0,1\},\{0\}}(t), \quad r_{\{0,1\},\{1\}}(s) = r_{\{0,1\},\{1\}}(t).$$

But it does not hold that $s = t$. •

The examples suggest that the gluing condition is the one that will fail most often, and there is a reason for this feeling.

7.1.9 Proposition (Presheaves of mappings are separated) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if X is a set, and if $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$ is a presheaf over $\mathcal{S}$ such that*

(i) *each element* $f \in F(\mathcal{U})$ *is a mapping from* $\mathcal{U}$ *to* X *and*

(ii) *if* $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ *are such that* $\mathcal{V} \subseteq \mathcal{U}$, *then the restriction map* $r_{\mathcal{U},\mathcal{V}}$ *is given by*

$$r_{\mathcal{U},\mathcal{V}}(f)(x) = f(x), \qquad x \in \mathcal{V},$$

then $\mathscr{F}$ *is separated.*

Proof Suppose that $\mathcal{U} \in \mathscr{O}$, that $(\mathcal{U}_a)_{a\in A}$ is an open cover of $\mathcal{U}$, and that $f, g \in F(\mathcal{U})$ satisfy $r_{\mathcal{U},\mathcal{U}_a}(f) = r_{\mathcal{U},\mathcal{U}_a}(g)$ for every $a \in A$. For $x \in \mathcal{U}$ let $a \in A$ be such that $x \in \mathcal{U}_a$. It follows immediately from the definition of the restriction maps that $f(x) = g(x)$. ■

In practice, one often wishes to patch together locally defined objects and have these be a sheaf. The following result shows how this can be done, the statement referring ahead to Section 7.1.6 for the notion of morphisms of sheaves.

7.1.10 Proposition (Building a sheaf from local constructions) *Let* $(\mathcal{S}, \mathscr{O})$ *be a topological space and let* $(\mathcal{U}_a)_{a\in A}$ *be an open cover for* $\mathcal{S}$. *Suppose that, for each* $a \in A$,

$$\mathscr{F}_a = (F_a(\mathcal{U}))_{\mathcal{U}\in\mathscr{O},\ \mathcal{U}\subseteq\mathcal{U}_a}.$$

is a sheaf (of sets, rings, or modules) over $\mathcal{U}_a$ *and denote the restriction maps for* $\mathscr{F}_a$ *by* $r^a_{\mathcal{U},\mathcal{V}}$ *for* $\mathcal{U}, \mathcal{V} \subseteq \mathcal{U}_a$ *open with* $\mathcal{V} \subseteq \mathcal{U}$. *If, for* $a_1, a_2 \in A$ *satisfying* $\mathcal{U}_{a_1} \cap \mathcal{U}_{a_2} \neq \emptyset$, *we have*

$$r^{a_1}_{\mathcal{U}_{a_1},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(F_{a_1}(\mathcal{U}_{a_1})) = r^{a_2}_{\mathcal{U}_{a_2},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(F_{a_2}(\mathcal{U}_{a_2})), \tag{7.1}$$

then there exists a sheaf $\mathscr{F}$ *over* $\mathcal{S}$, *unique up to isomorphism, with the property that* $F(\mathcal{U}_a)$ *is isomorphic to* $F_a(\mathcal{U}_a)$ *for each* $a \in A$.

Proof While this is the natural place to state this result, to prove the result we shall make use of the étalé space of a sheaf which is defined in the next section. We suppose, therefore, that the reader has understood this section.

Let $x \in \mathcal{S}$ and suppose that $x \in \mathcal{U}_a$ for some $a \in A$. Let $\mathscr{F}_x$ be the stalk of the sheaf $\mathscr{F}_a$ at x. Note that $\mathscr{F}_x$ does not depend on a by (7.1). We then take $E_{\mathcal{U}} = \mathring{\cup}_{x\in\mathcal{U}} \mathscr{F}_x$. Let us say that a map $\sigma\colon \mathcal{U} \to E_{\mathcal{U}}$ is a ***section*** of $E_{\mathcal{U}}$ if $\sigma(x) \in \mathscr{F}_x$ for each $x \in \mathcal{U}$ and if, for each $x \in \mathcal{U}$ there exists a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x, $a \in A$, and $s_a \in F_a(\mathcal{V} \cap \mathcal{U}_a)$ such that $\sigma(y) = [s_a]_y$ for all $y \in \mathcal{V}$. Again, the condition (7.1) ensures that if this condition holds for some $\mathcal{V}$, $a \in A$. and $s_a \in F_a(\mathcal{V} \cap \mathcal{U}_a)$, it will hold for all such. We let $F(\mathcal{U})$ be the set of sections of $E_{\mathcal{U}}$, and we claim that $(F(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ is a sheaf with the restriction maps being restriction of sections in the usual sense.

By Proposition 7.1.9 it follows that $\mathscr{F}$ as defined is separated. To verify the gluing property, let $\mathcal{U} \in \mathscr{O}$ and let $(\mathcal{V}_b)_{b\in B}$ be an open cover of $\mathcal{U}$. Suppose that sections $\sigma_b \in F(\mathcal{V}_b)$, $b \in B$, satisfy $r_{\mathcal{V}_{b_1},\mathcal{V}_{b_1}\cap\mathcal{V}_{b_2}}(\sigma) = r_{\mathcal{V}_{b_2},\mathcal{V}_{b_1}\cap\mathcal{V}_{b_2}}(\tau)$ for every $b_1, b_2 \in B$. For $x \in \mathcal{U}$, let $\sigma(x) = \sigma_b(x)$ where $b \in B$ satisfies $x \in \mathcal{V}_b$. This definition clearly does not depend on b. We must show that σ is a section of $E_{\mathcal{U}}$ as defined above. Let $x \in \mathcal{U}$. By assumption, $\sigma = \sigma_b$ agree in a neighbourhood of x. Since σ_b is a section of $E_{\mathcal{V}_b}$ there exists a neighbourhood $\mathcal{V} \subseteq \mathcal{V}_b$ of x, $a \in A$, and $s_a \in F_a(\mathcal{V} \cap \mathcal{U}_a)$ such that $\sigma_b(y) = [s_a]_y$ for every $y \in \mathcal{V}$. Since $\sigma(y) = [s_a]_y$ for every $y \in \mathcal{V}$.

Note that for each $a \in A$ we have that $F(\mathcal{U}_a)$ is actually the sheafification of $\mathscr{F}_a$. Since $\mathscr{F}_a$ is a sheaf, it follows from Propositions 7.1.27 and 7.1.29 that $\mathscr{F}_a$ is isomorphic to $F(\mathcal{U}_a)$. This gives the existence part of the result.

To show uniqueness, note that the condition that $F(\mathcal{U}_a)$ be isomorphic to $\mathscr{F}_a$ ensures that the stalk of $\mathscr{F}$ be isomorphic to $\mathscr{F}_x$, and so by Proposition 7.1.53 it follows that any sheaf satisfying the conclusions of the proposition must be isomorphic to the sheaf $\mathscr{F}$ constructed above. ■

7.1.3 The étalé space of a presheaf

The examples of presheaves we are most interested in, the presheaves $\mathscr{C}^r_{\mathsf{M}}$ and $\mathscr{G}^r_{\mathsf{E}}$, arise naturally as sections of some geometric object. However, there is nothing built into our definition of a presheaf that entails that it arises in this way. In this section we associate to a presheaf a space which realises sections of a presheaf as sections of some object, albeit a sort of peculiar one.

In Section 5.7.1 we saw the notions of germs of C^r-functions and germs of C^r-sections of a vector bundle. We begin our constructions of this section by understanding the germ construction for general presheaves. For the purposes of this discussion, we work with a presheaf $\mathscr{F}$ (of sets, rings, or modules) over a topological space $(\mathcal{S}, \mathscr{O})$. We let $x \in \mathcal{S}$ let $\mathscr{O}_x$ be the collection of open subsets of $\mathcal{S}$ containing x. This is a directed set using inclusion since, given $\mathcal{U}_1, \mathcal{U}_2 \in \mathscr{O}_x$, we have $\mathcal{U}_1 \cap \mathcal{U}_2 \in \mathscr{O}_x$ and $\mathcal{U}_1 \cap \mathcal{U}_2 \subseteq \mathcal{U}_1$ and $\mathcal{U}_1 \cap \mathcal{U}_2 \subseteq \mathcal{U}_2$. What we want is the direct limit in $(F(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}_x}$. This we define using the equivalence relation where, for $\mathcal{U}_1, \mathcal{U}_2 \in \mathscr{O}_x$, $s_1 \in F(\mathcal{U}_1)$ and $s_2 \in F(\mathcal{U}_2)$ are ***equivalent*** if there exists $\mathcal{V} \in \mathscr{O}_x$ such that $\mathcal{V} \subseteq \mathcal{U}_1$, $\mathcal{V} \subseteq \mathcal{U}_2$ and $r_{\mathcal{U}_1,\mathcal{V}}(s_1) = r_{\mathcal{U}_2,\mathcal{V}}(s_2)$. The equivalence class of a section $s \in F(\mathcal{U})$ we denote by $r_{\mathcal{U},x}(s)$, or simply by $[s]_x$ if we are able to forget about the neighbourhood on which s is defined.

The preceding constructions allow us to make the following definition.

7.1.11 Definition (Stalk, germ of a section) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$ be a presheaf (of sets, rings, or modules) over $\mathcal{S}$. For $x \in \mathcal{S}$, the ***stalk*** of $\mathscr{F}$ at x is the set of equivalence classes under the equivalence relation defined above, and is denoted by $\mathscr{F}_x$. The equivalence class $r_{\mathcal{U},x}(s)$ of a section $s \in F(\mathcal{U})$ is called the ***germ*** of s at x.

In case $\mathscr{F} = \mathscr{R}$ is a sheaf of rings over $\mathcal{S}$, we denote by $0_x \in \mathscr{R}_x$ and $1_x \in \mathscr{R}_x$ the germs of the sections $\zeta, \mu \in R(\mathcal{U})$ over some neighbourhood $\mathcal{U}$ of x given by $\zeta = 0$ and $\mu = 1$. Similarly, if $\mathscr{F} = \mathscr{E}$ is a sheaf of $\mathscr{R}$-modules, $0_x \in \mathscr{E}_x$ denotes the germ of a local section of $\mathscr{E}$ over a neighbourhood of x taking the value 0. •

If $\mathscr{F} = \mathscr{R}$ is a presheaf of rings let us define a ring operation on the set $\mathscr{R}_x$ of equivalence classes under this equivalence relation by

$$r_{\mathcal{U},x}(f) + r_{\mathcal{V},x}(g) = r_{\mathcal{U}\cap\mathcal{V},x} \circ r_{\mathcal{U},\mathcal{U}\cap\mathcal{V}}(f) + r_{\mathcal{U}\cap\mathcal{V},x} \circ r_{\mathcal{V},\mathcal{U}\cap\mathcal{V}}(g),$$
$$(r_{\mathcal{U},x}(f)) \cdot (r_{\mathcal{V},x}(g)) = (r_{\mathcal{U}\cap\mathcal{V},x} \circ r_{\mathcal{U},\mathcal{U}\cap\mathcal{V}}(f)) \cdot (r_{\mathcal{U}\cap\mathcal{V},x} \circ r_{\mathcal{V},\mathcal{U}\cap\mathcal{V}}(g)),$$

where $f \in R(\mathcal{U})$, $g \in R(\mathcal{V})$. One readily verifies, just as we did for germs of functions, mappings, and sections of vector bundles, that these ring operations is well-defined and satisfy the ring axioms. Similarly, if $\mathscr{F} = \mathscr{E}$ is a presheaf of $\mathscr{R}$-modules, we can define a module structure on the set $\mathscr{E}_x$ of equivalence class by

$$r_{\mathcal{U},x}(s) + r_{\mathcal{V},x}(t) = r_{\mathcal{U}\cap\mathcal{V},x} \circ r_{\mathcal{U},\mathcal{U}\cap\mathcal{V}}(s) + r_{\mathcal{U}\cap\mathcal{V},x} \circ r_{\mathcal{V},\mathcal{U}\cap\mathcal{V}}(t),$$
$$(r_{\mathcal{W},x}(f)) \cdot (r_{\mathcal{V},x}(s)) = (r_{\mathcal{W}\cap\mathcal{V},x} \circ r_{\mathcal{W},\mathcal{W}\cap\mathcal{V}}(f)) \cdot (r_{\mathcal{W}\cap\mathcal{V},x} \circ r_{\mathcal{V},\mathcal{W}\cap\mathcal{V}}(s)),$$

where $s \in E(\mathcal{U})$, $t \in E(\mathcal{V})$, and $f \in R(\mathcal{W})$. Again, these operations can be verified to make sense and provide the structure of a module over the ring $\mathscr{R}_x$.

For sheaves of rings or modules the notion of stalk makes it possible to define the notion of the support of a local section.

7.1.12 Definition (Support of a local section) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E} = (E(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a presheaf of $\mathscr{R}$-modules over $\mathcal{S}$. The ***support*** of a local section $s \in E(\mathcal{U})$ is

$$\operatorname{supp}(s) = \{x \in \mathcal{U} \mid [s]_x \neq 0_x\}. \qquad \bullet$$

Note that the support of a local section $s \in E(\mathcal{U})$ is necessarily closed since if $[s]_x = 0_x$ then $[s]_y = 0_y$ for y in some neighbourhood of x.

With stalks at hand, we can make another useful construction associated with a presheaf.

7.1.13 Definition (Étalé space of a presheaf) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a presheaf (of sets, rings, or modules). The ***étalé space*** of $\mathscr{F}$ is the disjoint union of the stalks of $\mathscr{F}$:

$$\operatorname{Et}(\mathscr{F}) = \mathring{\bigcup_{x\in\mathcal{S}}} \mathscr{F}_x.$$

The ***étalé topology*** on $\operatorname{Et}(\mathscr{F})$ is that topology whose basis consists of subsets of the form

$$\mathcal{B}(\mathcal{U}, s) = \{r_{\mathcal{U},x}(s) \mid x \in \mathcal{U}\}, \qquad \mathcal{U} \in \mathscr{O},\ s \in F(\mathcal{U}).$$

By $\pi_{\mathscr{F}} \colon \operatorname{Et}(\mathscr{F}) \to \mathcal{S}$ we denote the canonical projection $\pi_{\mathscr{F}}(r_{\mathcal{U},x}(s)) = x$ which we call the ***étalé projection***. •

Let us give some properties of étalé spaces, including the verification that the proposed basis we give for the étalé topology is actually a basis.

7.1.14 Proposition (Properties of the étalé topology) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space with $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ a presheaf (of sets, rings, or modules) over $\mathcal{S}$. The étalé topology on $\operatorname{Et}(\mathscr{F})$ has the following properties:*

(i) the sets $\mathcal{B}(\mathcal{U}, \mathrm{s})$, $\mathcal{U} \in \mathscr{O}$, $\mathrm{s} \in \mathrm{F}(\mathcal{U})$, form a basis for a topology;

(ii) the projection $\pi_{\mathscr{F}}$ is a local homeomorphism, i.e., about every $[\mathrm{s}]_{\mathrm{x}} \in \operatorname{Et}(\mathscr{F})$ there exists a neighbourhood $\mathcal{O} \subseteq \operatorname{Et}(\mathscr{F})$ such that $\pi_{\mathscr{F}}$ is a homeomorphism onto its image.

Proof (i) According to [Willard 1970, Theorem 5.3] this means that we must show that for sets $\mathcal{B}(\mathcal{U}_1, s_1)$ and $\mathcal{B}(\mathcal{U}_2, s_2)$ and for $[s]_x \in \mathcal{B}(\mathcal{U}_1, s_1) \cap \mathcal{B}(\mathcal{U}_2, s_2)$, there exists $\mathcal{B}(\mathcal{V}, t) \subseteq \mathcal{B}(\mathcal{U}_1, s_1) \cap \mathcal{B}(\mathcal{U}_2, s_2)$ such that $[s]_x \in \mathcal{B}(\mathcal{V}, t)$. We let $\mathcal{V} \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$ be a neighbourhood of x such that $s(y) = s_1(y) = s_2(y)$ for each $y \in \mathcal{V}$, this being possible since $[s]_x \in \mathcal{B}(\mathcal{U}_1, s_1) \cap \mathcal{B}(\mathcal{U}_2, s_2)$. We then clearly have $\mathcal{B}(\mathcal{V}, t) \subseteq \mathcal{B}(\mathcal{U}_1, s_1) \cap \mathcal{B}(\mathcal{U}_2, s_2)$ as desired.

(ii) By definition of the étalé topology, $\pi_{\mathscr{F}}|\mathcal{B}(\mathcal{U}, s)$ is a homeomorphism onto $\mathcal{U}$ (its inverse is s), and this suffices to show that $\pi_{\mathscr{F}}$ is a local homeomorphism. ■

The way in which one should think of the étalé topology is depicted in Figure 7.1. The point is that open sets in the étalé topology can be thought of as the "graphs" of

Figure 7.1 How to think of open sets in the étalé topology

local sections. A good example to illustrate the étalé topology is the constant sheaf.

7.1.15 Example (The étalé space of a constant sheaf) We let $(\mathcal{S}, \mathscr{O})$ be a topological space and let X be a set. By $\mathscr{F}_X = (F_X(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$ we denote the constant sheaf defined by $F_X(\mathcal{U}) = X$. Note that the stalk $\mathscr{F}_{X,x}$ is simply X. Thus $\mathrm{Et}(\mathscr{F}_X) = \cup_{x \in \mathcal{S}}(x, X)$ which we identity with $\mathcal{S} \times X$ in the natural way. Under this identification of $\mathrm{Et}(\mathscr{F}_X)$ with $\mathcal{S} \times X$, the étalé projection $\pi \colon \mathcal{S} \times X \to \mathcal{S}$ is identified with projection onto the first factor. Thus a section is, first of all, a map $\sigma \colon \mathcal{S} \to X$. It must also satisfy the criterion of continuity, and so we must understand the étalé topology on $\mathcal{S} \times X$. Let $\mathcal{U} \in \mathscr{O}$ and let $s \in F_X(\mathcal{U}) = X$. The associated basis set for the étalé topology is then

$$\mathcal{B}(\mathcal{U}, s) = \{(x, s) \mid x \in \mathcal{U}\}.$$

These are precisely the open sets for $\mathcal{S} \times X$ if we equip X with the discrete topology. Thus $\mathrm{Et}(\mathscr{F}_X)$ is identified with the product topological space $\mathcal{S} \times X$ where X has the discrete topology. •

Let us close this section by thinking about the étalé spaces for the examples of interest to us.

7.1.16 Examples (The étalé spaces for $\mathscr{C}^r_{\mathsf{M}}$ and $\mathscr{G}^r_{\mathsf{E}}$) Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$, let $r' \in \{\infty, \omega, \mathrm{hol}\}$ be as required, and let $\mathbb{F} = \mathbb{R}$ if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M be a manifold of class C^r and let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. It is rather apparent that the stalks of $\mathrm{Et}(\mathscr{C}^r_{\mathsf{M}})$ and $\mathrm{Et}(\mathscr{G}^r_{\mathsf{E}})$ are exactly the sets $\mathscr{C}^r_{x,\mathsf{M}}$ and $\mathscr{G}^r_{x,\mathsf{E}}$ of germs of functions and sections, respectively.

Let us examine some of the properties of these eťalé spaces.

1 **Lemma** *The étalé topology on both* $\mathrm{Et}(\mathscr{C}^{\mathrm{r}}_{\mathsf{M}})$ *and* $\mathrm{Et}(\mathscr{G}^{\mathrm{r}}_{\mathsf{E}})$ *is not Hausdorff when* $\mathrm{r} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$*.*

what?

Proof We shall prove this for $\mathrm{Et}(\mathscr{C}^{r}_{\mathsf{M}})$, the construction for $\mathscr{G}^{r}_{\mathsf{E}}$ being quite similar. Let $\mathcal{U} \subset \mathsf{M}$ be an open set and as in , let $f \in \mathrm{C}^{\infty}(\mathsf{M})$ be such that $f(x) \in \mathbb{R}_{>0}$ for $x \in \mathcal{U}$ and $f(x) = 0$ for $x \in \mathsf{M} \setminus \mathcal{U}$. Let $g \in \mathrm{C}^{\infty}(\mathsf{M})$ be the zero function. Now let $x \in \mathrm{bd}(\mathcal{U})$. We claim that any neighbourhoods of $[f]_x$ and $[g]_x$ in $\mathrm{Et}(\mathscr{C}^{r}_{\mathsf{M}})$ intersect. To see this, let $\mathcal{O}_f$ and $\mathcal{O}_g$ be neighbourhoods in the étalé topology of $[f]_x$ and $[g]_x$. Since any sufficiently small neighbourhood of $[f]_x$ and $[g]_x$ is homeomorphic to a neighbourhood of x under the étalé projection, let us suppose without loss of generality that $\mathcal{O}_f$ and $\mathcal{O}_g$ are both homeomorphic to a neighbourhood $\mathcal{V}$ of x under the projection. For $y \in \mathcal{V} \cap (\mathsf{M} \setminus \mathrm{cl}(\mathcal{U}))$, $[f]_y = [g]_y$. Since $\mathcal{O}_f$ and $\mathcal{O}_g$ are uniquely determined by the germs of f and g in $\mathcal{V}$, respectively, it follows that $[f]_y = [g]_y \in \mathcal{O}_f \cap \mathcal{O}_g$, giving the desired conclusion. ▼

2 **Lemma** *If* M *is Hausdorff, then the étalé topology on both* $\mathrm{Et}(\mathscr{C}^{\mathrm{r}}_{\mathsf{M}})$ *and* $\mathrm{Et}(\mathscr{G}^{\mathrm{r}}_{\mathsf{E}})$ *is Hausdorff when* $\mathrm{r} \in \{\omega, \mathrm{hol}\}$*.*

Proof We again prove the result for functions, leaving the very similar proof for vector bundles as an exercise. Let $[f]_x$ and $[g]_y$ be distinct. If $x \neq y$ then there are disjoint neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of x and y and then $\mathcal{B}(\mathcal{U}, f)$ and $\mathcal{B}(\mathcal{V}, g)$ are disjoint neighbourhoods of $[f]_x$ and $[g]_y$, respectively, since the étalé projection is a homeomorphism from the neighbourhoods in M to the neighbourhoods in $\mathrm{Et}(\mathscr{C}^{r}_{\mathsf{M}})$. If $x = y$ let $[f]_x$ and $[g]_x$ be distinct and suppose that every neighbourhood of $[f]_x$ and $[g]_x$ in the étalé topology intersect. This implies, in particular, that for every connected neighbourhood $\mathcal{U}$ of x the basic neighbourhoods $\mathcal{B}(\mathcal{U}, f)$ and $\mathcal{B}(\mathcal{U}, g)$ intersect. This implies by Lemma 7.1.20 below the existence of an open subset $\mathcal{V}$ of $\mathcal{U}$ such that f and g agree on $\mathcal{V}$. This, however, contradicts the identity principle, Theorem 4.1.5. Thus the étalé topology is indeed Hausdorff in the holomorphic or real analytic case. ▼

Readers who are annoyed by the notation $\mathrm{Et}(\mathscr{C}^{r}_{\mathsf{M}})$ and $\mathrm{Et}(\mathscr{G}^{r}_{\mathsf{E}})$ will be pleased to know that we will stop using this notation eventually. •

7.1.4 Étalé spaces

Let us now talk about étalé spaces in general. As with presheaves, we will give a few definitions associated with the various structures we shall use.

7.1.17 Definition (Étalé space of sets) If $(\mathcal{S}, \mathscr{O})$ is a topological space, an ***étalé space of sets*** over $\mathcal{S}$ is a topological space $\mathscr{S}$ with a surjective map $\pi \colon \mathscr{S} \to \mathcal{S}$, called the ***étalé projection***, such that π is a local homeomorphism. The the ***stalk*** at x is $\mathscr{S}_x = \pi^{-1}(x)$. •

Given étalé spaces $\pi \colon \mathscr{S} \to \mathcal{S}$ and $\rho \colon \mathscr{T} \to \mathcal{S}$ over $(\mathcal{S}, \mathscr{O})$, let us define

$$\mathscr{S} \times_{\mathcal{S}} \mathscr{T} = \{(\alpha, \beta) \in \mathscr{S} \times \mathscr{T} \mid \pi(\alpha) = \rho(\beta)\}.$$

This space is given the relative topology from $\mathscr{S} \times \mathscr{T}$.

7.1.18 Definition (Étalé space of rings) If $(\mathcal{S}, \mathscr{O})$ is a topological space, an ***étalé space of rings*** over $\mathcal{S}$ is a topological space $\mathscr{R}$ with a surjective map $\pi \colon \mathscr{R} \to \mathcal{S}$ such that

(i) $\mathscr{R}$ is an étalé space of sets,

(ii) the stalk $\mathscr{R}_x = \pi^{-1}(x)$ is a ring for each $x \in \mathcal{S}$,

(iii) the ring operations are continuous, i.e., the maps

$$\mathscr{R} \times_{\mathcal{S}} \mathscr{R} \ni (f,g) \mapsto f+g \in \mathscr{R}, \quad \mathscr{R} \times_{\mathcal{S}} \mathscr{R} \ni (f,g) \mapsto f \cdot g \in \mathscr{R}$$

are continuous. •

7.1.19 Definition (Étalé space of modules) If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{R}$ is an étalé space of rings over $\mathcal{S}$, an ***étalé space of $\mathscr{R}$-modules*** over $\mathcal{S}$ is a topological space $\mathscr{E}$ with a surjective map $\pi \colon \mathscr{E} \to \mathcal{S}$ such that

(i) $\mathscr{E}$ is an étalé space of sets,

(ii) the stalk $\mathscr{E}_x = \pi^{-1}(x)$ is an $\mathscr{R}_x$-module for each $x \in \mathcal{S}$,

(iii) the module operations are continuous, i.e., the maps

$$\mathscr{E} \times_{\mathcal{S}} \mathscr{E} \ni (\sigma,\tau) \mapsto \sigma+\tau \in \mathscr{E}, \quad \mathscr{R} \times_{\mathcal{S}} \mathscr{E} \ni (f,\sigma) \mapsto f \cdot \sigma \in \mathscr{E}$$

are continuous. •

A ***section*** of $\mathscr{S}$ over $\mathcal{U} \in \mathscr{O}$ is a continuous map $\sigma \colon \mathcal{U} \to \mathscr{S}$ for which $\pi \circ \sigma = \mathrm{id}_{\mathcal{S}}$. The set of sections of $\mathscr{S}$ over $\mathcal{U}$ is denoted by $\Gamma(\mathcal{U}; \mathscr{S})$. The following properties of sections are used often when proving statements about étalé spaces.

7.1.20 Lemma (Properties of sections of étalé spaces) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\pi \colon \mathscr{S} \to \mathcal{S}$ be an étalé space (of sets, rings, or modules) over $\mathcal{S}$, and let $\mathrm{x} \in \mathcal{S}$:*

(i) if $\alpha \in \mathscr{S}_{\mathrm{x}}$ then there exists a neighbourhood $\mathcal{U}$ of x and a section σ of $\mathscr{S}$ over $\mathcal{U}$ such that $\sigma(\mathrm{x}) = \alpha$;

(ii) if σ and τ are sections of $\mathscr{S}$ over neighbourhoods $\mathcal{U}$ and $\mathcal{V}$, respectively, of x for which $\sigma(\mathrm{x}) = \tau(\mathrm{x})$, then there exists a neighbourhood $\mathcal{W} \subseteq \mathcal{U}$ of x such that $\sigma|\mathcal{W} = \tau|\mathcal{W}$.

Proof (i) Let $\mathcal{O}$ be a neighbourhood of α in $\mathscr{S}$, and suppose, without loss of generality, that $\pi|\mathcal{O}$ is a homeomorphism onto its image. The inverse $\sigma \colon \pi(\mathcal{O}) \to \mathcal{O} \subseteq \mathscr{S}$ is continuous, and so it a section.

(ii) Let $\alpha = \sigma(x) = \tau(x)$ and let $\mathcal{O} \subseteq \mathscr{S}$ be a neighbourhood of α such that $\pi|\mathcal{O}$ is a homeomorphism onto its image. Let $\mathcal{U}' \subseteq \mathcal{U}$ and $\mathcal{V}' \subseteq \mathcal{V}$ be such that $\sigma(\mathcal{U}'), \tau(\mathcal{V}') \subseteq \mathcal{O}$, this by continuity of the sections. Let $\mathcal{W} = \mathcal{U}' \cap \mathcal{V}'$. Note that $\sigma|\mathcal{W}$ and $\tau|\mathcal{W}$ are continuous bijections onto their image and that they are further homeomorphisms onto their image, with the continuous inverse being furnished by π. Thus σ and τ are both inverse for π in the same neighbourhood of α, and so are, therefore, equal. ■

Most of our examples of étalé spaces will come from Proposition 7.1.22 below. Let us give a somewhat independent example.

7.1.21 Example (The constant étalé space) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let X be a set. We define $\mathscr{S}_X = \mathcal{S} \times X$ and we equip this set with the product topology inherited by using the discrete topology on X. One readily verifies that the projection $\pi \colon \mathcal{S} \times X \to \mathcal{S}$

given by projection onto the first factor then makes $\mathscr{S}_X$ into an étalé space. One also verifies that sections of $\mathscr{S}_X$ over $\mathcal{U} \in \mathscr{O}$ are regarded as locally constant maps from $\mathcal{U}$ to X. •

We should verify that the étalé space of a presheaf is an étalé space in the general sense.

7.1.22 Proposition (Et($\mathscr{F}$) is an étalé space) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ is a presheaf (of sets, rings, or modules) over $\mathcal{S}$, then $\pi_{\mathscr{F}} \colon \mathrm{Et}(\mathscr{F}) \to \mathcal{S}$ is an étalé space (of sets, rings, or modules) and $\mathrm{Et}(\mathscr{F})_x = \mathscr{F}_x$.*

Proof By Proposition 7.1.14 the étalé projection is a local homeomorphism. Let us show that the ring operations on $\mathrm{Et}(\mathscr{R})$ are continuous if $\mathscr{F} = \mathscr{R}$ is a presheaf of rings; the corresponding result for modules is proved in an entirely similar manner. Let $\mathcal{O} \subseteq \mathrm{Et}(\mathscr{R})$ be open and let $[f]_x, [g]_x \in \mathrm{Et}(\mathscr{R})$ be such that $[f]_x + [g]_x \in \mathcal{O}$. Without loss of generality, suppose that $f, g \in R(\mathcal{U})$ and, still without loss of generality, suppose that $\mathcal{B}(\mathcal{U}, f + g) \subseteq \mathcal{O}$. Then we have

$$\mathrm{Et}(\mathscr{R}) \times_{\mathcal{S}} \mathrm{Et}(\mathscr{R}) \supseteq \mathcal{B}(\mathcal{U}, f) \times_{\mathcal{S}} \mathcal{B}(\mathcal{U}, g) \ni ([f]_y, [g]_y) \mapsto [f + g]_y \in \mathcal{B}(\mathcal{U}, f + g) \subseteq \mathcal{O},$$

where, of course,

$$\mathcal{B}(\mathcal{U}, f) \times_{\mathcal{S}} \mathcal{B}(\mathcal{U}, g) = \{([f]_y, [g]_z) \in \mathcal{B}(\mathcal{U}, f) \times \mathcal{B}(\mathcal{U}, g) \mid y = z\}.$$

This gives the desired conclusion since $\mathcal{B}(\mathcal{U}, f) \times_{\mathcal{S}} \mathcal{B}(\mathcal{U}, g)$ is open in $\mathrm{Et}(\mathscr{R}) \times_{\mathcal{S}} \mathrm{Et}(\mathscr{R})$.

The final assertion of the proposition is just the definition. ■

7.1.23 Notation (Stalks) We shall write either $\mathscr{F}_x$ or $\mathrm{Et}(\mathscr{F})_x$ for the stalk, depending on what is most appropriate. •

Thus, associated to every presheaf is an étalé space. Moreover, associated to every étalé space is a natural presheaf.

7.1.24 Definition (The presheaf of sections of an étalé space) For a topological space $(\mathcal{S}, \mathscr{O})$ and an étalé space $\pi \colon \mathscr{S} \to \mathcal{S}$ (of sets, rings, or modules), the ***presheaf of sections*** $\mathscr{S}$ is the presheaf $\mathrm{Ps}(\mathscr{S})$ (of sets, rings, or modules) which assigns to $\mathcal{U} \in \mathscr{O}$ the set $\Gamma(\mathcal{U}; \mathscr{S})$ of sections of $\mathscr{S}$ over $\mathcal{U}$ and for which the restriction map for $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$ is given by $r_{\mathcal{U},\mathcal{V}}(\sigma) = \sigma|\mathcal{V}$.

In the case that $\mathscr{S} = \mathscr{R}$ is an étalé space of rings, the ring operations are

$$(f + g)(x) = f(x) + g(x), \ (f \cdot g)(x) = f(x) \cdot g(x) \qquad f, g \in \Gamma(\mathcal{U}; \mathscr{R}), \ x \in \mathcal{U}.$$

In the case that $\mathscr{S} = \mathscr{E}$ is an étalé space of $\mathscr{R}$-modules, the module operations are

$$(\sigma + \tau)(x) = \sigma(x) + \tau(x), \ (f \cdot \sigma)(x) = f(x) \cdot \sigma(x) \qquad \sigma, \tau \in \Gamma(\mathcal{U}; \mathscr{E}), \ f \in \Gamma(\mathcal{U}; \mathscr{R}), \ x \in \mathcal{U}. \ \bullet$$

It is readily seen that $\mathrm{Ps}(\mathscr{S})$ is indeed a presheaf. Moreover, it is a sheaf.

7.1.25 Proposition (Ps($\mathscr{S}$) is a sheaf) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{S}$ is an étalé space (of sets, rings, or modules) over $\mathcal{S}$, then the presheaf $\mathrm{Ps}(\mathscr{S})$ is a sheaf.*

Proof Let $\mathcal{U} \in \mathscr{O}$ and let $(\mathcal{U}_a)_{a\in A}$ be an open cover for $\mathcal{U}$. Suppose that $\sigma, \tau \in \Gamma(\mathcal{U}; \mathscr{S})$ satisfy $\sigma(x) = \tau(x)$ for every $x \in \mathcal{U}_a$ and every $a \in A$. It is obvious, then, that $\sigma = \tau$. Now again let $\mathcal{U} \in \mathscr{O}$ and let $(\mathcal{U}_a)_{a\in A}$ be an open cover for $\mathcal{U}$. Suppose that for each $a \in A$ there exists $\sigma_a \in \Gamma(\mathcal{U}_a; \mathscr{S})$ such that $\sigma_{a_1}(x) = \sigma_{a_2}(x)$ for every $x \in \mathcal{U}_{a_1} \cap \mathcal{U}_{a_2}$. Then, for $x \in \mathcal{U}$, define $\sigma(x) = \sigma_a(x)$ where $a \in A$ is such that $x \in \mathcal{U}_a$. This is clearly well-defined. We need only show that σ is continuous. But this follows since σ_a is continuous, and σ agrees with σ_a in a neighbourhood of x. ■

Now we have a process of starting with a presheaf $\mathscr{F}$ and constructing another presheaf $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$, and also a process of starting with an étalé space $\mathscr{S}$ and constructing another étalé space $\mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$. One anticipates that there is a relationship between these objects, and we shall explore this now. It is convenient at this point to refer ahead to the notion of an étalé morphism from Definition 7.1.32.

7.1.26 Proposition ($\mathrm{Et}(\mathrm{Ps}(\mathscr{S})) \simeq \mathscr{S}$) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{S}$ is an étalé space (of sets, rings, or modules) over $\mathcal{S}$, then the map $\alpha\colon \mathscr{S} \to \mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$ given by $\alpha(\sigma(\mathrm{x})) = [\sigma]_{\mathrm{x}}$, where $\sigma\colon \mathcal{U} \to \mathscr{S}$ is a section over $\mathcal{U}$, is an isomorphism of étalé spaces.*

Proof First, let us verify that α is well-defined. Suppose that local sections σ and τ of $\mathscr{S}$ agree at x. By Lemma 7.1.20 it follows that σ and τ agree in some neighbourhood of x. But this means that $[\sigma]_x = [\tau]_x$, giving well-definedness of α. To show that α is injective, suppose that $\alpha(\sigma(x)) = \alpha(\tau(x))$. Thus $[\sigma]_x = [\tau]_x$ and so σ and τ agree on some neighbourhood of x by Lemma 7.1.20. Thus $\sigma(x) = \tau(x)$, giving injectivity. To show that α is surjective, let $[\sigma]_x \in \mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$. Again since sections of $\mathscr{S}$ are local inverses for the étalé projection, it follows that $\alpha(\sigma(x)) = [\sigma]_x$, giving surjectivity. It is also clear that $\alpha(\mathscr{S}_x) \subseteq \mathrm{Et}(\mathrm{Ps}(\mathscr{S}))_x$. Let us verify that the ring operations are preserved by α when $\mathscr{S} = \mathscr{R}$ is an étalé space of rings; an entirely similar verification holds for étalé spaces of modules. The definition of the ring operation on stalks of $\mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$ ensures that

$$\alpha(\sigma(x) + \tau(x)) = [\sigma + \tau]_x = [\sigma]_x + [\tau]_x = \alpha(\sigma(x)) + \alpha(\tau(x))$$

and

$$\alpha(\sigma(x) \cdot \tau(x)) = [\sigma \cdot \tau]_x = [\sigma]_x \cdot [\tau]_x = \alpha(\sigma(x)) \cdot \alpha(\tau(x))$$

i.e., α is a ring homomorphism of stalks. It thus remains to show that α is continuous. Let $[\sigma]_x \in \mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$ and let $\mathcal{O}$ be a neighbourhood of $[\sigma]_x$ in $\mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$. By Lemma 7.1.20, there exists a neighbourhood $\mathcal{U}$ of x such that $\mathcal{B}(\mathcal{U}, [\sigma])$ is a neighbourhood of x contained in $\mathcal{O}$. Here $[\sigma]$ is the section of $\mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$ over $\mathcal{U}$ given by $[\sigma](y) = [\sigma]_y$. Since $\alpha(\sigma(y)) = [\sigma]_y$ for every $y \in \mathcal{U}$, it follows that $\alpha(\mathcal{B}(\mathcal{U}, \sigma)) = \mathcal{B}(\mathcal{U}, [\sigma])$, giving continuity as desired. ■

Now let us look at the relationship between a presheaf $\mathscr{F}$ and the presheaf $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$. We again refer ahead to Definition 7.1.32 for the notion of a morphism of presheaves.

7.1.27 Proposition ($\mathrm{Ps}(\mathrm{Et}(\mathscr{F})) \simeq \mathscr{F}$ if $\mathscr{F}$ is a sheaf) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ is a sheaf (of sets, rings, or modules) over $\mathcal{S}$, then the map which assigns to $\mathrm{s} \in \mathrm{F}(\mathcal{U})$ the section $\beta_{\mathcal{U}}(\mathrm{s}) \in \Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{F}))$ given by $\beta_{\mathcal{U}}(\mathrm{s})(\mathrm{x}) = [\mathrm{s}]_{\mathrm{x}}$ is an isomorphism of presheaves.*

Proof We must show that $\beta_{\mathcal{U}}$ is a bijection for each $\mathcal{U} \in \mathscr{O}$. To see that $\beta_{\mathcal{U}}$ is injective, suppose that $\beta_{\mathcal{U}}(s) = \beta_{\mathcal{U}}(t)$. Then $[s]_x = [t]_x$ for every $x \in \mathcal{U}$. Thus, for each $x \in \mathcal{U}$ there exists a neighbourhood $\mathcal{U}_x \subseteq \mathcal{U}$ of x such that $r_{\mathcal{U},\mathcal{U}_x}(s) = r_{\mathcal{U},\mathcal{U}_x}(t)$. By condition (i) of Definition 7.1.6 it follows that $s = t$. For surjectivity, let $\sigma \in \Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{F}))$. Let $x \in \mathcal{U}$ and let $\mathcal{U}_x$ be a neighbourhood of x and $s_x \in F(\mathcal{U}_x)$ be such that $\sigma(x) = [s_x]_x$. Since sections of $\mathrm{Et}(\mathscr{F})$ are local inverses for the local homeomorphism $\pi_{\mathscr{F}}$ (by definition of the étalé topology), sections of $\mathrm{Et}(\mathscr{F})$ agreeing at x must agree in a neighbourhood of x. In particular, there must exist a neighbourhood of x, $\mathcal{V}_x \subseteq \mathcal{U}_x$, such that $\sigma(y) = [s_x]_y$ for every $y \in \mathcal{V}_x$. It follows from Definition 7.1.6(i), therefore, that

$$r_{\mathcal{V}_{x_1},\mathcal{V}_{x_1}\cap\mathcal{V}_{x_2}}(s_{x_1}) = r_{\mathcal{V}_{x_2},\mathcal{V}_{x_1}\cap\mathcal{V}_{x_2}}(s_{x_2})$$

for every $x_1, x_2 \in \mathcal{U}$. By Definition 7.1.6(ii) it follows that there exists $s_\sigma \in F(\mathcal{U})$ such that $\sigma(x) = [s_x]_x = [s_\sigma]_x$ for every $x \in \mathcal{U}$, as desired.

Now we prove that $\beta_{\mathcal{U}}$ is a ring homomorphism if $\mathscr{F} = \mathscr{R}$ is a sheaf of rings. Indeed,

$$\beta_{\mathcal{U}}(f+g)(x) = [f+g]_x = [f]_x + [g]_x = \beta_{\mathcal{U}}(f)(x) + \beta_{\mathcal{U}}(g)(x)$$

and

$$\beta_{\mathcal{U}}(f \cdot g)(x) = [f \cdot g]_x = [f]_x \cdot [g]_x = (\beta_{\mathcal{U}}(f)(x)) \cdot (\beta_{\mathcal{U}}(g)(x)),$$

showing that $\beta_{\mathcal{U}}$ is a homomorphism of rings.

An entirely similar proof gives that $\beta_{\mathcal{U}}$ is a module homomorphism in the case that $\mathscr{F} = \mathscr{E}$ is a sheaf of $\mathscr{R}$-modules. ■

Thus, one of the nice things about the étalé space is that it allows one to realise a presheaf as a presheaf of sections of something, somehow making the constructions more concrete (although the étalé spaces themselves can be quite difficult to understand). This correspondence between sheaves and étalé spaces leads to a common abuse of notation and terminology, with the frequent and systematic confounding of a sheaf and its étalé space. Moreover, as we shall see in Section 7.1.8, there is a degree of inevitability to this, as some constructions with sheaves lead one naturally to building étalé spaces.

7.1.5 The sheafification of a presheaf

While it is true that many of the presheaves we will encounter are sheaves, cf. Proposition 7.1.7, it is also the case that some presheaves are not sheaves, and we saw a natural and not so natural example of this in Example 7.1.8. As we saw in those examples, a presheaf may fail to be a sheaf for two reasons: (1) the local behaviour of restrictions of sections does not accurately represent the local behaviour of sections (failure of the presheaf to be separated); (2) there are characteristics of global sections that are not represented by local characteristics (failure of the presheaf to satisfy the gluing conditions). The process of sheafification seeks to repair these defects by shrinking or enlarging the sets of sections as required by the sheaf axioms. The construction is as follows.

7.1.28 Definition (Sheafification) Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a presheaf (of sets, rings, or modules) over $\mathcal{S}$. The ***sheafification*** of $\mathscr{F}$ is the presheaf $\mathscr{F}^+ = (F^+(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ such that an element of $F^+(\mathcal{U})$ is comprised of the (not necessarily continuous) maps $\sigma\colon \mathcal{U} \to \mathrm{Et}(\mathscr{F})$ such that

(i) $\pi_{\mathscr{F}} \circ \sigma = \mathrm{id}_{\mathcal{U}}$,

(ii) for each $x \in \mathcal{U}$ there is a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x and $s \in F(\mathcal{V})$ such that $\sigma(y) = r_{\mathcal{V},y}(s)$ for every $y \in \mathcal{V}$, and

(iii) if $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$, then the restriction map $r^+_{\mathcal{U},\mathcal{V}}$ is defined by

$$r^+_{\mathcal{U},\mathcal{V}}(\sigma)(x) = \sigma(x)$$

for each $x \in \mathcal{V}$.

(iv) If $\mathscr{F} = \mathscr{R}$ is a presheaf of rings, the ring operations on $R^+(\mathcal{U})$ are defined by

$$[f]_x + [g]_x = [f + g]_x, \quad [f]_x \cdot [g]_x = [f \cdot g]_x,$$

where $[f]_x, [g]_x \in F(\mathcal{V})$ for some sufficiently small neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x and for $x \in \mathcal{U}$.

(v) If $\mathscr{F} = \mathscr{E}$ is a presheaf of $\mathscr{R}$-modules, the module operations on $E^+(\mathcal{U})$ are defined by

$$[s]_x + [t]_x = [s + t]_x, \quad [f]_x \cdot [s]_x = [f \cdot s]_x,$$

where $[s]_x, [t]_x \in E(\mathcal{V})$ and $[f]_x \in R(\mathcal{V})$ for some sufficiently small neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x and for $x \in \mathcal{U}$. •

As one hopes, the sheafification of a presheaf is a sheaf.

7.1.29 Proposition (The sheafification is a sheaf) *If $(\mathcal{S}, \mathscr{O})$ is topological space and if $\mathscr{F}$ is a presheaf (of sets. rings, or modules) over $\mathcal{S}$, then*

(i) $\mathscr{F}^+ = \mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$,

(ii) the sheafification $\mathscr{F}^+$ is a sheaf, and

(iii) if $\mathrm{x} \in \mathcal{S}$, the map $\iota_{\mathrm{x}}\colon \mathscr{F}_{\mathrm{x}} \to \mathscr{F}^+_{\mathrm{x}}$ defined by $\iota_{\mathrm{x}}([\mathrm{s}]_{\mathrm{x}}) = [\sigma_{\mathrm{s}}]_{\mathrm{x}}$ where $\sigma_{\mathrm{s}}(\mathrm{y}) = [\mathrm{s}]_{\mathrm{y}}$ for y in some neighbourhood of x, is a bijection.

Proof (i) It is clear that we have an inclusion from $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$ into $\mathscr{F}^+$, just by definition of $\mathscr{F}^+$. We shall show that this inclusion is a surjective mapping of presheaves. For surjectivity of the natural inclusion, let $\mathcal{U} \in \mathscr{O}$ and let $\tau \in F^+(\mathcal{U})$. For $x \in \mathcal{U}$ there exists a neighbourhood $\mathcal{U}_x \subseteq \mathcal{U}$ of x and $s_x \in F(\mathcal{U}_x)$ such that $\tau(y) = [s_x]_y$ for each $y \in \mathcal{U}_x$. Define $\sigma_x \in \Gamma(\mathcal{U}_x; \mathrm{Et}(\mathscr{F}))$ by $\sigma_x(y) = [s_x]_y$. Thus we have an open cover $(\mathcal{U}_x)_{x\in\mathcal{U}}$ of $\mathcal{U}$ and a corresponding family $(\sigma_x)_{x\in\mathcal{U}}$ of sections of $\mathrm{Et}(\mathscr{F})$. Since $\mathrm{Et}(\mathscr{F})$ is separated, it follows that

$$r_{\mathcal{U}_{x_1},\mathcal{U}_{x_1}\cap\mathcal{U}_{x_2}}(\sigma_{x_1}) = r_{\mathcal{U}_{x_2},\mathcal{U}_{x_1}\cap\mathcal{U}_{x_2}}(\sigma_{x_2}),$$

cf. the proof of surjectivity for Proposition 7.1.27. Now we use the gluing property of $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$ to assert the existence of $\sigma \in F(\mathcal{U})$ such that $r_{\mathcal{U},\mathcal{U}_x}(\sigma) = \sigma_x$ for every $x \in \mathcal{U}$. We clearly have $\sigma(x) = \tau(x)$ for every $x \in \mathcal{U}$, giving surjectivity.

(ii) This follows from the previous part of the result along with Proposition 7.1.25.

(iii) To prove injectivity of the map, suppose that $\iota_x([s]_x) = \iota_x([t]_x)$. Then there exists a neighbourhood $\mathcal{U}$ of x such that s and t restrict to $\mathcal{U}$ and agree on $\mathcal{U}$. Thus $\sigma_s = \sigma_t$ on $\mathcal{U}$. For surjectivity, let $[\sigma]_x \in \mathscr{F}_x^+$. Then there exists a neighbourhood $\mathcal{V}$ of x such that σ is defined on $\mathcal{V}$ and a section $s \in F(\mathcal{V})$ such that $\sigma(x) = [s]_x$. Thus $\iota_x([s]_x) = \sigma_s(x) = \sigma(x)$, giving surjectivity. ■

The sheafification has an important "universality" property. To state this property, we refer ahead to the notion of a morphism of presheaves in Definition 7.1.32.

7.1.30 Proposition (Universality of the sheafification) *If $(\mathcal{S}, \mathscr{O})$ is a topological space and if $\mathscr{F}$ is a presheaf (of sets, rings, or modules) over $\mathcal{S}$, then there exists a morphism of presheaves $(\iota_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\mathscr{F}$ to $\mathscr{F}^+$ such that, if $\mathscr{G}$ is a sheaf over $\mathcal{S}$ and if $(\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a morphism of presheaves from $\mathscr{F}$ to $\mathscr{G}$, then there exists a unique morphism of presheaves $(\Phi^+_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\mathscr{F}^+$ to $\mathscr{G}$ satisfying $\Phi_{\mathcal{U}} = \Phi^+_{\mathcal{U}} \circ \iota_{\mathcal{U}}$ for every $\mathcal{U} \in \mathscr{O}$.*

Moreover, if $\hat{\mathscr{F}}$ is a sheaf and if $(\hat{\iota}_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a morphism of presheaves (of sets, rings, or modules) from $\mathscr{F}$ to $\mathscr{F}^+$ having the above property, then there exists a unique isomorphism of presheaves from $\hat{\mathscr{F}}$ to $\mathscr{F}^+$.

Proof Let us define $\iota_{\mathcal{U}} \colon F(\mathcal{U}) \to F^+(\mathcal{U})$ by $\iota_{\mathcal{U}}(s)(x) = [s]_x$. Now, given a morphism $(\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ of presheaves from $\mathscr{F}$ to $\mathscr{G}$, define a morphism $(\Phi'^+_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ of presheaves from $\mathscr{F}^+$ to $\mathrm{Ps}(\mathrm{Et}(\mathscr{G}))$ by

$$\Phi'^+_{\mathcal{U}}([s]_x) = [\Phi_{\mathcal{U}}(s)]_x.$$

We should show that this definition is independent of s. That is to say, we should show that if $[s]_x = [t]_x$ for every $x \in \mathcal{U}$ then $\Phi_{\mathcal{U}}(s) = \Phi_{\mathcal{U}}(t)$. Since $[s]_x = [t]_x$ for every $x \in \mathcal{U}$, for each $x \in \mathcal{U}$ there exists a neighbourhood $\mathcal{U}_x$ such that $r_{\mathcal{U},\mathcal{U}_x}(s) = r_{\mathcal{U},\mathcal{U}_x}(t)$. Since $(\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a morphism of presheaves, we have

$$r_{\mathcal{U},\mathcal{U}_x}(\Phi_{\mathcal{U}}(s)) = r_{\mathcal{U},\mathcal{U}_x}(\Phi_{\mathcal{U}}(t)).$$

Since $\mathscr{G}$ is separable, we infer that $\Phi_{\mathcal{U}}(t) = \Phi_{\mathcal{U}}(s)$, as desired.

Recall from Proposition 7.1.27 the mapping $\beta_{\mathcal{U}}$ from $G(\mathcal{U})$ to $\Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{G}))$ and that the family of mappings $(\beta_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ defines a presheaf isomorphism by virtue of $\mathscr{G}$ being a sheaf. Sorting through the definitions gives $\Phi_{\mathcal{U}}(s) = \beta^{-1}_{\mathcal{U}} \circ \Phi'^+_{\mathcal{U}} \circ \iota_{\mathcal{U}}$, which gives the existence part of the first assertion by taking $\Phi^+_{\mathcal{U}} = \beta^{-1}_{\mathcal{U}} \circ \Phi'^+_{\mathcal{U}}$. For the uniqueness part of the assertion, note that the requirement that $\Phi_{\mathcal{U}}(s) = \Phi^+_{\mathcal{U}} \circ \iota_{\mathcal{U}}(s)$ implies that

$$\Phi^+_{\mathcal{U}}([s]_x) = \Phi_{\mathcal{U}}(s)(x) = \beta^{-1}_{\mathcal{U}} \circ \Phi'^+_{\mathcal{U}}([s]_x),$$

as desired.

Now we turn to the second assertion. Thus $\hat{\mathscr{F}} = (\hat{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ is a sheaf and for each $\mathcal{U} \in \mathscr{O}$ we have a mapping $\hat{\iota}_{\mathcal{U}} \colon F(\mathcal{U}) \to \hat{F}(\mathcal{U})$ such that, for any presheaf morphism $(\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\mathscr{F}$ to $\mathscr{G}$, there exists a unique presheaf morphism $(\hat{\Phi}_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\hat{\mathscr{F}}$ to $\mathscr{G}$ such that $\Phi_{\mathcal{U}} = \hat{\Phi}_{\mathcal{U}} \circ \hat{\iota}_{\mathcal{U}}$ for every $\mathcal{U} \in \mathscr{O}$. Applying this hypothesis to the presheaf morphism $(\iota_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\mathscr{F}$ to $\mathscr{F}^+$ gives a unique presheaf morphism $(\kappa_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ from $\hat{\mathscr{F}}$ to $\mathscr{F}^+$ such that $\iota_{\mathcal{U}} = \kappa_{\mathcal{U}} \circ \hat{\iota}_{\mathcal{U}}$ for every $\mathcal{U} \in \mathscr{O}$. We claim that, for every $\mathcal{U} \in \mathscr{O}$, $\kappa_{\mathcal{U}}$ is a bijection from $\hat{F}(\mathcal{U})$ to $F^+(\mathcal{U})$. Fix $\mathcal{U} \in \mathscr{O}$. In the same manner as we deduced the existence of $\kappa_{\mathcal{U}}$, we have a mapping

$\hat{\kappa}_{\mathcal{U}}\colon F^+(\mathcal{U}) \to \hat{F}(\mathcal{U})$ such that $\hat{\iota}_{\mathcal{U}} = \hat{\kappa}_{\mathcal{U}} \circ \iota_{\mathcal{U}}$. Thus $\hat{\iota}_{\mathcal{U}} = \hat{\kappa}_{\mathcal{U}} \circ \kappa_{\mathcal{U}} \circ \hat{\iota}_{\mathcal{U}}$. However, we also have $\hat{\iota}_{\mathcal{U}} = \mathrm{id}_{\hat{F}(\mathcal{U})} \circ \hat{\iota}_{\mathcal{U}}$ and so, by the uniqueness part of the first part of the proposition, we have $\hat{\kappa}_{\mathcal{U}} \circ \kappa_{\mathcal{U}} = \mathrm{id}_{\hat{F}(\mathcal{U})}$. In like manner, $\kappa_{\mathcal{U}} \circ \hat{\kappa}_{\mathcal{U}} = \mathrm{id}_{\hat{F}(\mathcal{U})}$, giving that $\hat{\kappa}_{\mathcal{U}}$ is the inverse of $\kappa_{\mathcal{U}}$. ■

To better get a handle on the sheafification of a presheaf, let us consider the sheafification of the presheaves from Example 7.1.8.

7.1.31 Examples (Sheafification)

1. We revisit Example 7.1.8–1 where we consider the presheaf $\mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R})$ of functions of class C^r on $\mathsf{M} = \mathbb{R}$ that were bounded on their domains. Here we claim that the sheafification of $\mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R})$ is simply $\mathrm{Ps}(\mathrm{Et}(\mathscr{C}^r_{\mathbb{R}}))$. By Proposition 7.1.29(i) we have $\mathrm{Ps}(\mathrm{Et}(\mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R}))) = (\mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R}))^+$. It is also clear that $\mathrm{Et}(\mathscr{C}^r_{\mathrm{bdd}}(\mathbb{R})) = \mathrm{Et}(\mathscr{C}^r_{\mathbb{R}})$ since the restriction of a function being bounded does not restrict stalks, and so we have our desired conclusion.
2. Let us determine the sheafification $\mathscr{F}^+_X$ of the constant sheaf $\mathscr{F}_X$ over a topological space $(\mathcal{S}, \mathscr{O})$ associated with a set X. As in Example 7.1.15 we have $\mathscr{F}_X \simeq \mathcal{S} \times X$ and so, first of all, sections of $\mathscr{F}^+_X$ over $\mathcal{U} \in \mathscr{O}$ are identified with maps from $\mathcal{U}$ to X. Let $\sigma\colon \mathcal{U} \to X$ be a section of $\mathscr{F}^+_X$ under this identification and let $x \in \mathcal{U}$. By definition of $\mathscr{F}^+_X$ there exists a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x and $s \in F_X(\mathcal{V})$ such that $\sigma(y) = s$ for every $y \in \mathcal{V}$. Thus σ is locally constant. Since any section of $\mathrm{Et}(\mathscr{F}_X)$ is, by our construction of the étalé topology on $\mathrm{Et}(\mathscr{F}_X)$ in Example 7.1.15 and by our definition of the constant étalé space $\mathscr{S}_X$ in Example 7.1.21, locally constant, the sheafification of $\mathscr{F}_X$ is exactly $\mathrm{Et}(\mathscr{F}_X)$.
3. Here we consider the case of Example 7.1.8–1 where $\mathcal{S} = \{0, 1\}$. Here, because of the discrete topology on $\mathcal{S}$ and because of the character of the restriction maps for the presheaf $\mathscr{F}$ under consideration, we have $\mathscr{F}_0 = [0_{\{0\}}]_0$ and $\mathscr{F}_1 = [0_{\{1\}}]_1$. Thus the sheafification $\mathscr{F}^+$ has zero stalks. In this case the presheaf has to shrink to obtain the sheafification, in order to account for the fact that the germs are trivial. •

Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$. Suppose that we have a presheaf $\mathscr{E}$ of $\mathscr{C}^r_{\mathsf{M}}$-modules, where M is a smooth, real analytic manifold or holomorphic manifold, as required. In these cases, the sheafification of $\mathscr{E}$ is also a presheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules by virtue of Proposition 7.1.29(iii).

7.1.6 Morphisms of presheaves and étalé spaces

Next we study maps between presheaves and étalé spaces.

7.1.32 Definition (Morphism of presheaves and étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{G} = (G(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{H} = (H(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be presheaves (of sets, rings, or modules) over $\mathcal{S}$, and let $\pi\colon \mathscr{S} \to \mathcal{S}$ and $\rho\colon \mathscr{T} \to \mathcal{S}$ be étalé spaces (of sets, rings, or modules) over $\mathcal{S}$.

(i) (a) A ***morphism*** of the presheaves $\mathscr{G}$ and $\mathscr{H}$ is an assignment to each $\mathcal{U} \in \mathscr{O}$ a mapping $\Phi_{\mathcal{U}}\colon G(\mathcal{U}) \to H(\mathcal{U})$ such that the diagram

$$\begin{array}{ccc} G(\mathcal{U}) & \xrightarrow{\Phi_{\mathcal{U}}} & H(\mathcal{U}) \\ {\scriptstyle r_{\mathcal{U},\mathcal{V}}}\downarrow & & \downarrow{\scriptstyle r_{\mathcal{U},\mathcal{V}}} \\ G(\mathcal{V}) & \xrightarrow[\Phi_{\mathcal{V}}]{} & H(\mathcal{V}) \end{array} \tag{7.2}$$

commutes for every $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ with $\mathcal{V} \subseteq \mathcal{U}$. We shall often use the abbreviation $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$. If $\mathscr{G}$ and $\mathscr{H}$ are sheaves, Φ is called a ***morphism*** of sheaves.

(b) If $\mathscr{G} = \mathscr{A}$ and $\mathscr{H} = \mathscr{B}$ are presheaves of rings, we additionally require that $\Phi_{\mathcal{U}}$ is a homomorphism of rings.

(c) If $\mathscr{G} = \mathscr{E}$ and $\mathscr{H} = \mathscr{F}$ are presheaves of $\mathscr{R}$-modules, we additionally require that $\Phi_{\mathcal{U}}$ be a homomorphism of $R(\mathcal{U})$-modules.

(ii) (a) An ***étalé morphism*** of $\mathscr{S}$ and $\mathscr{T}$ is a continuous map $\Phi\colon \mathscr{S} \to \mathscr{T}$ such that $\Phi(\mathscr{S}_x) \subseteq \mathscr{T}_x$.

(b) If $\mathscr{S} = \mathscr{A}$ and $\mathscr{T} = \mathscr{B}$ are étalé spaces of rings, we additionally require that $\Phi|\mathscr{A}_x$ is a homomorphism of rings for every $x \in \mathcal{S}$.

(c) If $\mathscr{S} = \mathscr{E}$ and $\mathscr{T} = \mathscr{F}$ are étalé spaces of $\mathscr{R}$-modules, we additionally require that $\Phi|\mathscr{R}_x$ is a homomorphism of $\mathscr{R}_x$-modules for every $x \in \mathcal{S}$. •

Let us show that the preceding notions are often in natural correspondence. To do so, let us first indicate how to associate an étalé morphism to a morphism of presheaves, and vice versa.

Let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be a morphism of presheaves (of sets, rings, or modules) $\mathscr{F}$ and $\mathscr{G}$ over $(\mathcal{S}, \mathscr{O})$. Define a mapping $\mathrm{Et}(\Phi)\colon \mathrm{Et}(\mathscr{F}) \to \mathrm{Et}(\mathscr{G})$ by

$$\mathrm{Et}(\Phi)([s]_x) = [\Phi_{\mathcal{U}}(s)]_x,$$

where $\mathcal{U}$ is such that $s \in F(\mathcal{U})$. We denote by $\mathrm{Et}(\Phi)_x$ the restriction of $\mathrm{Et}(\Phi)$ to $\mathrm{Et}(\mathscr{F})_x$. This construction is well-defined by virtue of the commuting of the diagram (7.2). That $\mathrm{Et}(\Phi)$ is a homomorphism of rings or modules when restricted to stalks follows from the commuting of the diagram (7.2) and the definition of the ring or module operations operation on stalks. Finally, the mapping $\mathrm{Et}(\Phi)$ is verified to be continuous as the basic neighbourhood $\mathcal{B}(\mathcal{U}, s)$ is mapped to the basic neighbourhood $\mathcal{B}(\mathcal{U}, \Phi_{\mathcal{U}}(s))$.

formalise what's below!

Let $\beta \in \mathrm{image}(\mathrm{Et}(\Phi))$ and write $\beta = [\Phi_{\mathcal{U}}(s)]_x$. Consider the open set $\mathcal{B}(\mathcal{U}, \Phi_{\mathcal{U}}(s))$ and let

$$[t]_x \in \mathrm{Et}(\Phi)^{-1}(\mathcal{B}(\mathcal{U}, \Phi_{\mathcal{U}}(s))).$$

Write $t \in F(\mathcal{V})$. Thus $[\Phi_{\mathcal{V}}(t)]_x = [\Phi_{\mathcal{U}}]_x$ and so $\Phi_{\mathcal{V}}(t)$ and $\Phi_{\mathcal{U}}(s)$ have equal restrict ion to some $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$. Thus

$$\mathcal{B}(\mathcal{W}, r_{\mathcal{V},\mathcal{W}}(t)) \subseteq \mathrm{Et}(\Phi)^{-1}(\mathcal{B}(\mathcal{U}, \Phi_{\mathcal{U}}(s))),$$

showing that $\mathrm{Et}(\Phi)^{-1}(\mathcal{B}(\mathcal{U}, \Phi_{\mathcal{U}}(s)))$ is open.

Conversely, if $\Phi\colon \mathscr{S} \to \mathscr{T}$ is an étalé morphism of étalé spaces (of sets, rings, or modules) over $(\mathcal{S}, \mathscr{O})$, if $\mathcal{U} \in \mathscr{O}$, and if $\sigma \in \Gamma(\mathcal{U}; \mathscr{S})$, then we define a presheaf morphism $\mathrm{Ps}(\Phi)$ from $\mathrm{Ps}(\mathscr{S})$ to $\mathrm{Ps}(\mathscr{T})$ by requiring that $\mathrm{Ps}(\Phi)_{\mathcal{U}}(\sigma) \in \Gamma(\mathcal{U}; \mathscr{T})$ is given by

$$\mathrm{Ps}(\Phi)_{\mathcal{U}}(\sigma)(x) = \Phi([\sigma]_x).$$

This construction is well-defined since Φ is continuous. It is also obvious that $\mathrm{Ps}(\Phi)$ commutes with restrictions. Moreover, it clearly defines a homomorphism of rings or modules when the étalé space possess these structures.

formalise what's above and below!

Moreover, one readily verifies, merely by sorting through definitions, that the diagram

$$\begin{array}{ccc} \mathrm{Ps}(\mathrm{Et}(\mathscr{F})) & \xrightarrow{\beta_{\mathscr{F}}} & \mathscr{F} \\ {\scriptstyle \mathrm{Ps}(\mathrm{Et}(\Phi))}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathrm{Ps}(\mathrm{Et}(\mathscr{G})) & \xrightarrow[\beta_{\mathscr{G}}]{} & \mathscr{G} \end{array} \tag{7.3}$$

commutes when $\mathscr{F}$ and $\mathscr{G}$ are sheaves, and where $\beta_{\mathscr{F}}$ and $\beta_{\mathscr{G}}$ are the presheaf isomorphisms from Proposition 7.1.27. In like manner, the diagram

$$\begin{array}{ccc} \mathrm{Et}(\mathrm{Ps}(\mathscr{S})) & \xrightarrow{\alpha_{\mathscr{S}}} & \mathscr{S} \\ {\scriptstyle \mathrm{Et}(\mathrm{Ps}(\Phi))}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \mathrm{Et}(\mathrm{Ps}(\mathscr{T})) & \xrightarrow[\alpha_{\mathscr{T}}]{} & \mathscr{T} \end{array} \tag{7.4}$$

always commutes, where $\alpha_{\mathscr{S}}$ and $\alpha_{\mathscr{T}}$ are the étalé isomorphisms from Proposition 7.1.26.

The following property of étalé morphisms is sometimes useful.

7.1.33 Proposition (Étalé morphisms are open) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\pi\colon \mathscr{S} \to \mathcal{S}$ and $\rho\colon \mathscr{T} \to \mathcal{S}$ are étalé spaces (of sets, rings, or modules) over $\mathcal{S}$, and if $\Phi\colon \mathscr{S} \to \mathscr{T}$ is an étalé morphism, then Φ is an open mapping.*

Proof Let $\mathcal{O} \subseteq \mathscr{S}$ be open and, for $[\sigma]_x \in \mathcal{O}$ let $\mathcal{U}$ be a neighbourhood of x such that the basic neighbourhood $\mathcal{B}(\mathcal{U}, \sigma|\mathcal{U})$ is contained in $\mathcal{O}$. Note that, by continuity, Φ maps $\mathcal{B}(\mathcal{U}, \sigma|\mathcal{U})$ to $\mathcal{B}(\mathcal{U}, \Phi \circ \sigma|\mathcal{U})$. Thus this latter neighbourhood is contained in $\Phi(\mathcal{O})$. Moreover, $\Phi(\mathcal{O})$ is the union of these neighbourhood, showing that it is open. ■

In closing, let us understand the morphisms of sheaves can be themselves organised into a sheaf. Suppose that $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$ is a presheaf of rings over a topological space $(\mathcal{S}, \mathscr{O})$ and let $\mathscr{E} = (E(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$ and $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$ be presheaves of $\mathscr{R}$-modules over $\mathcal{S}$. For $\mathcal{U} \in \mathscr{O}$ we then have the restrictions $\mathscr{E}|\mathcal{U}$ and $\mathscr{F}|\mathcal{U}$ which are presheaves of $\mathscr{R}|\mathcal{U}$-modules. Let us define a presheaf $\mathscr{H}om(\mathscr{E}; \mathscr{F})$ by assigning to $\mathcal{U} \in \mathscr{O}$ the collection of presheaf morphisms from $\mathscr{E}|\mathcal{U}$ to $\mathscr{F}|\mathcal{U}$. Thus a section of $\mathscr{H}om(\mathscr{E}; \mathscr{F})$ over $\mathcal{U}$ is a family $(\Phi_{\mathcal{V}})_{\mathcal{U} \supseteq \mathcal{V}\ \text{open}}$ where $\Phi_{\mathcal{V}} \in \mathrm{Hom}_{R(\mathcal{V})}(E(\mathcal{V}); F(\mathcal{V}))$. If $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$, the restriction map $r_{\mathcal{U},\mathcal{V}}$ maps the section $(\Phi_{\mathcal{W}})_{\mathcal{U} \supseteq \mathcal{W}\ \text{open}}$ over $\mathcal{U}$ to the section $(\Phi_{\mathcal{W}})_{\mathcal{V} \supseteq \mathcal{W}\ \text{open}}$ over $\mathcal{V}$. We render $\mathscr{H}om(\mathscr{E};\mathscr{F})$ a sheaf of $\mathscr{R}$-modules as follows. Let $\mathcal{U}\in\mathscr{O}$, consider sections $(\Phi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}$ and $(\Psi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}$ over $\mathcal{U}$, and let $f\in R(\mathcal{U})$. Then we define

$$(\Phi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}+(\Psi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}=(\Phi_{\mathcal{V}}+\Psi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}},$$
$$f\cdot\Big((\Phi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}\Big)=(r_{\mathcal{U},\mathcal{V}}(f)\cdot\Phi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}.$$

One readily verifies that this does indeed provide the structure of an $\mathscr{R}$-module. Let us give a useful property of the presheaf $\mathscr{H}om(\mathscr{E};\mathscr{F})$.

7.1.34 Lemma (The presheaf of morphisms of sheaves is a sheaf) *Let $(\mathcal{S},\mathscr{O})$ be a topological space, let $\mathscr{R}=(\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{E}=(\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{F}=(\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be sheaves of $\mathscr{R}$-modules over $\mathcal{S}$. Then $\mathscr{H}om(\mathscr{E};\mathscr{F})$ is a sheaf.*

Proof Let $\mathcal{U}\in\mathscr{O}$ and let $(\mathcal{U}_a)_{a\in A}$ be an open cover of $\mathcal{U}$. Let $(\Phi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}$ and $(\Psi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}$ be sections over $\mathcal{U}$ whose restrictions to each of the open sets $\mathcal{U}_a$, $a\in A$, agree. Let $\mathcal{V}\subseteq\mathcal{U}$ be open and let $s\in E(\mathcal{V})$. By hypothesis, $\Phi_{\mathcal{V}\cap\mathcal{U}_a}(s_a)=\Psi_{\mathcal{V}\cap\mathcal{U}_a}(s_a)$ for every $a\in A$ and $s_a\in E(\mathcal{V}\cap\mathcal{U}_a)$. This implies that

$$\Phi_{\mathcal{V}}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(s))=\Psi_{\mathcal{V}}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(s))$$

for every $a\in A$, and so

$$r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(\Phi_{\mathcal{V}}(s))=r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(\Psi_{\mathcal{V}}(s))$$

for every $a\in A$. Since $\mathscr{F}$ is separated, this implies that $\Phi_{\mathcal{V}}(s)=\Psi_{\mathcal{V}}(s)$. We conclude, therefore, that $\mathscr{H}om(\mathscr{E};\mathscr{F})$ is separated.

Now again let $\mathcal{U}\in\mathscr{O}$ and let $(\mathcal{U}_a)_{a\in A}$ be an open cover for $\mathcal{U}$. For each $a\in A$ let $(\Phi_{a,\mathcal{V}})_{\mathcal{U}_a\supseteq\mathcal{V}\text{ open}}$ be a section of $\mathscr{H}om(\mathscr{E};\mathscr{F})$ over $\mathcal{U}_a$ and suppose that the restrictions of the sections over $\mathcal{U}_a$ and $\mathcal{U}_b$ agree on the intersection $\mathcal{U}_a\cap\mathcal{U}_b$ for every $a,b\in A$. Let $\mathcal{V}\subseteq\mathcal{U}$ be open and let $s\in E(\mathcal{V})$. By hypothesis

$$\Phi_{a,\mathcal{V}\cap\mathcal{U}_a\cap\mathcal{U}_b}(r_{\mathcal{V}\cap\mathcal{U}_a,\mathcal{V}\cap\mathcal{U}_a\cap\mathcal{U}_b}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(s)))=\Phi_{b,\mathcal{V}\cap\mathcal{U}_a\cap\mathcal{U}_b}(r_{\mathcal{V}\cap\mathcal{U}_b,\mathcal{V}\cap\mathcal{U}_a\cap\mathcal{U}_b}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_b}(s))$$

for every $a,b\in A$. Thus

$$r_{\mathcal{V}\cap\mathcal{U}_a,\mathcal{V}\cap\mathcal{U}_a\cap\mathcal{U}_b}(\Phi_{a,\mathcal{V}\cap\mathcal{U}_a}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(s)))=r_{\mathcal{V}\cap\mathcal{U}_b,\mathcal{V}\cap\mathcal{U}_a\cap\mathcal{U}_b}(\Phi_{b,\mathcal{V}\cap\mathcal{U}_b}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_b}(s)))$$

for every $a,b\in A$. Since $\mathscr{F}$ is has the gluing property, we infer the existence of $t\in F(\mathcal{V})$ such that

$$r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(t)=\Phi_{a,\mathcal{V}\cap\mathcal{U}_a}(r_{\mathcal{V},\mathcal{V}\cap\mathcal{U}_a}(s))$$

for every $a\in A$. We define $\Phi_{\mathcal{V}}$ by asking that $\Phi_{\mathcal{V}}(s)=t$. One can check directly, if tediously, that $\Phi_{\mathcal{V}}\in\mathrm{Hom}_{R(\mathcal{V})}(E(\mathcal{V});F(\mathcal{V}))$. Thus the section $(\Phi_{\mathcal{V}})_{\mathcal{U}\supseteq\mathcal{V}\text{ open}}$ of $\mathscr{H}om(\mathscr{E};\mathscr{F})$ over $\mathcal{U}$ so defined has the property that it restricts to $(\Phi_{a,\mathcal{V}})_{\mathcal{U}_a\supseteq\mathcal{V}\text{ open}}$ for each $a\in A$. ■

Let is give a few examples of morphisms of sheaves.

7.1.35 Examples (Sheaf morphisms)

1. Let $r \in \{\infty, \omega\}$ and let M be a smooth or real analytic manifold. Let us consider the sheaf $\mathscr{G}^k_{\bigwedge^r(\mathsf{T}^*\mathsf{M})}$ of germs of sections of the bundle of k-forms. Since the exterior derivative d commutes with restrictions to open sets, d induces a morphism of sheaves:
$$\mathrm{d}\colon \mathscr{G}^r_{\bigwedge^k(\mathsf{T}^*\mathsf{M})} \to \mathscr{G}^r_{\bigwedge^{k+1}(\mathsf{T}^*\mathsf{M})}.$$
This is a morphism of sheaves of Abelian groups, but not a morphism of sheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules, since d is not linear with respect to multiplication by C^r-functions.

2. We let M be a holomorphic manifold and consider the sheaf $\mathscr{G}^\infty_{\bigwedge^{r,s}(\mathsf{T}^{\mathbb{C}}\mathsf{M})}$ of germs of sections of the bundles of forms of bidegree (r,s), $r, s \in \mathbb{Z}_{\geq 0}$. This is a sheaf of $\mathscr{C}^\infty(\mathsf{M};\mathbb{C})$-modules, of course. The mappings ∂ and $\bar{\partial}$ of Section 4.4.4 commute with restrictions to open sets, and so define morphisms of sheaves
$$\partial\colon \mathscr{G}^\infty_{\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})} \to \mathscr{G}^\infty_{\bigwedge^{r+1,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})}, \quad \bar{\partial}\colon \mathscr{G}^\infty_{\bigwedge^{r,s}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})} \to \mathscr{G}^\infty_{\bigwedge^{r,s+1}(\mathsf{T}^{*\mathbb{C}}\mathsf{M})}.$$
These are morphisms of sheaves of Abelian groups, but neither of these are morphisms of $\mathscr{C}^\infty_{\mathsf{M}}\mathbb{C}$-modules, since neither ∂ nor $\bar{\partial}$ are linear with respect to multiplication by smooth functions.

3. If in the preceding example we instead regard $\mathscr{G}^\infty_{\bigwedge^{r,s}(\mathsf{T}^{\mathbb{C}}\mathsf{M})}$ as sheaves of $\mathscr{C}^{\mathrm{hol}}_{\mathsf{M}}$-modules, then, by Proposition 4.4.11(v) and because $\bar{\partial}$ annihilates holomorphic functions, $\bar{\partial}$ is a morphism of $\mathscr{C}^{\mathrm{hol}}_{\mathsf{M}}$-modules. •

7.1.7 Subpresheaves and étalé subspaces

We wish to talk about some standard algebraic constructions in the sheaf setting, and this requires that we know what a subsheaf is.

7.1.36 Definition (Subpresheaf, étalé subspace) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{G} = (G(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be presheaves (of sets, rings, or modules) over $\mathcal{S}$, and let $\pi\colon \mathscr{S} \to \mathcal{S}$ and $\rho\colon \mathscr{T} \to \mathcal{S}$ be étalé spaces (of sets, rings, or modules) over $\mathcal{S}$.

(i) The presheaf $\mathscr{F}$ is a ***subpresheaf*** of $\mathscr{G}$ if, for each $\mathcal{U} \in \mathscr{O}$, $F(\mathcal{U}) \subseteq G(\mathcal{U})$ and if the inclusion maps $i_{\mathscr{F},\mathcal{U}}\colon F(\mathcal{U}) \to G(\mathcal{U})$, $\mathcal{U} \in \mathscr{O}$, define a morphism $i_{\mathscr{F}} = (i_{\mathscr{F},\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ of presheaves (of sets, rings, or modules). If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, we say that $\mathscr{F}$ is a ***subsheaf*** of $\mathscr{G}$.

(ii) The étalé space $\mathscr{S}$ is a ***étalé subspace*** of $\mathscr{T}$ if $\mathscr{S}_x \subseteq \mathscr{T}_x$ and if the inclusion map from $\mathscr{S}$ into $\mathscr{T}$ is a étalé morphism (of sets, rings, or modules). •

As with morphisms, we can often freely go between subpresheaves and étalé subspaces. Let us spell this out. Suppose that $\mathscr{F}$ is a subpresheaf of $\mathscr{G}$. The commuting of the diagram (7.2) ensures that the mapping $\mathrm{Et}(i_{\mathscr{F}})\colon [s]_x \mapsto [i_{\mathscr{F},\mathcal{U}}(s)]_x$ from $\mathrm{Et}(\mathscr{F})_x$ to $\mathrm{Et}(\mathscr{G})_x$ is injective (and is a ring or module homomorphism, when appropriate), with $\mathcal{U}$ being such that $s \in F(\mathcal{U})$. As we saw above, this injection of $\mathrm{Et}(\mathscr{F})$ into $\mathrm{Et}(\mathscr{G})$ is an étalé morphism and so $\mathrm{Et}(\mathscr{F})$ is a étalé subspace of $\mathrm{Et}(\mathscr{G})$. Conversely, if $\mathscr{S}$ is a étalé subspace of $\mathscr{T}$ then we obviously have $\Gamma(\mathcal{U};\mathscr{S}) \subseteq \Gamma(\mathcal{U};\mathscr{T})$. We can see that

$(\Gamma(\mathcal{U};\mathscr{S}))_{\mathcal{U}\in\mathscr{O}}$ is a subpresheaf of $(\Gamma(\mathcal{U};\mathscr{T}))_{\mathcal{U}\in\mathscr{O}}$, using the definition of the ring or module operations on sections of étalé spaces when appropriate.

As for the passing to and from these constructions, our observations above ensure that, when $\mathscr{F}$ and $\mathscr{G}$ are sheaves, the presheaf $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$ corresponds under the restriction of $\beta_{\mathscr{G}}$ to $\mathrm{Ps}(\mathrm{Et}(\mathscr{F}))$ to the subpresheaf $\mathscr{F}$. And conversely, the étalé space $\mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$ always corresponds, under the restriction of $\alpha_{\mathscr{S}}$ to $\mathrm{Et}(\mathrm{Ps}(\mathscr{S}))$, to $\mathscr{S}$.

In order to illustrate that the preceding discussion has some content, let us give an explicit example showing when one has to exercise some care.

7.1.37 Example (Distinct presubsheaves with the same stalks) Let us consider the presheaf $\mathscr{C}^r_{\mathbb{R}}$ of functions of class C^r on $\mathbb{R}$. This is obviously a subpresheaf of itself. Moreover, in Example 7.1.8–1 we considered the subpresheaf $\mathscr{C}^R_{\mathrm{bdd}}(\mathbb{R})$ of bounded functions of class C^r. These étalé subspaces have the same stalks since the condition of boundedness places no restrictions on the germs. However, the presheaves are different. Thus the character of a presubsheaf is only ensured to be characterised by its stalks when the presheaf and the presubsheaf are sheaves. •

The following characterisation of étalé subspaces is sometimes useful.

7.1.38 Proposition (Étalé subspaces are open sets) *If $(\mathcal{S},\mathscr{O})$ is a topological space, if $\rho\colon\mathscr{T}\to\mathcal{S}$ is an étalé space of sets, rings, or modules over $\mathcal{S}$, and if $\mathscr{S}\subseteq\mathscr{T}$ is such that $\mathscr{S}_x\triangleq\mathscr{S}\cap\mathscr{T}_x\neq\emptyset$ for each $x\in\mathcal{S}$, then the following statements are equivalent:*

(i) $\mathscr{S}$ is an étalé subspace of $\mathscr{T}$;

(ii) $\mathscr{S}$ is an open subset of $\mathscr{T}$ and, when appropriate, $\mathscr{S}_x=\mathscr{S}\cap\mathscr{T}_x$ is a subring or submodule of $\mathscr{T}_x$.

Proof The implication (i) $\Longrightarrow$ (ii) follows from Proposition 7.1.33. For the converse implication, we need only show that the inclusion of $\mathscr{S}$ into $\mathscr{T}$ is continuous. Let $[s]_x\in\mathscr{S}$ and let $\mathcal{O}$ be a neighbourhood of $[s]_x$ in $\mathscr{T}$. Let $\mathcal{U}$ be a neighbourhood of x such that $\mathcal{B}(\mathcal{U},s)$ is contained in $\mathcal{O}$. Since $\mathcal{B}(\mathcal{U},s)$ is a neighbourhood of $[s]_x$ in $\mathscr{S}$ the continuity of the inclusion follows. ■

7.1.8 Kernel, image, etc., of morphisms

One can expect that it is possible to assign the usual algebraic constructions of kernels, images, quotients, etc., to morphisms of presheaves and étalé spaces. The story turns out to have some hidden dangers that one must carefully account for. In this section we work with sheaves of modules over a prescribed sheaf of rings.

7.1.39 Definition (Kernel, image, quotient, cokernel, coimage presheaves) Let $(\mathcal{S},\mathscr{O})$ be a topological space, let $\mathscr{R}=(R(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E}=(E(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{F}=(F(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be presheaves of $\mathscr{R}$-modules over $\mathcal{S}$. Let $\Phi=(\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be a presheaf morphism from $\mathscr{E}$ to $\mathscr{F}$.

(i) The ***kernel presheaf*** of Φ is the presheaf of $\mathscr{R}$-modules defined by

$$\ker_{\mathrm{pre}}(\Phi)(\mathcal{U})=\ker(\Phi_{\mathcal{U}}).$$

(ii) The ***image presheaf*** of Φ is the presheaf of $\mathscr{R}$-modules defined by

$$\mathrm{image}_{\mathrm{pre}}(\Phi)(\mathcal{U}) = \mathrm{image}(\Phi_{\mathcal{U}}).$$

(iii) If $\mathscr{E}$ is a subpresheaf of $\mathscr{F}$, the ***quotient presheaf*** of $\mathscr{F}$ by $\mathscr{E}$ is the presheaf of $\mathscr{R}$-modules defined by

$$\mathscr{F}/_{\mathrm{pre}}\mathscr{E}(\mathcal{U}) = \mathscr{F}(\mathcal{U})/\mathscr{E}(\mathcal{U}).$$

(iv) The ***cokernel presheaf*** of Φ is the presheaf of $\mathscr{R}$-modules defined by

$$\mathrm{coker}_{\mathrm{pre}}(\Phi)(\mathcal{U}) = \mathrm{coker}(\Phi_{\mathcal{U}}) = F(\mathcal{U})/\,\mathrm{image}(\Phi_{\mathcal{U}}).$$

(v) The ***coimage presheaf*** of Φ is the presheaf of $\mathscr{R}$-modules defined by

$$\mathrm{coimage}_{\mathrm{pre}}(\Phi)(\mathcal{U}) = \mathrm{coimage}(\Phi_{\mathcal{U}}) = E(\mathcal{U})/\ker(\Phi_{\mathcal{U}}).$$

In all cases, the restriction maps are the obvious ones, induced by the restriction maps $r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}}$ and $r^{\mathscr{F}}_{\mathcal{U},\mathcal{V}}$ for $\mathscr{E}$ and $\mathscr{F}$, respectively. Thus, for example, the restriction map for $\ker(\Phi)$ is

$$\ker_{\mathrm{pre}}(\Phi)(\mathcal{U}) \ni s \mapsto r^{\mathscr{E}}_{\mathcal{U},\mathcal{V}}(s) \in \ker_{\mathrm{pre}}(\Phi)(\mathcal{V}),$$

the restriction map for $\mathrm{image}_{\mathrm{pre}}(\Phi)$ is

$$\mathrm{image}_{\mathrm{pre}}(\Phi)(\mathcal{U}) \ni t \mapsto r^{\mathscr{F}}_{\mathcal{U},\mathcal{V}}(t) \in \mathrm{image}_{\mathrm{pre}}(\Phi)(\mathcal{V}),$$

and the restriction map for $\mathscr{F}/_{\mathrm{pre}}\mathscr{E}$ is

$$\mathscr{F}/_{\mathrm{pre}}\mathscr{E}(\mathcal{U}) \ni s + E(\mathcal{U}) \mapsto r^{\mathscr{F}}_{\mathcal{U},\mathcal{V}}(s) + E(\mathcal{V}) \in \mathscr{F}/_{\mathrm{pre}}\mathscr{E}(\mathcal{V}). \qquad \bullet$$

Using the properties of presheaf morphisms and subpresheaves, one readily verifies that the given definitions of the restrictions maps make sense.

Let us first see that the stalks of the presheaves just defined are what one expects.

7.1.40 Proposition (Stalks of algebraic constructions are algebraic constructions of stalks) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a presheaf of rings over $\mathcal{S}$, let $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be presheaves of $\mathscr{R}$-modules over $\mathcal{S}$, and let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be a presheaf morphism from $\mathscr{E}$ to $\mathscr{F}$. Then the following statements hold:*

(i) $\ker_{\mathrm{pre}}(\Phi)_{\mathrm{x}} = \ker(\mathrm{Et}(\Phi)_{\mathrm{x}})$ *for every* $\mathrm{x} \in \mathcal{S}$*;*

(ii) $\mathrm{image}_{\mathrm{pre}}(\Phi)_{\mathrm{x}} = \mathrm{image}(\mathrm{Et}(\Phi)_{\mathrm{x}})$ *for every* $\mathrm{x} \in \mathcal{S}$*;*

(iii) if $\mathscr{E}$ is a subpresheaf of $\mathscr{F}$, then $\mathrm{Et}(\mathscr{F}/_{\mathrm{pre}}\mathscr{E})_{\mathrm{x}} = \mathrm{Et}(\mathscr{F})_{\mathrm{x}}/\mathrm{Et}(\mathscr{E})_{\mathrm{x}}$ *for every* $\mathrm{x} \in \mathcal{S}$*;*

(iv) $\mathrm{coker}_{\mathrm{pre}}(\Phi)_{\mathrm{x}} = \mathrm{coker}(\mathrm{Et}(\Phi)_{\mathrm{x}})$ *for every* $\mathrm{x} \in \mathcal{S}$*;*

(v) $\mathrm{coimage}_{\mathrm{pre}}(\Phi)_{\mathrm{x}} = \mathrm{coimage}(\mathrm{Et}(\Phi)_{\mathrm{x}})$ *for every* $\mathrm{x} \in \mathcal{S}$.

Proof (i) Note that $\alpha \in \ker_{\text{pre}}(\Phi)_x$ if and only if there exists a neighbourhood $\mathcal{U}$ of x and $s \in \ker(\Phi_{\mathcal{U}})$ such that $\alpha = r_{\mathcal{U},x}(s)$. Since $\text{Et}(\Phi)_x(\alpha) = r_{\mathcal{U},x}(\Phi_{\mathcal{U}}(s))$ we conclude that $\alpha \in \ker_{\text{pre}}(\Phi)_x$ if and only if $\text{Et}(\Phi)_x(\alpha) = 0$.

(ii) Note that $\beta \in \text{image}_{\text{pre}}(\Phi)_x$ if and only if there exists a neighbourhood $\mathcal{U}$ of x and $s \in E(\mathcal{U})$ such that $\beta = r_{\mathcal{U},x}(\Phi_{\mathcal{U}}(s))$. Let $\alpha = r_{\mathcal{U},x}(s)$. Since $\text{Et}(\Phi)_x(\alpha) = r_{\mathcal{U},x}(\Phi_{\mathcal{U}}(s))$ we conclude that $\beta \in \text{image}_{\text{pre}}(\Phi)_x$ if and only if $\beta \in \text{image}(\text{Et}(Phi)_x)$.

(iii) We have $\beta \in \text{Et}(\mathscr{F}/_{\text{pre}}\mathscr{E})_x$ if and only if there exists a neighbourhood $\mathcal{U}$ of x and $t \in F(\mathcal{U})$ such that $\beta = r_{\mathcal{U},x}(t + E(\mathcal{U}))$. Since the restriction maps are group homomorphisms, one directly verifies that

$$r_{\mathcal{U},x}(t + E(\mathcal{U})) = r_{\mathcal{U},x}(t) + r_{\mathcal{U},x}(E(\mathcal{U}))$$

and since $r_{\mathcal{U},x}(E(\mathcal{U})) = \text{Et}(\mathscr{E})_x$ (again, this is directly verified), this part of the result follows.

(iv) and (v) follow from the first three assertions. ■

As we are about to see, not all parts of the preceding definition are on an equal footing. In fact, what we shall see is that the kernel presheaf is pretty nicely behaved, while the other constructions need more care if one is to give them the interpretations one normally gives to these sorts of algebraic constructions. Here is one good property of the kernel.

7.1.41 Proposition (The kernel presheaf is often a sheaf) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a presheaf of rings over $\mathcal{S}$, let $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be sheaves of $\mathscr{R}$-modules over $\mathcal{S}$, and let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be a presheaf morphism from $\mathscr{E}$ to $\mathscr{F}$. Then* $\ker_{\text{pre}}(\Phi)$ *is a sheaf.*

Proof Let $\mathcal{U} \in \mathscr{O}$, let $(\mathcal{U}_a)_{a\in A}$ be an open cover for $\mathcal{U}$, let $s, t \in \ker_{\text{pre}}(\Phi)(\mathcal{U})$, and suppose that $r^{\mathscr{E}}_{\mathcal{U},\mathcal{U}_a}(s) = r^{\mathscr{E}}_{\mathcal{U},\mathcal{U}_a}(t)$ for every $a \in A$. Since $\mathscr{E}$ is a sheaf, $s = t$, and so $\ker_{\text{pre}}(\Phi)$ is separated.

Next let $\mathcal{U} \in \mathscr{O}$, let $(\mathcal{U}_a)_{a\in A}$ be an open cover for $\mathcal{U}$, let $s_a \in \ker_{\text{pre}}(\Phi)(\mathcal{U}_a)$, $a \in A$, satisfy $r^{\mathscr{E}}_{\mathcal{U}_{a_1},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{a_1}) = r^{\mathscr{E}}_{\mathcal{U}_{a_2},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_1}}(s_{a_2})$ for every $a \in A$. Since $\mathscr{E}$ is a sheaf, there exists $s \in E(\mathcal{U})$ such that $r^{\mathscr{E}}_{\mathcal{U},\mathcal{U}_a}(s) = s_a$ for each $a \in A$. Moreover,

$$r^{\mathscr{F}}_{\mathcal{U},\mathcal{U}_a}(\Phi_{\mathcal{U}}(s)) = \Phi_{\mathcal{U}_a}(s_a) = 0,$$

and since $\mathscr{F}$ is separated we have $\Phi_{\mathcal{U}}(s) = 0$ and so $s \in \ker_{\text{pre}}(\Phi)(\mathcal{U})$, as desired. ■

By example, let us illustrate that the image presheaf is not generally a sheaf, even when the domain and range are sheaves.

7.1.42 Example (The image of a presheaf morphism may not be a sheaf) Let $\mathcal{S} = \mathbb{S}^1$, let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\mathscr{E} = \mathscr{C}^r_{\mathbb{S}^1}$ be the sheaf of functions of class C^r on $\mathbb{S}^1$, and let $\mathscr{F}$ be the presheaf of nowhere zero $\mathbb{C}$-valued functions of class C^r on $\mathbb{S}^1$. We consider both $\mathscr{E}$ and $\mathscr{F}$ to be presheaves of Abelian groups, with the group structure being addition in the former case and multiplication in the latter case. One may verify easily that $\mathscr{F}$ is also a sheaf. Let us consider the presheaf morphism exp from $\mathscr{E}$ to $\mathscr{F}$ defined by asking that

$$\exp_{\mathcal{U}}(f)(x, y) = \mathrm{e}^{2\pi \mathrm{i} f(x,y)}, \qquad (x, y) \in \mathcal{U}.$$

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be the open subsets covering $\mathbb{S}^1$ defined by

$$\mathcal{U}_1 = \{(x, y) \in \mathbb{S}^1 \mid y < \tfrac{1}{\sqrt{2}}\}, \qquad \mathcal{U}_2 = \{(x, y) \in \mathbb{S}^1 \mid y > -\tfrac{1}{\sqrt{2}}\}.$$

Let $f_1 \in \mathrm{C}^r(\mathcal{U}_1)$ be defined by asking that $f_1(x, y)$ be the angle of the point (x, y) from the positive x-axis; thus $f_1(x, y) \in (-\frac{5\pi}{4}, \frac{\pi}{4})$. In like manner, let $f_2 \in \mathrm{C}^r(\mathcal{U}_2)$ be the function defined by asking that $f_2(x, y)$ be the angle of the point (x, y) measured from the positive x-axis; thus $f_2(x, y) \in (-\frac{\pi}{4}, \frac{5\pi}{4})$. Note that $\exp_{\mathcal{U}_1}(f_1)$ and $\exp_{\mathcal{U}_2}(f_2)$ agree on $\mathcal{U}_1 \cap \mathcal{U}_2$. However, there exists no $f \in \mathrm{C}^r(\mathbb{S}^1)$ such that $\exp_{\mathbb{S}^1}(f)$ agrees with $\exp_{\mathcal{U}_1}(f_1)$ on $\mathcal{U}_1$ and with $\exp_{\mathcal{U}_2}(f_2)$ on $\mathcal{U}_2$. Thus $\mathrm{image}_{\mathrm{pre}}(\exp)$ is not a sheaf. •

The example shows that, in order to achieve a useful theory, we need to modify our definitions to make sure we are dealing with objects where the stalks capture the behaviour of the presheaf. The following definition illustrates how to do this.

7.1.43 Definition (Kernel, image, quotient, cokernel, coimage for presheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a presheaf of rings over $\mathcal{S}$, let $\mathscr{E} = (E(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be presheaves of $\mathscr{R}$-modules over $\mathcal{S}$, and let $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be a presheaf morphism from $\mathscr{E}$ to $\mathscr{F}$.

(i) The ***kernel*** of Φ is the sheaf $\ker(\Phi)$ of sections of $\mathrm{Et}(\ker_{\mathrm{pre}}(\Phi))$.

(ii) The ***image*** of Φ is the sheaf $\mathrm{image}(\Phi)$ of sections of the sheafification of $\mathrm{image}_{\mathrm{pre}}(\Phi)$.

(iii) If $\mathscr{E}$ is a subpresheaf of $\mathscr{F}$, the ***quotient*** of $\mathscr{F}$ by $\mathscr{E}$ is the sheaf $\mathscr{F}/\mathscr{E}$ of sections of the sheafification of $\mathscr{F}/_{\mathrm{pre}}\mathscr{E}$.

(iv) The ***cokernel*** of Φ is the sheaf $\mathrm{coker}(\Phi)$ of sections of the sheafification of $\mathrm{coker}_{\mathrm{pre}}(\Phi)$.

(v) The ***coimage*** of Φ is the sheaf $\mathrm{coimage}(\Phi)$ of sections of the sheafification of $\mathrm{coimage}_{\mathrm{pre}}(\Phi)$. •

Note that $\ker(\Phi)$ and $\ker_{\mathrm{pre}}(\Phi)$ are in natural correspondence by Propositions 7.1.41 and 7.1.27. We think of $\ker(\Phi)$ as the presheaf of sections of the étalé space of $\ker_{\mathrm{pre}}(\Phi)$ in order to be consistent with the other algebraic constructions. While these algebraic constructions involve a distracting use of sheafification, it is important to note that, at the stalk level, the constructions have the hoped for properties.

7.1.44 Proposition (Agreement of stalks of algebraic constructions) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ is a presheaf of rings over $\mathcal{S}$, if $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ are presheaves of $\mathscr{R}$-modules over $\mathcal{S}$, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a presheaf morphism from $\mathscr{E}$ to $\mathscr{F}$, then the following statements hold:*

(i) $\ker_{\mathrm{pre}}(\Phi)_{\mathrm{x}} \simeq \ker(\Phi)_{\mathrm{x}}$;

(ii) $\mathrm{image}_{\mathrm{pre}}(\Phi)_{\mathrm{x}} \simeq \mathrm{image}(\Phi)_{\mathrm{x}}$;

(iii) if $\mathscr{E}$ is a subpresheaf of $\mathscr{F}$, then $(\mathscr{F}/_{\mathrm{pre}}\mathscr{E})_{\mathrm{x}} \simeq (\mathscr{F}/\mathscr{E})_{\mathrm{x}}$;

(iv) $\mathrm{coker}_{\mathrm{pre}}(\Phi)_{\mathrm{x}} \simeq \mathrm{coker}(\Phi)_{\mathrm{x}}$;

(v) $\mathrm{coimage}_{\mathrm{pre}}(\Phi)_x \simeq \mathrm{coimage}(\Phi)_x$.

(In all cases, "$\simeq$" stands for the isomorphism from a presheaf to its sheafification from part (iii) of Proposition 7.1.29.)

Proof All of these assertions follow from Proposition 7.1.29(iii) and Proposition 7.1.40. ■

While the presheaf image $\mathrm{image}_{\mathrm{pre}}(\Phi)$ of a morphism of sheaves $\mathscr{E}$ and $\mathscr{F}$ is not necessarily a sheaf, we can nonetheless naturally identity the image with a subpresheaf of $\mathscr{F}$.

7.1.45 Proposition (The image presheaf is a subpresheaf of the codomain) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ is a presheaf of rings over $\mathcal{S}$, if $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ are sheaves of $\mathscr{R}$-modules over $\mathcal{S}$, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a presheaf morphism from $\mathscr{E}$ to $\mathscr{F}$, then there exists a natural injective presheaf morphism from* $\mathrm{image}(\Phi)$ *into $\mathscr{F}$.*

Proof By Proposition 7.1.30, since we have an inclusion $i_\Phi = (i_\Phi(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ of $\mathrm{image}_{\mathrm{pre}}(\Phi)$ in $\mathscr{F}$, we have a natural induced morphism $i^+_\Phi = (i^+_{\Phi,\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ of presheaves from $\mathrm{image}(\Phi)$ into $\mathscr{F}$. We need only show that this induced morphism is injective. To do this, we recall the notation from the proof of Proposition 7.1.30. Thus we have $i^+_{\Phi,\mathcal{U}} = \beta^{-1}_{\mathcal{U}} \circ i'^+_{\Phi,\mathcal{U}}$, where $\beta_{\mathcal{U}}$ is as in Proposition 7.1.27 (for the presheaf $\mathscr{F}$) and where

$$i'^+_{\Phi,\mathcal{U}}([s]_x) = [i_{\Phi,\mathcal{U}}(s)]_x.$$

Since $\mathscr{F}$ is a sheaf, $\beta_{\mathcal{U}}$ is an isomorphism, and so is injective. So we need only show that $i'^+_{\Phi,\mathcal{U}}$ is injective. Suppose that $[i_{\Phi,\mathcal{U}}(s)]_x = 0$. Thus there exists a neighbourhood $\mathcal{V}$ of x such that

$$r_{\mathcal{U},\mathcal{V}}(i_{\Phi,\mathcal{U}}(s)) = i_{\Phi,\mathcal{V}}(r_{\mathcal{U},\mathcal{V}}(s)) = 0,$$

using the commuting diagram (7.2). Injectivity of $i_{\Phi,\mathcal{V}}$ gives $r_{\mathcal{U},\mathcal{V}}(s) = 0$ and so $[s]_x = 0$, which gives the desired injectivity of $i'^+_{\Phi,\mathcal{U}}$. ■

We now turn our attention to algebraic constructions associated to étalé morphisms of étalé spaces of modules.

7.1.46 Definition (Kernel, image, quotient, cokernel, coimage for étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be an étalé space of rings over $\mathcal{S}$, let $\pi\colon \mathscr{S} \to \mathcal{S}$ and $\rho\colon \mathscr{T} \to \mathcal{S}$ be étalé spaces of $\mathscr{R}$-modules over $\mathcal{S}$, and let $\Phi\colon \mathscr{S} \to \mathscr{T}$ be an étalé morphism.

(i) The ***kernel*** of Φ is the étalé subspace $\ker(\Phi)$ of $\mathscr{S}$ given by $\ker(\Phi)_x = \ker(\Phi|\mathscr{S}_x)$.

(ii) The ***image*** of Φ is the étalé subspace $\mathrm{image}(\Phi)$ of $\mathscr{T}$ given by $\mathrm{image}(\Phi)_x = \mathrm{image}(\Phi|\mathscr{S}_x)$.

(iii) If $\mathscr{S}$ is a étalé subspace of $\mathscr{T}$, the ***quotient*** of $\mathscr{T}$ by $\mathscr{S}$ is the étalé space $\mathscr{T}/\mathscr{S}$ over $\mathcal{S}$ given by $(\mathscr{T}/\mathscr{S})_x = \mathscr{T}_x/\mathscr{S}_x$, with the quotient topology induced by the projection from $\mathscr{T}$ to $\mathscr{T}/\mathscr{S}$.

(iv) The ***cokernel*** of Φ is the étalé space $\mathrm{coker}(\Phi) = \mathscr{T}/\mathrm{image}(\Phi)$.

(v) The ***coimage*** of Φ is the étalé space $\mathrm{coimage}(\Phi) = \mathscr{S}/\ker(\Phi)$. •

Let us verify that the above étalé spaces are indeed étalé spaces.

7.1.47 Proposition (Kernels, images, and quotients of étalé spaces are étalé spaces) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R}$ is an étalé space of rings over $\mathcal{S}$, if $\pi\colon \mathscr{S} \to \mathcal{S}$ and $\rho\colon \mathscr{T} \to \mathcal{S}$ be étalé spaces of $\mathscr{R}$-modules over $\mathcal{S}$, and if $\Phi\colon \mathscr{S} \to \mathscr{T}$ is an étalé morphism, then the following statements hold:*

(i) $\ker(\Phi)$ *is an étalé subspace of* $\mathscr{S}$*;*

(ii) $\operatorname{image}(\Phi)$ *is an étalé subspace of* $\mathscr{T}$*;*

(iii) if $\mathscr{S}$ *is a étalé subspace of* $\mathscr{T}$*, then* $\mathscr{T}/\mathscr{S}$ *is an étalé space;*

(iv) $\operatorname{coker}(\Phi)$ *is an étalé space;*

(v) $\operatorname{coimage}(\Phi)$ *is an étalé space.*

Proof (i) Let $\zeta\colon \mathcal{S} \to \mathscr{S}$ be the zero section. Thus $\zeta(x)$ is the zero element in $\mathscr{S}_x$. We claim that ζ is continuous. Let $\mathcal{O}$ be a neighbourhood of $\zeta(x)$. Since the group operation is continuous and since $\zeta(x) + \zeta(x) = \zeta(x)$, there exist neighbourhoods $\mathcal{O}_1$ and $\mathcal{O}_2$ of $\zeta(x)$ such that

$$\{\alpha + \beta \mid (\alpha, \beta) \in \mathcal{O}_1 \times \mathcal{O}_2 \cap \mathscr{S} \times_{\mathcal{S}} \mathscr{S}\} \subseteq \mathcal{O}.$$

Let $\mathcal{P} = \mathcal{O} \cap \mathcal{O}_2 \cap \mathcal{O}_2$, noting that $\mathcal{P}$ is a neighbourhood of $\zeta(x)$. By shrinking $\mathcal{O}_1$ and $\mathcal{O}_2$ if necessary, we may suppose that $\pi|\mathcal{P}$ is a homeomorphism onto $\pi(\mathcal{P})$. Let $\alpha \in \mathcal{P}$ and let $y = \pi(\alpha)$. Note that $\pi(\alpha + \alpha) = \pi(\alpha) = y$, and since $\pi|\mathcal{P}$ is a homeomorphism we have $\alpha + \alpha = \alpha$, giving $\alpha = \zeta(y)$. Thus $\mathcal{P} = \zeta(\pi(\mathcal{P}))$, showing that $\zeta(\mathcal{P}) \subseteq \mathcal{O}$, giving the desired continuity of ζ. Since sections are local homeomorphisms (they are locally inverses of the étalé projection), it follows that $\operatorname{image}(\zeta)$ is open. Since Φ is continuous, $\ker(\Phi) = \Phi^{-1}(\operatorname{image}(\zeta))$ is open and by Proposition 7.1.38 it follows that $\ker(\Phi)$ is a étalé subspace.

(ii) This follows from Propositions 7.1.33 and 7.1.38.

(iii) Let us denote by $\pi_{\mathscr{S}}\colon \mathscr{T} \to \mathscr{T}/\mathscr{S}$ the mapping which, when restricted to fibres, is the canonical projection and let us denote by $\rho_{\mathscr{S}}\colon \mathscr{T}/\mathscr{S} \to \mathcal{S}$ the canonical projection. We must show that $\rho_{\mathscr{S}}$ is a local homeomorphism. Since $\rho_{\mathscr{S}} = \rho \circ \pi_{\mathscr{S}}$ and since compositions of local homeomorphisms are local homeomorphisms (this is directly verified), it suffices to show that $\pi_{\mathscr{S}}$ is a local homeomorphism. Clearly $\pi_{\mathscr{S}}$ is continuous by the definition of the quotient topology. We claim that $\pi_{\mathscr{S}}$ is also open. Let $\mathcal{B}(\mathcal{U}, \tau)$ be a basic neighbourhood in $\mathscr{T}$. Note that

$$\pi_{\mathscr{S}}(\mathcal{B}(\mathcal{U}, \tau)) = \mathcal{B}(\mathcal{U}, \tau + \mathscr{S}),$$

where $\tau + \mathscr{S}$ means the section (not necessarily continuous, since we are still trying to understand this) of $\mathscr{T}/\mathscr{S}$ over $\mathcal{U}$ given by $(\tau + \mathscr{S})(x) = \tau(x) + \mathscr{S}_x$. Thus a typical point in $\pi_{\mathscr{S}}^{-1}(\pi_{\mathscr{S}}(\mathcal{B}(\mathcal{U}, \tau)))$ has the form $\tau(x) + \sigma(x)$ for $x \in \mathcal{U}$ and where σ is a section of $\mathscr{S}$ defined on some neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x. Thus $\mathcal{B}(\mathcal{V}, \tau|\mathcal{V} + \sigma)$ is a basic neighbourhood of $\tau(x) + \sigma(x)$ in $\pi_{\mathscr{S}}^{-1}(\pi_{\mathscr{S}}(\mathcal{B}(\mathcal{U}, \tau)))$ showing that the latter set is open, and hence $\pi_{\mathscr{S}}(\mathcal{B}(\mathcal{U}, \tau))$ is open in the quotient topology. This shows that basic open sets in $\mathscr{T}$ are mapped to open sets in $\mathscr{T}/\mathscr{S}$, showing that $\pi_{\mathscr{S}}$ is open, as claimed. To complete this part of the proof it suffices to show that $\pi_{\mathscr{S}}|\mathcal{B}(\mathcal{U}, \tau)$ is a bijection. For injectivity, suppose that $\tau(x) + \mathscr{S}_x = \tau(y) + \mathscr{S}_y$ for $x, y \in \mathcal{U}$. Clearly this implies that $x = y$, giving injectivity. Surjectivity is equally clear.

Parts (iv) and (v) follow from the first three parts. ■

7.1.9 Exact sequences of presheaves and étalé spaces

We are interested in looking at exact sequences of presheaves and étalé spaces. Let us give the definitions so that we first know what we are talking about.

7.1.48 Definition (Exact sequence of presheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a presheaf of rings over $\mathcal{S}$, let $J \subseteq \mathbb{Z}$ be of one of the following forms:

$$J = \{0, 1, \dots, n\}, \quad J = \mathbb{Z}_{\geq 0}, \quad J = \mathbb{Z},$$

let $\mathscr{E}_j = (E_j(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$, $j \in J$, be a family of presheaves of $\mathscr{R}$-modules, and let $\Phi_j = (\Phi_{j,\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ be a presheaf morphism from $\mathscr{E}_j$ to $\mathscr{E}_{j+1}$, whenever $j, j+1 \in J$. If $j_0 \in J$ is such that $j_0 - 1, j_0, j_0 + 1 \in J$ then the sequence

$$\cdots \longrightarrow \mathscr{E}_{j_0-1} \xrightarrow{\Phi_{j_0-1}} \mathscr{E}_{j_0} \xrightarrow{\Phi_{j_0}} \mathscr{E}_{j_0+1} \xrightarrow{\Phi_{j_0+1}} \cdots$$

is ***exact*** at j_0 if $\ker(\Phi_{j_0,\mathcal{U}}) = \operatorname{image}(\Phi_{j_0-1,\mathcal{U}})$ for every $\mathcal{U} \in \mathscr{O}$. •

7.1.49 Definition (Exact sequence of étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be an étalé space of rings over $\mathcal{S}$, let $J \subseteq \mathbb{Z}$ be of the form

$$J = \{0, 1, \dots, n\}, \quad J = \mathbb{Z}_{\geq 0}, \quad J = \mathbb{Z},$$

let $\mathscr{S}_j$, $j \in J$, be a family of étalé spaces of $\mathscr{R}$-modules, and let $\Phi_j \colon \mathscr{S}_j \to \mathscr{S}_{j+1}$ be an étalé morphism, whenever $j, j+1 \in J$. If $j_0 \in J$ is such that $j_0 - 1, j_0, j_0 + 1 \in J$ then the sequence

$$\cdots \longrightarrow \mathscr{S}_{j_0-1} \xrightarrow{\Phi_{j_0-1}} \mathscr{S}_{j_0} \xrightarrow{\Phi_{j_0}} \mathscr{S}_{j_0+1} \xrightarrow{\Phi_{j_0+1}} \cdots$$

is ***exact*** at j_0 if $\ker(\Phi_{j_0}) = \operatorname{image}(\Phi_{j_0-1})$. •

In order to investigate the relationships between the presheaf kernel and the kernel étalé space, let us state the following result which essentially says that, for sheaves, the two notions are equivalent.

7.1.50 Proposition (Characterisations of the kernel presheaf) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ is a presheaf of rings over $\mathcal{S}$, if $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ are presheaves of $\mathscr{R}$-modules, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a presheaf morphism from $\mathscr{E}$ to $\mathscr{F}$, then the following statements are equivalent:*

(i) $\ker_{\mathrm{pre}}(\Phi)(\mathcal{U})$ *is the zero section of* $\mathrm{E}(\mathcal{U})$ *for each* $\mathcal{U} \in \mathscr{O}$;

(ii) $\Phi_{\mathcal{U}}$ *is injective for each* $\mathcal{U} \in \mathscr{O}$.

Furthermore, the preceding conditions imply that

(iii) $\mathrm{Et}(\Phi)_{\mathrm{x}}$ *is injective for every* $\mathrm{x} \in \mathcal{S}$,

and this last condition implies the first two if $\mathscr{E}$ is separated.

Proof The equivalence of (i) and (ii) is an immediate consequence of the usual statement that a morphism of modules is injective if and only if it has trivial kernel.

(ii) $\Longrightarrow$ (iii) Let $\alpha \in \mathrm{Et}(\mathscr{E})_x$ and suppose that $\mathrm{Et}(\Phi)_x(\alpha) = 0$. Suppose that $\alpha = r_{\mathcal{U},x}(s)$ for some neighbourhood $\mathcal{U}$ of x. It follows from Lemma 7.1.20 that there exists a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x such that $r_{\mathcal{U},\mathcal{V}}(\Phi_{\mathcal{U}}(s)) = 0$. Using the commuting of the diagram (7.2) and the hypothesis that $\Phi_{\mathcal{V}}$ is injective we conclude that $r_{\mathcal{U},\mathcal{V}}(s) = 0$, giving $\alpha = 0$.

(iii) $\Longrightarrow$ (ii) Here we need to make the additional assumption that $\mathscr{E}$ is separated. Suppose that $s \in E(\mathcal{U})$ is such that $\Phi_{\mathcal{U}}(s)$ is the zero section of $F(\mathcal{U})$. Thus

$$\mathrm{Et}(\Phi)_x(r_{\mathcal{U},x}(s)) = r_{\mathcal{U},x}(\Phi_{\mathcal{U}}(s)) = 0$$

for every $x \in \mathcal{U}$ and so by hypothesis we have $r_{\mathcal{U},x}(s) = 0$ for every $x \in \mathcal{U}$. By Lemma 7.1.20, for each $x \in \mathcal{U}$ there exists a neighbourhood $\mathcal{U}_x \subseteq \mathcal{U}$ of x such that $r_{\mathcal{U},\mathcal{U}_x}(s) = 0$, and an application of the fact that $\mathscr{E}$ is separated gives $s = 0$. ■

The same sort of thing can be carried out for cokernels, but with one important difference.

7.1.51 Proposition (Characterisations of the cokernel étalé space) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ is a presheaf of rings over $\mathcal{S}$, if $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ are presheaves of $\mathscr{R}$-modules, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a presheaf morphism from $\mathscr{E}$ to $\mathscr{F}$, then the following statements are equivalent:*

(i) $\mathrm{coker}_{\mathrm{pre}}(\Phi)(\mathcal{U})$ *is the zero section of* $\mathrm{F}(\mathcal{U})$ *for each* $\mathcal{U} \in \mathscr{O}$*;*

(ii) $\Phi_{\mathcal{U}}$ *is surjective for each* $\mathcal{U} \in \mathscr{O}$*.*

Furthermore, the preceding conditions imply that

(iii) $\mathrm{Et}(\Phi)_{\mathrm{x}}$ *is surjective for every* $\mathrm{x} \in \mathcal{S}$*.*

Proof The equivalence of (i) and (ii) follows from the usual assertion that a morphism of modules is an epimorphism if and only if its cokernel is trivial. We shall prove that (ii) implies (iii). Let $\beta \in \mathrm{Et}(\mathscr{F})_x$ and write $\beta = r_{\mathcal{U},x}(t)$ for $t \in F(\mathcal{U})$. The hypothesised surjectivity of $\Phi_{\mathcal{U}}$ ensures that $t = \Phi_{\mathcal{U}}(s)$ for some $s \in E(\mathcal{U})$. Thus

$$\beta = r_{\mathcal{U},x}(t) = r_{\mathcal{U},x}(\Phi_{\mathcal{U}}(s)) = \mathrm{Et}(\Phi)_x(r_{\mathcal{U},x}(s)),$$

which gives the result. ■

The important distinction to make here, compared to the corresponding result for kernels, is that the third assertion is not equivalent to the first two, even when $\mathscr{E}$ and $\mathscr{F}$ are sheaves. Let us give an example to illustrate this.

7.1.52 Example (Surjectivity on stalks does not imply surjectivity) Let $r\{\infty, \omega\}$. We shall work with the manifold $\mathbb{S}^1$. Note that we have a canonical one-form, which we denote by $\mathrm{d}\theta$, on $\mathbb{S}^1$ arising from the trivialisation $\mathsf{T}^*\mathbb{S}^1 \simeq \mathbb{S}^1 \times \mathbb{R}$. Moreover, any C^r-one-form α on an open subset $\mathcal{U} \subseteq \mathbb{S}^1$ can be written as $\alpha = g\mathrm{d}\theta|\mathcal{U}$ for some C^r-function g on $\mathcal{U}$, and so we identify C^r-one-forms with C^r-functions. We consider the sheaf $\mathscr{C}^r_{\mathbb{S}^1}$ of functions of class C^r on $\mathbb{S}^1$. For $f \in \mathscr{C}^r_{\mathbb{S}^1}(\mathcal{U})$ let $\mathrm{d}f = f'\mathrm{d}\theta|\mathcal{U}$. We let Φ be the presheaf morphism from $\mathscr{C}^r_{\mathbb{S}^1}$ to $\mathscr{C}^r_{\mathbb{S}^1}$ defined by $\Phi_{\mathcal{U}}(f) = f'$ for $f \in \mathrm{C}^r(\mathcal{U})$. (Here we are thinking of $\mathscr{C}^r_{\mathbb{S}^1}$ as being a sheaf of Abelian groups with the group operation of addition.) We claim that the induced map on stalks is surjective. Indeed, if $(x, y) \in \mathbb{S}^1$, if $\mathcal{U}$ is a connected and simply connected neighbourhood of (x, y) in $\mathbb{S}^1$, and if $g \in \mathrm{C}^r(\mathcal{U})$, we can define $f \in \mathrm{C}^r(\mathcal{U})$ such that $\mathrm{d}f = g$ by taking f to be the indefinite integral of g, with the variable of integration being the usual angle variable. Since the germ $\mathrm{Et}(\mathscr{C}^r_{\mathbb{S}^1})_{(x,y)}$ is determined by the value of functions on connected and simply connected neighbourhoods of (x, y), it follows that $\mathrm{Et}(\Phi)_{(x,y)}$ is surjective. However, $\Phi_{\mathbb{S}^1}$ is not surjective since, for example, $\mathrm{d}\theta \notin \mathrm{image}(\Phi_{\mathbb{S}^1})$. •

The preceding example notwithstanding, it is true that surjectivity on stalks, combined with injectivity on stalks, does imply surjectivity globally.

7.1.53 Proposition (Correspondence of isomorphisms and stalk-wise isomorphisms) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ is a presheaf of rings over $\mathcal{S}$, if $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ are sheaves of $\mathscr{R}$-modules, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a presheaf morphism from $\mathscr{E}$ to $\mathscr{F}$, then the following statements are equivalent:*

(i) $\Phi_{\mathcal{U}} : \mathrm{E}(\mathcal{U}) \to \mathrm{F}(\mathcal{U})$ *is an isomorphism for every* $\mathcal{U} \in \mathscr{O}$*;*

(ii) $\mathrm{Et}(\Phi)_{\mathrm{x}} : \mathrm{Et}(\mathscr{E})_{\mathrm{x}} \to \mathrm{Et}(\mathscr{F})_{\mathrm{x}}$ *is an isomorphism for every* $\mathrm{x} \in \mathcal{S}$*.*

Proof That (i) implies (ii) follows from Propositions 7.1.50 and 7.1.51. It follows from Proposition 7.1.50 that injectivity of $\mathrm{Et}(\Phi)_x$ for each $x \in \mathcal{S}$ implies injectivity of $\Phi_{\mathcal{U}}$ for every $\mathcal{U} \in \mathscr{O}$. So suppose that $\mathrm{Et}(\Phi)_x$ is bijective for every $x \in \mathcal{S}$. Let $\mathcal{U} \in \mathscr{O}$ and let $t \in F(\mathcal{U})$. For $x \in \mathcal{U}$ let $\alpha \in \mathrm{Et}(\mathscr{E})_x$ be such that $\mathrm{Et}(\Phi)_x(\alpha) = r_{\mathcal{U},x}(t)$. Let $\alpha = r_{\mathcal{U},x}(s_x)$ for some $s_x \in E(\mathcal{U})$. By Lemma 7.1.20 let $\mathcal{U}_x \subseteq \mathcal{U}$ be a neighbourhood of x such that $r_{\mathcal{U},\mathcal{U}_x}(t) = r_{\mathcal{U},\mathcal{U}_x}(\Phi_{\mathcal{U}}(s_x))$. Now let $x, y \in \mathcal{U}$ and note that

$$\Phi_{\mathcal{U}_x\cap\mathcal{U}_y}(r_{\mathcal{U}_x,\mathcal{U}_x\cap\mathcal{U}_y}(s_x)) = \Phi_{\mathcal{U}_x\cap\mathcal{U}_y}(r_{\mathcal{U}_y,\mathcal{U}_x\cap\mathcal{U}_y}(s_y)),$$

since both expressions are equal to $r_{\mathcal{U},\mathcal{U}_x\cap\mathcal{U}_y}(t)$. By injectivity of $\Phi_{\mathcal{U}_x\cap\mathcal{U}_y}$ (which follows since we are assuming that $\mathrm{Et}(\Phi)_x$ is injective for every $x \in \mathcal{S}$), it follows that

$$r_{\mathcal{U}_x,\mathcal{U}_x\cap\mathcal{U}_y}(s_x) = r_{\mathcal{U}_y,\mathcal{U}_x\cap\mathcal{U}_y}(s_y).$$

Thus, since $\mathscr{E}$ is a sheaf, there exists $s \in E(\mathcal{U})$ such that $r_{\mathcal{U},\mathcal{U}_x}(s) = s_x$ for every $x \in \mathcal{U}$. Finally, we claim that $\phi(s) = t$. This follows from separability of $\mathscr{F}$ since we have $r_{\mathcal{U},\mathcal{U}_x}(t) = r_{\mathcal{U},\mathcal{U}_x}(\Phi_{\mathcal{U}}(s_x))$ for every $x \in \mathcal{U}$. ■

The preceding three results and example indicate that exactness of sequences of presheaves and étalé spaces will not necessarily correspond. We will be interested mainly in looking at thing at the level of stalks, so let us consider carefully the implications of properties holding at the stalk level.

7.1.54 Proposition (Characterisations of the kernel) *If $(\mathcal{S}, \mathscr{O})$ is a topological space, if $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ is a presheaf of rings over $\mathcal{S}$, if $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ and $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ are sheaves of $\mathscr{R}$-modules, and if $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ is a presheaf morphism from $\mathscr{E}$ to $\mathscr{F}$, then the following statements are equivalent:*

(i) $\mathrm{image}(\mathrm{Et}(\Phi))$ *is the zero section of* $\mathrm{Et}(\mathscr{F})$ *over* $\mathcal{S}$*;*

(ii) $\ker_{\mathrm{pre}}(\Phi)_x = 0$ *for every* $x \in \mathcal{S}$*;*

(iii) $\ker(\Phi)_x = 0$ *for every* $x \in \mathcal{S}$*;*

(iv) $\Phi_{\mathcal{U}}$ *is injective for every* $\mathcal{U} \in \mathscr{O}$*;*

(v) $\mathrm{Et}(\Phi)_x$ *is injective for every* $x \in \mathcal{S}$*;*

(vi) $\mathrm{Et}(\Phi)$ *is injective.*

Proof These equivalences were either already proved, or follow immediately from definitions. ■

The same sort of thing can be carried out for cokernels, but with one important difference.

7.1.55 Proposition (Characterisations of cokernel) *If* $(\mathcal{S}, \mathscr{O})$ *is a topological space, if* $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ *is a presheaf of rings over* $\mathcal{S}$*, if* $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ *and* $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ *are presheaves of* $\mathscr{R}$*-modules, and if* $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\in\mathscr{O}}$ *is a presheaf morphism from* $\mathscr{E}$ *to* $\mathscr{F}$*, then the following statements are equivalent:*

(i) $\mathrm{image}(\mathrm{Et}(\Phi)) = \mathrm{Et}(\mathscr{F})$*;*

(ii) $\mathrm{coker}_{\mathrm{pre}}(\Phi)_x = 0$ *for every* $x \in \mathcal{S}$*;*

(iii) $\mathrm{coker}(\Phi)_x = 0$ *for every* $x \in \mathcal{S}$*;*

(iv) $\mathrm{Et}(\Phi)_x$ *is surjective for every* $x \in \mathcal{S}$*;*

(v) $\mathrm{Et}(\Phi)$ *is surjective.*

Proof As with the preceding result, these equivalences were either already proved, or follow immediately from definitions. ■

Once again, we point out the missing assertion from the statement about cokernels as compared to the statement about kernels.

7.1.56 Punchline The above lengthy sequence of more or less elementary statements is really meant to point out that exact sequences of presheaves are not the same as exact sequences of the corresponding étalé spaces, even when the presheaves are sheaves. This distinction is important, and essentially lies at the heart of some parts of sheaf theory. We shall explore this further in Chapter 8. But, for the moment, let us simply merely say that, when we use the words “exact sequence of sheaves,” we shall always refer to the exact sequence of the étalé spaces, since it is these that are easy to understand, since they can be understood stalk-wise. •

7.1.10 Direct sums, direct products, and tensor products of sheaves

In this section we give a few additional, more or less straightforward, constructions with presheaves and sheaves.

7.1.57 Definition (Direct sums and direct products of presheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E}_a = (E_a(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$, $a \in A$, be a family of presheaves of $\mathscr{R}$-modules over $\mathcal{S}$.

(i) The ***direct product presheaf*** of the presheaves $\mathscr{E}_a$, $a \in A$, is the presheaf $\prod_{a\in A} \mathscr{E}_a$ over $\mathcal{S}$ defined by

$$\Big(\prod_{a\in A} \mathscr{E}_a\Big)(\mathcal{U}) = \prod_{a\in A} E_a(\mathcal{U}) = \Big\{\phi\colon A \to \cup_{a\in A} E_a(\mathcal{U}) \;\Big|\; \phi(a) \in E_a(\mathcal{U}) \text{ for all } a \in A\Big\}.$$

If $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$ the restriction map $r_{\mathcal{U},\mathcal{V}}$ for $\oplus_{a\in A}\mathscr{E}_a$ is defined by $r_{\mathcal{U},\mathcal{V}}(\phi)(a) = r^a_{\mathcal{U},\mathcal{V}}(\phi(a))$, where $r^a_{\mathcal{U},\mathcal{V}}$ is the restriction map for $\mathscr{E}_a$, $a \in A$.

(ii) The ***direct sum presheaf*** of the presheaves $\mathscr{E}_a$, $a \in A$, is the presheaf $\oplus_{a\in A}\mathscr{E}_a$ over $\mathcal{S}$ defined by

$$\begin{aligned}\Big(\oplus_{a\in A}\mathscr{E}_a\Big)(\mathcal{U}) &= \oplus_{a\in A}\, E_a(\mathcal{U}) \\ &= \Big\{\phi \in \prod_{a\in A} E_a(\mathcal{U}) \;\Big|\; \phi(a) = 0 \text{ for all but finitely many } a \in A\Big\}.\end{aligned}$$

The restriction maps are the same as for the direct product. •

7.1.58 Definition (Direct sums and direct products of étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be an étalé space of rings over $\mathcal{S}$, and let $\pi_a\colon \mathscr{S}_a \to \mathcal{S}$, $a \in A$, be a family of étalé spaces of $\mathscr{R}$-modules over $\mathcal{S}$.

(i) The ***direct product*** of the étalé spaces $\mathscr{S}_a$, $a \in A$, is the set $\prod_{a\in A} \mathscr{S}_a$ defined by

$$\prod_{a\in A} \mathscr{S}_a = \Big\{\phi\colon A \to \cup_{a\in A}\mathscr{S}_a \;\Big|\; \phi(a) \in \mathscr{S}_a \text{ for all } a \in A \text{ and } \pi_{a_1}(\sigma(a_1)) = \pi_{a_2}(\sigma(a_2)) \text{ for all } a_1, a_2 \in A\Big\},$$

together with the étalé projection Π defined by $\Pi(\phi) = \pi_a(\phi(a))$ for some (and so for all) $a \in A$.

(ii) The ***direct sum*** of the étalé spaces $\mathscr{S}_a$, $a \in A$, is the subset $\oplus_{a\in A}\mathscr{S}_a$ of $\prod_{a\in A} \mathscr{S}_a$ defined by

$$\oplus_{a\in A}\mathscr{S}_a = \Big\{\phi \in \prod_{a\in A} \mathscr{S}_a \;\Big|\; \phi(a) = \{0\} \text{ for all but finitely many } a \in A\Big\},$$

and with the étalé projection being the restriction of that for the direct product. •

In order for the definition of the direct sum of étalé spaces to be itself an étalé space, we need to assign an appropriate topology to the set. This is more or less easily done. Recall that the product topology on $\prod_{a\in A} \mathscr{S}_a$ is that topology generated by sets of the form $\prod_{a\in A} \mathcal{O}_a$, where the set

$$\{a \in A \mid \mathcal{O}_a \neq \mathscr{S}_a\}$$

is finite. The product topology is the initial topology associated with the family of canonical projections $\mathrm{pr}_a\colon \prod_{a'\in A} \mathscr{S}_{a'} \to \mathscr{S}_a$, i.e., the coarsest topology for which all of what?

these projections is continuous (see below). The topology on $\oplus_{a\in A}\mathscr{S}_a$ is that induced by the product topology on $\prod_{a\in A}\mathscr{S}_a$. One concludes that sections of $\prod_{a\in A}\mathscr{S}_a$ over $\mathcal{U}$ are precisely the maps $\sigma\colon \mathcal{U}\to\prod_{a\in A}\mathscr{S}_a$ such that $\mathrm{pr}_a\circ\sigma$ is a section of $\mathscr{S}_a$ over $\mathcal{U}$ for each $a\in A$. Sections σ of $\oplus_{a\in A}\mathscr{S}_a$ over $\mathcal{U}$ have the property that there exists $a_1,\dots,a_k\in A$ and sections $\sigma_1,\dots,\sigma_k$ of $\mathscr{S}_{a_1},\dots,\mathscr{S}_{a_k}$, respectively, such that

$$\mathrm{pr}_a\circ\sigma(x)=\begin{cases}\sigma_{a_j}(x), & a=a_j\in\{a_1,\dots,a_k\},\\ 0, & a\notin\{a_1,\dots,a_k\}\end{cases}$$

for each $x\in\mathcal{U}$.

As direct sums are of most interest to us, let us record some properties of these.

7.1.59 Proposition (Properties of the direct sum presheaf) *Let $(\mathcal{S},\mathscr{O})$ be a topological space, let $\mathscr{R}=(\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E}_{\mathrm{a}}=(\mathrm{E}_{\mathrm{a}}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$, $\mathrm{a}\in\mathrm{A}$, be a family of presheaves of $\mathscr{R}$-modules over $\mathcal{S}$. Then*

(i) $\oplus_{\mathrm{a}\in\mathrm{A}}\mathscr{E}_{\mathrm{a}}$ is a sheaf if A is finite and if $\mathscr{E}_{\mathrm{a}}$ is a sheaf for each $\mathrm{a}\in\mathrm{A}$ and

(ii) $\oplus_{\mathrm{a}\in\mathrm{A}}\mathrm{Et}(\mathscr{E}_{\mathrm{a}})=\mathrm{Et}(\oplus_{\mathrm{a}\in\mathrm{A}}\mathscr{E}_{\mathrm{a}})$.

Proof (i) Let $\mathcal{U}\in\mathscr{O}$ and let $(\mathcal{U}_b)_{b\in B}$ be an open cover for $\mathcal{U}$. Suppose that $\phi,\phi'\in\oplus_{a\in A}E_a(\mathcal{U})$ satisfy $r_{\mathcal{U},\mathcal{U}_b}(\phi)=r_{\mathcal{U},\mathcal{U}_b}(\phi')$ for each $b\in B$. Then $r^a_{\mathcal{U},\mathcal{U}_b}(\phi(a))=r^a_{\mathcal{U},\mathcal{U}_b}(\phi'(a))$ for each $a\in A$ and $b\in B$. From this we deduce that $\phi(a)=\phi'(a)$ for each $a\in A$, giving separatedness of $\oplus_{a\in A}\mathscr{E}_a$. Next suppose that we have $\phi_b\in\oplus_{a\in A}E_a(\mathcal{U}_b)$ for each $b\in B$ satisfying

$$r_{\mathcal{U}_{b_1},\mathcal{U}_{b_1}\cap\mathcal{U}_{b_2}}(\phi_{b_1})=r_{\mathcal{U}_{b_2},\mathcal{U}_{b_1}\cap\mathcal{U}_{b_2}}(\phi_{b_2})$$

for every $b_1,b_2\in B$. This implies that

$$r^a_{\mathcal{U}_{b_1},\mathcal{U}_{b_1}\cap\mathcal{U}_{b_2}}(\phi_{b_1}(a))=r^a_{\mathcal{U}_{b_2},\mathcal{U}_{b_1}\cap\mathcal{U}_{b_2}}(\phi_{b_2}(a))$$

for every $a\in A$ and $b_1,b_2\in B$. Thus, for each $a\in A$, there exists $\phi_a\in\mathscr{E}_a$ such that

$$r^a_{\mathcal{U},\mathcal{U}_b}(\phi_a)=\phi_b(a)$$

for each $b\in B$. Now define $\phi\colon A\to\cup_{a\in A}\mathscr{E}_a$ by $\phi(a)=\phi_a$, and note that $\phi\in\oplus_{a\in A}\mathscr{E}_a$ since A is finite.

(ii) We need to show that $(\oplus_{a\in A}\mathscr{E}_a)_x=\oplus_{a\in A}\mathscr{E}_{a,x}$ for each $x\in\mathcal{S}$. First let $[\phi]_x\in(\oplus_{a\in A}\mathscr{E}_a)_x$. Then there exists a neighbourhood $\mathcal{U}$ of x and $a_1,\dots,a_k\in A$ such that ϕ is a section over $\mathcal{U}$ and $\phi(a)\neq 0$ if and only if $a\in\{a_1,\dots,a_k\}$. Thus $[\phi]_x$, as a map from A to $\cup_{a\in A}\mathscr{E}_{a,x}$, is given by $[\phi]_x(a)=[\phi(a)]_x$ and so is an element of $\oplus_{a\in A}\mathscr{E}_{a,x}$. Conversely, if $[\phi]_x\in\oplus_{a\in A}\mathscr{E}_{a,x}$ then there exists a neighbourhood $\mathcal{U}$ of x and $a_1,\dots,a_k\in A$ such that ϕ is a section over $\mathcal{U}$ and $\phi(a)\neq 0$ if and only if $a\in\{a_1,\dots,a_k\}$. Thus $[\phi]_x\in(\oplus_{a\in A}\mathscr{E}_a)_x$. ■

Note that in the first statement of the previous result, it is generally necessary that A be finite as the following example shows. Note that this example also gives an example of a presheaf of $\mathscr{C}^r_{\mathrm{M}}$-modules that is not a sheaf.

7.1.60 Example (Infinite direct sums of sheaves are not generally sheaves) Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \text{hol}\}$, let $r' \in \{\infty, \omega, \text{hol}\}$ be as required, and let $\mathbb{F} = \mathbb{R}$ if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \text{hol}$. We take $\mathsf{M} = \mathbb{F}$ and consider the presheaf $\oplus_{k \in \mathbb{Z}_{>0}} \mathscr{C}^r_{\mathbb{F}}$. Let

$$\mathcal{U} = \mathbb{F} \setminus \cup_{j \in \mathbb{Z}_{\geq 0}} \{x \in \mathbb{F} \mid |x| = j\}$$

and let

$$\mathcal{U}_j = \mathsf{D}^1(0, j) \setminus \overline{\mathsf{D}}^1(0, j-1), \qquad j \in \mathbb{Z}_{>0},$$

so that $(\mathcal{U}_j)_{j \in \mathbb{Z}_{>0}}$ is an open cover for $\mathcal{U}$. Define $\phi_j \in \oplus_{k \in \mathbb{Z}_{>0}} \mathrm{C}^r(\mathcal{U}_j)$ by

$$\phi_j(k)(x) = \begin{cases} 1, & k \in \{1, \dots, j\}, \\ 0, & \text{otherwise.} \end{cases}$$

Note, however, that there is no section $\phi \in \oplus_{k \in \mathbb{Z}_{>0}} \mathrm{C}^r(\mathcal{U})$ which restricts to ϕ_j for each $j \in \mathbb{Z}_{>0}$ since any such section ϕ has the property that, for any $k \in \mathbb{Z}_{>0}$, $\phi(k)$ is nonzero, being nonzero restricted to $\mathcal{U}_k$. •

Now we turn to tensor products.

7.1.61 Definition (Tensor products of presheaves) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$ be a presheaf of rings over $\mathcal{S}$, and let $\mathscr{E}_a = (E_a(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$, $a \in \{1, 2\}$, be presheaves of $\mathscr{R}$-modules over $\mathcal{S}$. The ***tensor product presheaf*** of the presheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ is the presheaf $\mathscr{E}_1 \otimes \mathscr{E}_2 = (E_1 \otimes E_2(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$ defined by

$$E_1 \otimes E_2(\mathcal{U}) = E_1(\mathcal{U}) \otimes E_2(\mathcal{U}).$$

If $\mathcal{U}, \mathcal{V} \in \mathscr{O}$ satisfy $\mathcal{V} \subseteq \mathcal{U}$ the restriction map $r_{\mathcal{U},\mathcal{V}}$ for $\mathscr{E}_1 \otimes \mathscr{E}_2$ is defined by

$$r_{\mathcal{U},\mathcal{V}}(\alpha_1 \otimes \alpha_2) = r^1_{\mathcal{U},\mathcal{V}}(\alpha_1) \otimes r^2_{\mathcal{U},\mathcal{V}}(\alpha_2),$$

where $r^a_{\mathcal{U},\mathcal{V}}$ is the restriction map for $\mathscr{E}_a$, $a \in \{1, 2\}$, and where $\alpha_a \in E_a(\mathcal{U})$, $a \in \{1, 2\}$. •

7.1.62 Definition (Tensor products of étalé spaces) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R}$ be an étalé space of rings over $\mathcal{S}$, and let $\pi_a \colon \mathscr{S}_a \to \mathcal{S}$, $a \in \{1, 2\}$, be étalé spaces of $\mathscr{R}$-modules over $\mathcal{S}$. The ***tensor product*** of the étalé spaces $\mathscr{S}_1$ and $\mathscr{S}_2$ is $\mathscr{S}_1 \otimes \mathscr{S}_2 = \mathrm{Et}(\mathrm{Ps}(\mathscr{S}_1) \otimes \mathrm{Ps}(\mathscr{S}_2))$. •

Taking tensor products does not preserve sheaves.

7.1.63 Example (Tensor products of sheaves may not be sheaves) Let $\mathcal{X} = [0, 1] \times \mathbb{Z}$ and define an equivalence relation $\sim_1$ in $\mathcal{X}$ by declaring that $(x_1, k_1) \sim_1 (x_2, k_2)$ if either

1. $(x_1, k_1) = (x_2, k_2)$ and $x_1, x_2 \notin \{0, 1\}$,
2. $x_1 = 0$, $x_2 = 1$, and $k_1 = -k_2$, or
3. $x_1 = 1$, $x_2 = 0$, and $k_1 = -k_2$.

We also let $\mathcal{A} = [0,1]$ and define an equivalence relation $\sim_0$ in $\mathcal{A}$ by declaring that $x_1 \sim_0 x_2$ if either

1. $x_1 = x_2$ and $x_1, x_2 \notin \{0,1\}$,
2. $x_1 = 0$ and $x_2 = 1$, or
3. $x_1 = 1$ and $x_2 = 0$.

We denote $\mathcal{Y} = \mathcal{X}/\sim_1$ and $\mathcal{B} = \mathcal{A}/\sim_0$ and denote by $\pi_1\colon \mathcal{X} \to \mathcal{Y}$ and $\pi_0\colon \mathcal{A} \to \mathcal{B}$ the canonical projections. Define a projection $\pi\colon \mathcal{Y} \to \mathcal{B}$ by $\pi([x,k]) = [x]$. This can be thought of as a discrete version of the Möbius vector bundle. By $\mathscr{E}$ we denote the presheaf over $\mathcal{B}$ whose sections over $\mathcal{U} \subseteq \mathcal{B}$ are continuous sections of $\pi\colon \mathcal{Y} \to \mathcal{B}$. This presheaf can be easily verified to be a sheaf.

Define $\mathcal{U}_1, \mathcal{U}_2$ by

$$\mathcal{U}_1 = \pi_0((\tfrac{1}{8}, \tfrac{7}{8})), \quad \mathcal{U}_2 = \pi_0([0, \tfrac{1}{4}) \cup (\tfrac{3}{4}, 1]).$$

Define sections $s_1, t_1 \in E(\mathcal{U}_1)$ by $s_1([x]) = 1$ and $t_1([x]) = -1$. Define sections $s_2, t_2 \in E(\mathcal{U}_2)$ by

$$s_2([x]) = \begin{cases} 1, & x \in [0, \frac{1}{5}), \\ -1, & x \in (\frac{7}{8}, 1] \end{cases}$$

and

$$t_2([x]) = \begin{cases} -1, & x \in [0, \frac{1}{5}), \\ 1, & x \in (\frac{7}{8}, 1]. \end{cases}$$

On $(\frac{1}{8}, \frac{1}{4}) \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$ we have

$$s_1 \otimes t_1 = 1 \otimes -1 = s_2 \otimes t_2$$

and on $(\frac{3}{4}, \frac{7}{8}) \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$ we have

$$s_1 \otimes t_1 = 1 \otimes -1 = -1 \otimes 1 = s_2 \otimes t_2.$$

Note that $\mathcal{U}_1 \cup \mathcal{U}_2 = \mathcal{B}$ and that the only continuous section over $\mathcal{B}$ is the zero section. Thus there can be no sections $s, t \in E(\mathcal{B})$ such that $r_{\mathcal{B},\mathcal{U}_1}(s \otimes t) = s_1 \otimes t_1$ and $r_{\mathcal{B},\mathcal{U}_2}(s \otimes t) = s_2 \otimes t_2$. •

7.2 Vector bundles and sheaves

In this section we consider some relationships between vector bundles and sheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules. The purpose of studying these relationships is twofold. On the one hand, one gets some useful intuition about sheaves of modules by understanding how they relate to vector bundles. On the other hand, the tools of sheaf theory provide a means to say some useful, and sometimes nontrivial, things about vector bundles.

7.2.1 From stalks of a sheaf to fibres

Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. As we have seen in Example 7.1.5, this gives rise in a natural way to a sheaf, the sheaf $\mathscr{G}^r_{\mathsf{E}}$ of sections of E. The stalk of this sheaf at $x \in \mathsf{M}$ is the set $\mathscr{G}^r_{x,\mathsf{E}}$ of germs of sections which is a module over the ring $\mathscr{C}^r_{x,\mathsf{M}}$ of germs of functions. The stalk is *not* the same as the fibre E_x, however, the fibre can be obtained from the stalk, and in this section we see how this is done. We shall couch this in a brief general algebraic construction, just to add colour.

Recall that if R is a commutative unit ring, if $\mathsf{I} \subseteq \mathsf{R}$ is an ideal, and if A is a unital R-module, IA is the submodule of A generated by elements of the form rv where $r \in \mathsf{I}$ and $v \in \mathsf{A}$.

7.2.1 Proposition (Vector spaces from modules over local rings) *Let R be a commutative unit ring that is local, i.e., possess a unique maximal ideal $\mathfrak{m}$, and let A be a unital R-module. Then $\mathsf{A}/\mathfrak{m}\mathsf{A}$ is a vector space over $\mathsf{R}/\mathfrak{m}$. Moreover, this vector space is naturally isomorphic to $(\mathsf{R}/\mathfrak{m}) \otimes_{\mathsf{R}} \mathsf{A}$.*

Proof We first prove that $\mathsf{R}/\mathfrak{m}$ is a field. Denote by $\pi_{\mathfrak{m}} \colon \mathsf{R} \to \mathsf{R}/\mathfrak{m}$ the canonical projection. Let $\mathsf{I} \subseteq \mathsf{R}/\mathfrak{m}$ be an ideal. We claim that

$$\tilde{\mathsf{I}} = \{r \in \mathsf{R} \mid \pi_{\mathfrak{m}}(r) \in \mathsf{I}\}$$

is an ideal in R. Indeed, let $r_1, r_2 \in \tilde{\mathsf{I}}$ and note that $\pi_{\mathfrak{m}}(r_1 - r_2) = \pi_{\mathfrak{m}}(r_1) - \pi_{\mathfrak{m}}(r_2) \in \mathsf{I}$ since $\pi_{\mathfrak{m}}$ is a ring homomorphism and since I is an ideal. Thus $r_1 - r_2 \in \tilde{\mathsf{I}}$. Now let $r \in \tilde{\mathsf{I}}$ and $s \in \mathsf{R}$ and note that $\pi_{\mathfrak{m}}(sr) = \pi_{\mathfrak{m}}(s)\pi_{\mathfrak{m}}(r) \in \mathsf{I}$, again since $\pi_{\mathfrak{m}}$ is a ring homomorphism and since I is an ideal. Thus $\tilde{\mathsf{I}}$ is an ideal. Clearly $\mathfrak{m} \subseteq \tilde{\mathsf{I}}$ so that either $\tilde{\mathsf{I}} = \mathfrak{m}$ or $\tilde{\mathsf{I}} = \mathsf{R}$. In the first case $\mathsf{I} = \{0_{\mathsf{R}} + \mathfrak{m}\}$ and in the second case $\mathsf{I} = \mathsf{R}/\mathfrak{m}$. Thus the only ideals of $\mathsf{R}/\mathfrak{m}$ are $\{0_{\mathsf{R}} + \mathfrak{m}\}$ and $\mathsf{R}/\mathfrak{m}$. To see that this implies that $\mathsf{R}/\mathfrak{m}$ is a field, let $r + \mathfrak{m} \in \mathsf{R}/\mathfrak{m}$ be nonzero and consider the ideal $(r + \mathfrak{m})$. Since $(r + \mathfrak{m})$ is nontrivial we must have $(r + \mathfrak{m}) = \mathsf{R}/\mathfrak{m}$. In particular, $1 = (r + \mathfrak{m})(s + \mathfrak{m})$ for some $s + \mathfrak{m} \in \mathsf{R}/\mathfrak{m}$, and so $r + \mathfrak{m}$ is a unit.

Now we show that $\mathsf{A}/\mathfrak{m}\mathsf{A}$ is a vector space over $\mathsf{R}/\mathfrak{m}$. This amounts to showing that the natural vector space operations

$$(u + \mathfrak{m}\mathsf{A}) + (v + \mathfrak{m}\mathsf{A}) = u + v + \mathfrak{m}\mathsf{A}, \quad (r + \mathfrak{m})(u + \mathfrak{m}\mathsf{A}) = ru + \mathfrak{m}\mathsf{A}$$

make sense. The only possible issue is with scalar multiplication, so suppose that

$$r + \mathfrak{m} = s + \mathfrak{m}, \quad u + \mathfrak{m}\mathsf{A} = v + \mathfrak{m}\mathsf{A}$$

so that $s = r + a$ for $a \in \mathfrak{m}$ and $v = u + w$ for $w \in \mathfrak{m}\mathsf{A}$. Then

$$sv = (r + a)(u + w) = ru + au + rw + aw,$$

and we observe that $au, rw, aw \in \mathfrak{m}\mathsf{A}$, and so the sensibility of scalar multiplication is proved.

For the last assertion, note that we have the exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow \mathsf{R} \longrightarrow \mathsf{R}/\mathfrak{m} \longrightarrow 0$$

By right exactness of the tensor product [Hungerford 1980, Proposition IV.5.4] this gives the exact sequence

$$\mathfrak{m} \otimes_{\mathsf{R}} \mathsf{A} \longrightarrow \mathsf{A} \longrightarrow (\mathsf{R}/\mathfrak{m}) \otimes_{\mathsf{R}} \mathsf{A} \longrightarrow 0$$

noting that $\mathsf{R} \otimes_{\mathsf{R}} \mathsf{A} \simeq \mathsf{A}$. By this isomorphism, the image of $\mathfrak{m} \otimes_{\mathsf{R}} \mathsf{A}$ in A is simply generated by elements of the form rv for $r \in \mathfrak{m}$ and $v \in \mathsf{A}$. That is to say, the image of $\mathfrak{m} \otimes_{\mathsf{R}} \mathsf{A}$ in A is simply $\mathfrak{m}\mathsf{A}$. Thus we have the induced commutative diagram

$$\begin{array}{ccccccccc} & & \mathfrak{m} \otimes_{\mathsf{R}} \mathsf{A} & \longrightarrow & \mathsf{A} & \longrightarrow & (\mathsf{R}/\mathfrak{m}) \otimes_{\mathsf{R}} \mathsf{A} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{m}\mathsf{A} & \longrightarrow & \mathsf{A} & \longrightarrow & \mathsf{A}/\mathfrak{m}\mathsf{A} & \longrightarrow & 0 \end{array}$$

with exact rows. We claim that there is an induced mapping as indicated by the dashed arrow, and that this mapping is an isomorphism. To define the mapping, let $\alpha \in (\mathsf{R}/\mathfrak{m}) \otimes_{\mathsf{R}} \mathsf{A}$ and let $v \in \mathsf{A}$ project to α. The image of β is then taken to be $v + \mathfrak{m}\mathsf{A}$. It is a straightforward exercise to show that this mapping is well-defined and is an isomorphism, using exactness of the diagram. ■

With this simple algebraic construction as background, we can then indicate how to recover the fibres of a vector bundle from the stalks of its sheaf of sections.

7.2.2 Proposition (From stalks to fibres) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* $\pi\colon \mathsf{E} \to \mathsf{M}$ *be a vector bundle of class* C^{r}. *For* $\mathrm{x} \in \mathsf{M}$ *let* $\mathfrak{m}_{\mathrm{x}}$ *denote the unique maximal ideal in* $\mathscr{C}^{\mathrm{r}}_{\mathrm{x},\mathsf{M}}$. *Then the following statements hold:*

(i) *the field* $\mathscr{C}^{\mathrm{r}}_{\mathrm{x},\mathsf{M}}/\mathfrak{m}_{\mathrm{x}}$ *is isomorphic to* $\mathbb{F}$ *via the isomorphism*

$$[\mathrm{f}]_{\mathrm{x}} + \mathfrak{m}_{\mathrm{x}} \mapsto \mathrm{f}(\mathrm{x});$$

(ii) *the* $\mathscr{C}^{\mathrm{r}}_{\mathrm{x},\mathsf{M}}/\mathfrak{m}_{\mathrm{x}}$*-vector space* $\mathscr{G}^{\mathrm{r}}_{\mathrm{x},\mathsf{E}}/\mathfrak{m}_{\mathrm{x}}\mathscr{G}^{\mathrm{r}}_{\mathrm{x},\mathsf{E}}$ *is isomorphic to* E_{x} *via the isomorphism*

$$[\xi]_{\mathrm{x}} + \mathfrak{m}_{\mathrm{k}}\mathscr{G}^{\mathrm{r}}_{\mathrm{x},\mathsf{E}} \mapsto \xi(\mathrm{x});$$

(iii) *the map from* $(\mathscr{C}^{\mathrm{r}}_{\mathrm{x},\mathsf{M}}/\mathfrak{m}_{\mathrm{x}}) \otimes_{\mathscr{C}^{r}_{\mathrm{x},\mathsf{M}}} \mathscr{G}^{\mathrm{r}}_{\mathrm{x},\mathsf{E}}$ *to* E_{x} *defined by*

$$([\mathrm{f}]_{\mathrm{x}} + \mathfrak{m}_{\mathrm{x}}) \otimes [\xi]_{\mathrm{x}} \mapsto \mathrm{f}(\mathrm{x})\xi(\mathrm{x})$$

is an isomorphism of $\mathbb{F}$*-vector spaces.*

Proof (i) The map is clearly a homomorphism of fields. To show that it is surjective, if $a \in \mathbb{F}$ then a is the image of $[f]_x + \mathfrak{m}_x$ for any germ $[f]_x$ for which $f(x) = a$. To show injectivity, if $[f]_x + \mathfrak{m}_x$ maps to 0 then clearly $f(x) = 0$ and so $f \in \mathfrak{m}_x$.

(ii) The map is clearly linear, so we verify that it is an isomorphism. Let $v_x \in \mathsf{E}_x$. Then v_x is the image of $[\xi]_x + \mathfrak{m}_x\mathscr{G}^r_{x,\mathsf{E}}$ for any germ $[\xi]_x$ for which $\xi(x) = v_x$. Also suppose that $[\xi]_x + \mathfrak{m}_x\mathscr{G}^r_{x,\mathsf{E}}$ maps to zero. Then $\xi(x) = 0$. Since $\mathscr{G}^r_{\mathsf{E}}$ is locally free (see the next section in case the meaning here is not patently obvious), it follows that we can write

$$\xi(y) = f_1(y)\eta_1(y) + \cdots + f_m(y)\eta_m(y)$$

for sections $\eta_1, \dots, \eta_m$ of class C^r in a neighbourhood of x and for functions $f_1, \dots, f_m$ of class C^r in a neighbourhood of x. Moreover, the sections may be chosen such that $(\eta_1(y), \dots, \eta_m(y))$ is a basis for E_y for every y in some suitably small neighbourhood of x. Thus

$$\xi(x) = 0 \quad \Longrightarrow \quad f_1(x) = \dots = f_m(x) = 0,$$

giving $\xi \in \mathfrak{m}_x \mathscr{G}^r_{x,\mathsf{E}}$, as desired.

(iii) The $\mathbb{F}$-linearity of the stated map is clear, and the fact that the map is an isomorphism follows from the final assertion of Proposition 7.2.1. ■

This result relates stalks to fibres. In the next section, specifically in Theorem 7.2.7, we shall take a more global view towards relating vector bundles and sheaves.

In the preceding result we were able to rebuild the fibre of a vector bundle from the germs of sections. There is nothing keeping one from making this construction for a general sheaf.

7.2.3 Definition (Fibres for sheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let M be a manifold of class C^r, and let $\mathscr{F}$ be a sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. The ***fibre*** of $\mathscr{F}$ is the $\mathbb{F}$-vector space $\mathsf{E}(\mathscr{F})_x = \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$. •

This definition of fibre agrees (or more precisely is isomorphic to), of course, with the usual notion of the fibre of a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ when $\mathscr{F} = \mathscr{G}^r_{\mathsf{E}}$; this is the content of the proof of Proposition 7.2.2. Let us look, therefore, at a case of a sheaf which is not equivalent to a vector bundle in this sense.

7.2.4 Example (Fibres for a non-vector bundle sheaf) Let $r \in \{\infty, \omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r \in \{\infty, \omega\}$ and let $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Let us take $\mathsf{M} = \mathbb{F}$ and define a presheaf $\mathscr{I}^r_0 = (I^r_0(\mathcal{U}))_{\mathcal{U}\ \text{open}}$ by

$$I^r_0(\mathcal{U}) = \begin{cases} C^r(\mathcal{U}), & 0 \notin \mathcal{U}, \\ \{f \in C^r(\mathcal{U}) \mid f(0) = 0\}, & 0 \in \mathcal{U}. \end{cases}$$

One directly verifies that $\mathscr{I}^r_0$ is a sheaf. Moreover, $\mathscr{I}^r_0$ is a sheaf of $\mathscr{C}^r_{\mathbb{F}}$-modules; this too is easily verified. Let us compute the fibres associated with this sheaf. The germs of this sheaf at $x \in \mathbb{F}$ are readily seen to be given by

$$\mathscr{I}^r_{0,x} = \begin{cases} \mathscr{C}^r_{x,\mathbb{F}}, & x \neq 0, \\ \mathfrak{m}_0 = \{[f]_0 \in \mathscr{C}^r_{0,\mathbb{F}} \mid f(x) = 0\}, & x = 0. \end{cases}$$

Thus we have

$$\mathsf{E}(\mathscr{I}^r_0)_x = \begin{cases} \mathscr{C}^r_{x,\mathbb{F}}/\mathfrak{m}_x\mathscr{C}^r_{x,\mathbb{F}} \simeq \mathbb{F}, & x \neq 0, \\ \mathfrak{m}_0/\mathfrak{m}_0^2 \simeq \mathbb{F}, & x = 0. \end{cases}$$

Note that the fibre at 0 is “bigger” than we expect it to be. We shall address this shortly.

Let us expand on this example a little further. Let us consider the morphism $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\text{ open}}$ of $\mathscr{C}^r_{\mathbb{F}}$-modules given by

$$\Phi_{\mathcal{U}}(f)(x) = xf(x),$$

i.e., Φ is multiplication by the function "x." Note that $\mathscr{I}^r_0$ is the image presheaf of Φ since, as we showed in the proof of Lemma 1 from the proof of Proposition 4.3.4, if f is a function defined in a neighbourhood $\mathcal{U}$ of 0, we can write $f(x) = xg(x)$ for some $g \in \mathrm{C}^r(\mathcal{U})$. By Proposition 7.1.41 the kernel presheaf for Φ is a sheaf. If $g \in \ker(\Phi_{\mathcal{U}})$ then it is clear that $g(x) = 0$ for $x \neq 0$, and then continuity requires that $g(x) = 0$ for $x = 0$. That is to say, $\ker(\Phi)$ is the zero sheaf and so the fibres are also zero. •

7.2.2 Locally finitely generated and locally free sheaves

In this section we introduce a class of sheaves that correspond exactly with vector bundles. We give an initial definition for general sheaves of modules since it adds a little context to the discussion, a context that will be best appreciated in Section 7.3.

7.2.5 Definition (Locally finitely generated sheaf, locally free sheaf) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a sheaf of $\mathscr{R}$-modules. The sheaf $\mathscr{F}$ is

(i) ***locally finitely generated*** if, for each $x_0 \in \mathcal{S}$, there exists a neighbourhood $\mathcal{U}$ of x_0 and sections $s_1, \dots, s_k \in F(\mathcal{U})$ such that $[s_1]_x, \dots, [s_k]_x$ generate the $\mathscr{R}_x$-module $\mathscr{F}_x$ for every $x \in \mathcal{U}$ and is

(ii) ***locally free*** if, for each $x_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{U}$ of x_0 such that $F(\mathcal{U})$ is a free $R(\mathcal{U})$-module. •

The following elementary result shows that, in the locally finitely generated case, the local generators can be selected from the generators for a particular stalk.

7.2.6 Lemma (Local generators for locally finitely generated sheaves) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a locally finitely generated sheaf of $\mathscr{R}$-modules. If, for $\mathrm{x}_0 \in \mathcal{S}$, $[\mathrm{s}_1]_{\mathrm{x}_0}, \dots, [\mathrm{s}_k]_{\mathrm{x}_0}$ are generators for the $\mathscr{R}_{\mathrm{x}_0}$-module $\mathscr{F}_{\mathrm{x}_0}$, then there exists a neighbourhood $\mathcal{U}$ of x_0 such that $[\mathrm{s}_1]_\mathrm{x}, \dots, [\mathrm{s}_k]_\mathrm{x}$ are generators for $\mathscr{F}_\mathrm{x}$ for each $\mathrm{x} \in \mathcal{U}$.*

Proof By hypothesis, there exists a neighbourhood $\mathcal{V}$ of x_0 and sections $t_1, \dots, t_k \in F(\mathcal{V})$ such that $[t_1]_x, \dots, [t_m]_x$ generate $\mathscr{F}_x$ for all $x \in \mathcal{V}$. Since $[s_1]_{x_0}, \dots, [s_k]_{x_0}$ generate $\mathscr{F}_{x_0}$,

$$[t_l]_{x_0} = \sum_{j=1}^{k} [a^j_l]_{x_0} [s_j]_{x_0}, \qquad l \in \{1, \dots, m\},$$

for germs $[a^j_l]_{x_0} \in \mathscr{C}^r_{x_0,\mathsf{M}}$. By definition of germ, there exists a neighbourhood $\mathcal{U}$ such that

$$t_l(x) = \sum_{j=1}^{k} a^j_l(x) s_j(x), \qquad l \in \{1, \dots, m\},\ x \in \mathcal{U}.$$

Taking germs shows that the generators $[t_1]_x, \dots, [t]_m$ of $\mathscr{F}_x$ for $x \in \mathcal{U}$ are linear combinations of $[s_1]_x, \dots, [s_k]_x$, as desired. ■

With some general definitions and a basic result under our belts, let us consider locally free, locally finitely generated sheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules, as these are what are of principle interest for us here.

7.2.7 Theorem (Correspondence between vector bundles and locally free, locally finitely generated sheaves) *Let* $\mathrm{r} \in \{\infty, \omega, \mathrm{hol}\}$ *and let* $\mathbb{F} = \mathbb{R}$ *if* $\mathrm{r} \in \{\infty, \omega\}$ *and let* $\mathbb{F} = \mathbb{C}$ *if* $\mathrm{r} = \mathrm{hol}$. *Let* $\pi\colon \mathsf{E} \to \mathsf{M}$ *be a vector bundle of class* C^{r}. *Then* $\mathscr{G}^{\mathrm{r}}_{\mathsf{E}}$ *is a locally free, locally finitely generated sheaf of* $\mathscr{C}^{\mathrm{r}}_{\mathsf{M}}$*-modules.*

Conversely, if $\mathscr{F}$ *is a locally free, locally finitely generated sheaf of* $\mathscr{C}^{\mathrm{r}}_{\mathsf{M}}$*-modules, then there exists a vector bundle* $\pi\colon \mathsf{E} \to \mathsf{M}$ *of class* C^{r} *such that* $\mathscr{F}$ *is isomorphic to* $\mathscr{G}^{\mathrm{r}}_{\mathsf{E}}$.

Proof First let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r and let $x_0 \in \mathsf{M}$. Let $(\mathcal{V}, \psi)$ be a vector bundle chart such that the corresponding chart $(\mathcal{U}, \phi)$ for M contains x_0. Suppose that $\psi(\mathcal{V}) = \phi(\mathcal{U}) \times \mathbb{F}^m$ and let $\eta_1, \dots, \eta_m \in \Gamma^r(\mathsf{E}|\mathcal{U})$ satisfy $\psi(\eta_j(x)) = (\phi(x), \boldsymbol{e}_j)$ for $x \in \mathcal{U}$ and $j \in \{1, \dots, m\}$. Let us arrange the components η^k_j, $j, k \in \{1, \dots, m\}$, of the sections $\eta_1, \dots, \eta_m$ in an $m \times m$ matrix:

$$\boldsymbol{\eta}(x) = \begin{bmatrix} \eta^1_1(x) & \cdots & \eta^1_m(x) \\ \vdots & \ddots & \vdots \\ \eta^m_1(x) & \cdots & \eta^m_m(x) \end{bmatrix}.$$

Now let $\xi \in \Gamma^r(\mathsf{E}|\mathcal{U})$, let the components of ξ be ξ^k, $k \in \{1, \dots, k\}$, and arrange the components in a vector

$$\boldsymbol{\xi}(x) = \begin{bmatrix} \xi^1(x) \\ \vdots \\ \xi^m(x) \end{bmatrix}.$$

Now fix $x \in \mathcal{U}$. We wish to solve the equation

$$\xi(x) = f^1(x)\eta_1(x) + \cdots + f^m(x)\eta_m(x)$$

for $f^1(x), \dots, f^m(x) \in \mathbb{F}$. Let us write

$$\boldsymbol{f}(x) = \begin{bmatrix} f^1(x) \\ \vdots \\ f^m(x) \end{bmatrix}.$$

Writing the equation we wish to solve as a matrix equation we have

$$\boldsymbol{\xi}(x) = \boldsymbol{\eta}(x)\boldsymbol{f}(x).$$

Therefore,

$$\boldsymbol{f}(x) = \boldsymbol{\eta}^{-1}(x)\boldsymbol{\xi}(x),$$

noting that $\boldsymbol{\eta}(x)$ is invertible since the vectors $\eta_1(x), \dots, \eta_m(x)$ are linearly independent. By Cramer's Rule, or some such, the components of $\boldsymbol{\eta}^{-1}$ are C^r-functions of $x \in \mathcal{U}$, and so ξ is a $\mathrm{C}^r(\mathcal{U})$-linear combination of $\eta_1, \dots, \eta_m$, showing that $\Gamma^r(\mathsf{E}|\mathcal{U})$ is finitely generated. To show that this module is free, it suffices to show that $(\eta_1, \dots, \eta_m)$ is linearly independent over $\mathrm{C}^r(\mathcal{U})$. Suppose that there exists $f^1, \dots, f^m \in \mathrm{C}^r(\mathcal{U})$ such that

$$f^1\eta_1 + \dots + f^m\eta_m = 0_{\Gamma^r(\mathsf{E})}.$$

Then, for every $x \in \mathcal{U}$,

$$f^1(x)\eta_1(x) + \dots + f^m(x)\eta_m(x) = 0_x \quad \Longrightarrow \quad f^1(x) = \dots = f^m(x) = 0,$$

giving the desired linear independence.

Next suppose that $\mathscr{F}$ is a locally free, locally finitely generated sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. Let us first define the total space of our vector bundle. For $x \in \mathsf{M}$ define

$$\mathsf{E}_x = \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x.$$

By Propositions 7.2.1 and 7.2.2, E_x is a $\mathbb{F}$-vector space. We take $\mathsf{E} = \mathring{\cup}_{x\in\mathsf{M}} \mathsf{E}_x$. Let $x \in \mathsf{M}$ and let $\mathcal{U}_x$ be a neighbourhood of x such that $F(\mathcal{U}_x)$ is a free $\mathrm{C}^r(\mathcal{U}_x)$-module. By shrinking $\mathcal{U}_x$ if necessary, we suppose that it is the domain of a coordinate chart $(\mathcal{U}_x, \phi_x)$. Let $s_1, \dots, s_m \in F(\mathcal{U}_x)$ be such that $(s_1, \dots, s_m)$ is a basis for $F(\mathcal{U}_x)$. Note that $([s_1]_y, \dots, [s_m]_x)$ is a basis for $\mathscr{F}_y$ for each $y \in \mathcal{U}_x$. It is straightforward to show that

$$([s_1]_y + \mathfrak{m}_y\mathscr{F}_y, \dots, [s_m]_y + \mathfrak{m}_y\mathscr{F}_y)$$

is then a basis for E_y. For $y \in \mathcal{U}_x$ the map

$$a^1([s_1]_y + \mathfrak{m}_y) + \dots + a^m([s_m]_y + \mathfrak{m}_y) \mapsto (a^1, \dots, a^m)$$

is clearly an isomorphism. Now define $\mathcal{V}_x = \mathring{\cup}_{y\in\mathcal{U}_x} \mathsf{E}_y$ and define $\psi_x \colon \mathcal{V}_x \to \phi_x(\mathcal{U}_x) \times \mathbb{F}^m$ by

$$\psi_x(a^1([s_1]_y + \mathfrak{m}_y) + \dots + a^m([s_m]_y + \mathfrak{m}_y)) = (\psi_x(y), (a^1, \dots, a^m)).$$

This is clearly a vector bundle chart for E. Moreover, this construction furnishes a covering of E by vector bundle charts.

It remains to show that two overlapping vector bundle charts satisfy the appropriate overlap condition. Thus let $x, y \in \mathsf{M}$ be such that $\mathcal{U}_x \cap \mathcal{U}_y$ is nonempty. Let $(s_1, \dots, s_m)$ and $(t_1, \dots, t_m)$ be bases for $F(\mathcal{U}_x)$ and $F(\mathcal{U}_y)$, respectively. (Note that the cardinality of these bases agrees since, for $z \in \mathcal{U}_x \cap \mathcal{U}_y$, $([s_1]_z, \dots, [s_m]_z)$ and $([t_1]_z, \dots, [t_m]_z)$ are both bases for $\mathscr{F}_z$, cf. [Hungerford 1980, Corollary IV.2.12].) Note that

$$r_{\mathcal{U}_x,\mathcal{U}_x\cap\mathcal{U}_x}(s_j) = \sum_{k=1}^m f_j^k r_{\mathcal{U}_y,\mathcal{U}_x\cap\mathcal{U}_y}(t_k)$$

for $f_j^k \in \mathrm{C}^r(\mathcal{U}_x \cap \mathcal{U}_y)$, $j, k \in \{1, \dots, m\}$. At the stalk level we have

$$[s_j]_z = \sum_{k=1}^m [f_j^k]_z[t_k]_z,$$

from which we conclude that

$$([s_j]_z + \mathfrak{m}_z \mathscr{F}_z) = \sum_{k=1}^{m} f_j^k(z)([t_k]_z + \mathfrak{m}_z \mathscr{F}_z),$$

From this we conclude that the matrix

$$\boldsymbol{f}(z) = \begin{bmatrix} f_1^1(z) & \cdots & f_m^1(z) \\ \vdots & \ddots & \vdots \\ f_1^m(z) & \cdots & f_m^m(z) \end{bmatrix}$$

is invertible, being the change of basis matrix for the two bases for E_z. Moreover, the change of basis formula gives

$$\psi_y \circ \psi_x^{-1}(\boldsymbol{z}, (a^1, \dots, a^m)) = \Big(\phi_y \circ \phi_x^{-1}(\boldsymbol{z}), \Big(\sum_{j=1}^{m} a^j f_j^1(z), \dots, \sum_{j=1}^{m} a^j f_j^m(z)\Big)\Big)$$

for every $z \in \mathcal{U}_x \cap \mathcal{U}_y$, where $\boldsymbol{z} = \phi_x(z)$. Thus we see that the covering by vector bundle charts has the proper overlap condition to define a vector bundle structure for E.

It remains to show that $\mathscr{G}^r_{\mathsf{E}}$ is isomorphic to $\mathscr{F}$. Let $\mathcal{U} \subseteq \mathsf{M}$ be open and define $\Phi_{\mathcal{U}} : F(\mathcal{U}) \to \Gamma^r(\mathsf{E}|\mathcal{U})$ by

$$\Phi_{\mathcal{U}}(s)(x) = [s]_x + \mathfrak{m}_x \mathscr{F}_x.$$

For this definition to make sense, we must show that $\Phi_{\mathcal{U}}(s)$ is of class C^r. Let $y \in \mathcal{U}$ and, using the above constructions, let $(s_1, \dots, s_m)$ be a basis for $F(\mathcal{U}_y)$. Let us abbreviate $\mathcal{V} = \mathcal{U} \cap \mathcal{U}_y$. Note that $(r_{\mathcal{U},\mathcal{V}}(s_1), \dots, r_{\mathcal{U},\mathcal{V}}(s_m))$ is a basis for $F(\mathcal{V})$. (To see that this is so, one can identify $F(\mathcal{U})$ with $\Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{F}))$ using Proposition 7.1.27, and having done this the assertion is clear.) We thus write

$$r_{\mathcal{U},\mathcal{V}}(s) = f^1 r_{\mathcal{U},\mathcal{V}}(s_1) + \dots + f^m r_{\mathcal{U},\mathcal{V}}(s_m).$$

In terms of stalks we thus have

$$[s]_z = [f^1]_z [s_1]_z + \dots + [f^m]_z [s_m]_z$$

for each $z \in \mathcal{V}$. Therefore,

$$\Phi_{\mathcal{U}}(s)(z) = f^1(z)([s_1]_z + \mathfrak{m}_z \mathscr{F}_z) + \dots + f^m(z)([s_m]_z + \mathfrak{m}_z \mathscr{F}_z),$$

which (recalling that $\mathcal{U}_y$, and so also $\mathcal{V}$, is a chart domain) gives the local representative of $\Phi_{\mathcal{U}}(s)$ on $\mathcal{V}$ as

$$\boldsymbol{z} \mapsto (\boldsymbol{z}, (f^1 \circ \phi_y^{-1}(\boldsymbol{z}), \dots, f^m \circ \phi_y^{-1}(\boldsymbol{z}))).$$

Since this local representative is of class C^r and since this construction can be made for any $y \in \mathcal{U}$, we conclude that $\Phi_{\mathcal{U}}(s)$ is of class C^r.

Now, to show that the family of mappings $(\Phi_{\mathcal{U}})_{\mathcal{U}\text{ open}}$ is an isomorphism, by Proposition 7.1.53 it suffices to show that the induced mapping on stalks is an isomorphism. Let us denote the mapping of stalks at x by Φ_x. We again use our constructions from the first part of this part of the proof and let $(s_1, \dots, s_m)$ be a basis for $F(\mathcal{U}_x)$. Let us show that Φ_x

is surjective. Let $[\xi]_x \in \mathscr{G}^r_{x,\mathsf{M}}$, supposing that $\xi \in \Gamma^r(\mathsf{E}|\mathcal{U})$. Let $\mathcal{V} = \mathcal{U} \cap \mathcal{U}_x$. Let the local representative of ξ on $\mathcal{V}$ in the chart $(\mathcal{V}_x, \psi_x)$ be given by

$$\boldsymbol{y} \mapsto (\boldsymbol{y}, (f^1 \circ \phi_x^{-1}(\boldsymbol{y}), \dots, f^m \circ \phi_x^{-1}(\boldsymbol{y})))$$

for $f^1, \dots, f^m \in \mathrm{C}^r(\mathcal{V})$. Then, if

$$[s]_x = [f^1]_x[s_1]_x + \cdots + [f^m]_x[s_m]_x,$$

we have $\Phi_x([s]_x) = [\xi]_x$. To prove injectivity of Φ_x, suppose that $\Phi_x([s_x]) = 0_x$. This means that $\Phi_x([s]_x)$ is the germ of a section of E over some neighbourhood $\mathcal{U}$ of x that is identically zero. We may without loss of generality assume that $\mathcal{U} \subseteq \mathcal{U}_x$. We also assume without loss of generality (by restriction of necessary) that $s \in F(\mathcal{U})$. We thus have

$$\Phi_{\mathcal{U}}(s)(y) = 0, \qquad y \in \mathcal{U}.$$

Since $(r_{\mathcal{U}_x,\mathcal{U}}(s_1), \dots, r_{\mathcal{U}_x,\mathcal{U}}(s_m))$ is a basis for $F(\mathcal{U})$ we write

$$s = f^1 r_{\mathcal{U}_x,\mathcal{U}}(s_1) + \cdots + f^m r_{\mathcal{U}_x,\mathcal{U}}(s_m).$$

for some uniquely defined $f^1, \dots, f^m \in \mathrm{C}^r(\mathcal{U})$. We have

$$\Phi_{\mathcal{U}}(s)(y) = f^1(y)([s_1]_y + \mathfrak{m}_y \mathscr{F}_y) + \cdots + f^m(y)([s_m]_y + \mathfrak{m}_y \mathscr{F}_y)$$

for each $y \in \mathcal{U}$. Since

$$([s_1]_y + \mathfrak{m}_y \mathscr{F}_y, \dots, [s_m]_y + \mathfrak{m}_y \mathscr{F}_y)$$

is a basis for E_y, we must have $f^1(y) = \cdots = f^m(y) = 0$ for each $y \in \mathcal{U}$, giving $[s]_x = 0$. ■

7.2.3 Sheaf morphisms induced by vector bundle mappings

Having seen how sheaves and vector bundles are related, let us consider how mappings of vector bundles give rise to morphisms of the corresponding sheaves. We begin by considering the situation of morphisms of vector bundles. Thus we let $r \in \{\infty, \omega\}$ and consider vector bundles $\pi\colon \mathsf{E} \to \mathsf{M}$ and $\tau\colon \mathsf{F} \to \mathsf{M}$ of class C^r. We let $\Phi\colon \mathsf{E} \to \mathsf{F}$ be a vector bundle mapping over id_{M}. Thus $\Phi(\mathsf{E}_x) \subseteq \mathsf{F}_x$ and $\Phi|\mathsf{E}_x$ is linear for each $x \in \mathsf{M}$. We *do not* require that the rank of Φ be locally constant as some authors do. We then define a morphism $\hat{\Phi}$ of presheaves of the $\mathscr{C}^r_{\mathsf{M}}$-modules $\mathscr{G}^r_{\mathsf{E}}$ and $\mathscr{G}^r_{\mathsf{F}}$ by defining $\hat{\Phi}_{\mathcal{U}}\colon \Gamma^r(\mathsf{E}|\mathcal{U}) \to \Gamma^r(\mathsf{F}|\mathcal{U})$ by

$$\hat{\Phi}_{\mathcal{U}}(\xi)(x) = \Phi \circ \xi(x), \qquad x \in \mathcal{U}.$$

Also, given a morphism Ψ of the sheaves $\mathscr{G}^r_{\mathsf{E}}$ and $\mathscr{G}^r_{\mathsf{F}}$ of $\mathscr{C}^r_{\mathsf{M}}$-modules, we can associate a vector bundle mapping $\check{\Psi}\colon \mathsf{E} \to \mathsf{F}$ over id_{M} by

$$\check{\Psi}(e_x) = \Psi(r_{\mathcal{U},x}(s))(x),$$

where $s \in \Gamma^r(\mathsf{E}|\mathcal{U})$ is such that $s(x) = e_x$. Such a local section s exists, for example, by constructing it in a vector bundle chart about x. One can also easily verify that this vector bundle mapping is well-defined, independently of the choice of s.

7.3 Coherent holomorphic and real analytic sheaves

As we saw in Sections 2.3.2, 4.1.3, and 4.2.3, germs of holomorphic and real analytic functions and sections of vector bundles are finitely generated. Moreover, since these are also Noetherian, the ideals (resp. submodules) of the ring of germs of holomorphic or real analytic functions (resp. module of germs of holomorphic or real analytic sections) are also finitely generated. In this section we see that these properties at the germ level can, in a certain sense, be extended to corresponding local properties. As we shall see, this requires quite a lot of work, although at the core lies the Weierstrass Preparation Theorem.

7.3.1 Motivation, definitions, and basic properties

As we stated above, the idea of this section is to extend properties of germs to local properties. To show that this is meaningful, we must first suggest that there is a problem to solve here. First we show that the Noetherian property is *not* one that is global, even in the most structured situation, that of holomorphic or real analytic functions.

7.3.1 Example (The Noetherian property of germs does not extend to a global property[1]) Let $r \in \{\omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $r = \omega$ and $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. Note that $\mathscr{C}^r_{x,\mathbb{F}}$ is a Noetherian ring by Theorem 2.3.4. We will show that $\mathrm{C}^r(\mathbb{F})$ is not Noetherian. Recall that $\sin \in \mathrm{C}^r(\mathbb{F})$ and also recall Euler's product representation for sin:

$$\sin(\pi x) = \pi x \prod_{j=1}^{\infty}\Big(1 - \frac{x^2}{j^2}\Big),$$

with convergence being uniform on compact subsets. (We refer to Chapter 6 of [Ullrich 2008] for a discussion of this formula.) Let f_k, $k \in \{1, \dots, k\}$, be defined by

$$f_k(x) = \pi x \prod_{j=k}^{\infty}\Big(1 - \frac{x^2}{j^2}\Big),$$

and let I_k be the ideal in $\mathrm{C}^r(\mathbb{F})$ generated by f_k. Thus

$$\mathsf{I}_k = \{f f_k \mid f \in \mathrm{C}^r(\mathbb{F})\}.$$

Note that

$$f_k(x) = f_{k+1}(x)\Big(1 - \frac{x^2}{k^2}\Big),$$

showing that $\mathsf{I}_k \subseteq \mathsf{I}_{k+1}$, $k \in \mathbb{Z}_{>0}$. Note that if $f \in \mathsf{I}_k$ then $f(k) = 0$. Since $f_{k+1}(k) \neq 0$, we conclude that $f_{k+1} \notin \mathsf{I}_k$ and so we in fact have $\mathsf{I}_k \subset \mathsf{I}_{k+1}$, $k \in \mathbb{Z}_{>0}$. Thus we have a chain

$$\mathsf{I}_1 \subset \mathsf{I}_2 \subset \cdots$$

[1] The author would like to thank Mike Roth for suggesting this example.

that is not finite. Thus $C^r(\mathbb{F})$ is not Noetherian, as claimed. Note that the ideal $\mathsf{I} = \cup_{k\in\mathbb{Z}_{>0}}\mathsf{I}_k$ is not finitely generated. Indeed, were I to be generated by analytic functions $g_1, \dots, g_m$, there would necessarily be some $k \in \mathbb{Z}_{>0}$ such that

$$\mathsf{I} = (g_1, \dots, g_m) \subseteq \mathsf{I}_k.$$

But this implies that $\mathsf{I}_j = \mathsf{I}_k$ for all $j \geq k$, contradicting what we have already shown. •

The example is perhaps not as compelling as one would like, given its global nature. The next example shows that the property of being locally finitely generated is not one that can be extended from stalks to a local property.

7.3.2 Example (The finite generation property of germs does not extend to a local property) Let $r \in \{\omega, \text{hol}\}$ and take $\mathbb{F} = \mathbb{R}$ if $r = \omega$ and $\mathbb{F} = \mathbb{C}$ if $r = \text{hol}$. We consider $\mathsf{M} = \mathbb{F}$ and let

$$S = \{\tfrac{1}{j} \mid j \in \mathbb{Z}_{>0}\} \cup \{0\}.$$

Consider the presheaf $\mathscr{I}_S = (I_S(\mathcal{U}))_{\mathcal{U}\,\text{open}}$ given by

$$I_S(\mathcal{U}) = \{f \in C^r(\mathcal{U}) \mid f(x) = 0 \text{ for } x \in \mathcal{U} \cap S\}.$$

One can easily verify that $\mathscr{I}_S$ is a sheaf. We claim $\mathscr{I}_S$ is not locally finitely generated. The easiest way to see this is through the following observation. Note that $\mathscr{I}_{S,0} = \{0\}$ since any function of class C^r in a neighbourhood of 0 and vanishing on S must be zero by Proposition 1.1.18. However, note that if $x \neq 0$ then $\mathscr{I}_{S,x} \neq \{0\}$ and so, by Lemma 7.2.6, it follows that $\mathscr{I}_S$ cannot be locally finitely generated. •

Said otherwise, the example suggests that, even for locally finitely generated sheaves, one needs additional properties in order to ensure "Noetherian behaviour." In Definition 7.2.5 we provided the definition for a locally finitely generated sheaf. Let us turn to the supplementary conditions needed to get the desired behaviour. It is convenient in the early stages of the discussion to work with sheaves of modules over topological spaces. Thus we let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{E} = (E(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a sheaf of $\mathscr{R}$-modules. Let $\mathcal{U} \in \mathscr{O}$ and let $s_1, \dots, s_k \in E(\mathcal{U})$. We define a morphism $\varrho(s_1, \dots, s_k)$ of sheaves from $(\mathscr{R}|\mathcal{U})^k$ to $\mathscr{E}|\mathcal{U}$ by defining it stalkwise:

$$\varrho(s_1, \dots, s_k)_x([f^1]_x, \dots, [f^k]_x) = \sum_{j=1}^{k} [f^j]_x [s_j]_x, \qquad x \in \mathcal{U}.$$

The kernel $\ker(\varrho(s_1, \dots, s_k))$ of this morphism we call the ***sheaf of relations*** of the sections $s_1, \dots, s_k$ over $\mathcal{U}$.

With the preceding construction, we now make the following definition.

7.3.3 Definition (Coherent sheaf) Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{E} = (E(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a sheaf of $\mathscr{R}$-modules. The sheaf $\mathscr{E}$ is ***coherent***

(i) if it is locally finitely generated and
(ii) if, for every $\mathcal{U} \in \mathscr{O}$ and $s_1, \dots, s_k \in E(\mathcal{U})$, $\ker(\varrho(s_1, \dots, s_k))$ is locally finitely generated.

The sheaf of rings $\mathscr{R}$ is ***coherent*** if it is coherent as a sheaf of $\mathscr{R}$-modules in the obvious way. •

One might legitimately wonder whether coherent sheaves exist. We shall address this in Section 7.3.2 where we prove Oka's Coherence Theorem which gives a large
what? class of coherent sheaves. Another important class of sheaves will be developed in . For the moment, however, let us prove some basic properties of coherent sheaves.

The following characterisation of coherent sheaves is insightful.

7.3.4 Proposition (Coherent modules are cokernels of morphisms) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space, let $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a sheaf of rings over $\mathcal{S}$, and let $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a sheaf of $\mathscr{R}$-modules. If $\mathscr{E}$ is coherent then, for every $\mathrm{x} \in \mathcal{S}$, there exists a neighbourhood $\mathcal{U}$ of x, nonnegative integers k, m, and a morphism Ψ of sheaves $(\mathscr{R}|\mathcal{U})^{\mathrm{m}}$ and $(\mathscr{R}|\mathcal{U})^{\mathrm{k}}$ such that $\mathscr{E}|\mathcal{U}$ is isomorphic to $\mathrm{coker}(\Psi)$. The converse holds if $\mathscr{R}$ is a coherent sheaf of rings.*

Proof Since $\mathscr{E}$ is locally finitely generated, by [Hungerford 1980, Corollary IV.2.2], for every $x \in \mathcal{S}$ there exists a neighbourhood $\mathcal{U}$ of x, $k \in \mathbb{Z}_{>0}$, and a morphism Φ such that the sequence

$$(\mathscr{R}|\mathcal{U})^k \xrightarrow{\Phi} \mathscr{E}|\mathcal{U} \longrightarrow 0$$

is exact. Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k \in (\mathscr{R}|\mathcal{U})^k$ be defined by

$$[\boldsymbol{e}_j]_x = (0_x, \dots, 1_x, \dots, 0_x).$$

Thus $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k$ mimics the standard basis, as one would expect. In any case, if we define $s_1, \dots, s_k \in \mathscr{E}|\mathcal{U}$ by $[s_j]_x = \Phi_x([\boldsymbol{e}_j]_x)$, then $\Phi = \varrho(s_1, \dots, s_k)$. If $\mathscr{E}$ is additionally coherent, then $\ker(\Phi)$ is finitely generated. Thus, again by [Hungerford 1980, Corollary IV.2.2], (possibly by shrinking $\mathcal{U}$) there exists $m \in \mathbb{Z}_{>0}$ and a morphism Ψ' such that the sequence

$$(\mathscr{R}|\mathcal{U})^m \xrightarrow{\Psi'} \ker(\Phi) \longrightarrow 0$$

is exact. If ι denotes the inclusion of $\ker(\Phi)$ in $(\mathscr{R}|\mathcal{U})^k$ and if we define $\Psi = \iota \circ \Psi'$, then we have the commutative diagram

$$\begin{array}{ccccccc}
(\mathscr{R}|\mathcal{U})^m & \xrightarrow{\Psi} & (\mathscr{R}|\mathcal{U})^k & \xrightarrow{\Phi} & \mathscr{E}|\mathcal{U} & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \\
(\mathscr{R}|\mathcal{U})^m & \xrightarrow[\Psi]{} & (\mathscr{R}|\mathcal{U})^k & \longrightarrow & \mathrm{coker}(\Psi) & \longrightarrow & 0
\end{array}$$

with exact rows. One deduces the existence of an isomorphism as indicated by the dashed arrow, cf. the final part of the proof of Proposition 7.2.1. Thus $\mathscr{E}|\mathcal{U}$ is isomorphic to $\mathrm{coker}(\Psi)$.

For the proof of the converse, we rely on some results below that we have not yet proved. Since $\mathscr{R}|\mathcal{U}$ is coherent, the sheaves $(\mathscr{R}|\mathcal{U})^m$ and $(\mathscr{R}|\mathcal{U})^k$ are coherent by Proposition 7.3.6(vi). Thus we have an exact sequence

$$(\mathscr{R}|\mathcal{U})^m \xrightarrow{\Psi_{\mathcal{U}}} (\mathscr{R}|\mathcal{U})^k \longrightarrow \mathscr{E}|\mathcal{U} \longrightarrow 0$$

By Proposition 7.3.6(iii) it follows that $\mathscr{E}|\mathcal{U}$ is coherent. As this holds for a neighbourhood of every point in $\mathcal{S}$, it follows that $\mathscr{E}$ is coherent. ■

The following result was proved during the course of proving the proposition, and provides a useful means of checking the coherence of a given sheaf.

7.3.5 Corollary (A characterisation of coherent sheaves) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a coherent sheaf of rings over $\mathcal{S}$. A sheaf $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ of $\mathscr{R}$-modules is coherent if and only if*

(i) for each $x \in \mathcal{S}$, there exists $k \in \mathbb{Z}_{>0}$, a neighbourhood $\mathcal{U}$ of x, and an $\mathscr{R}|\mathcal{U}$-module homomorphism Φ for which the sequence

$$(\mathscr{R}|\mathcal{U})^k \xrightarrow{\Phi} \mathscr{E}|\mathcal{U} \longrightarrow 0$$

is exact and

(ii) for x, k, $\mathcal{U}$, and Φ as in (i), there exists $m \in \mathbb{Z}_{>0}$, a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x, and an $\mathscr{R}|\mathcal{V}$-module homomorphism Ψ for which the sequence

$$(\mathscr{R}|\mathcal{V})^m \xrightarrow{\Psi} (\mathscr{R}|\mathcal{V})^k \xrightarrow{\Phi|\mathcal{V}} \mathscr{E}|\mathcal{V} \longrightarrow 0$$

is exact.

The following result summarises some of the essential properties of coherent sheaves.

7.3.6 Proposition (Properties of coherent sheaves) *Let $(\mathcal{S}, \mathscr{O})$ be a topological space and let $\mathscr{R} = (\mathrm{R}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a sheaf of rings over $\mathcal{S}$. If $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$, $\mathscr{F} = (\mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$, and $\mathscr{G} = (\mathrm{G}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ are sheaves of $\mathscr{R}$-modules over $\mathcal{S}$, then the following statements hold:*

(i) if $\mathscr{E}$ is coherent, every finitely generated subsheaf of $\mathscr{E}$ is coherent;

(ii) if $\mathscr{E}$ is a subsheaf of $\mathscr{F}$ and if $\mathscr{F}$ is coherent, then $\mathscr{F}/\mathscr{E}$ is coherent;

(iii) if $\mathscr{E}$ and $\mathscr{F}$ are coherent and if $\Phi\colon \mathscr{E} \to \mathscr{F}$ is morphism of $\mathscr{R}$-modules, then $\ker(\Phi)$, $\mathrm{image}(\Phi)$ and $\mathrm{coker}(\Phi)$ are coherent;

(iv) if

$$0 \longrightarrow \mathscr{E} \xrightarrow{\Phi} \mathscr{F} \xrightarrow{\Psi} \mathscr{G} \longrightarrow 0$$

is an exact sequence of sheaves of $\mathscr{R}$-modules and if any two of the sheaves $\mathscr{E}$, $\mathscr{F}$, and $\mathscr{G}$ are coherent, then all three sheaves are coherent;

(v) *if $\mathscr{G}$ is coherent and if $\mathscr{E}$ and $\mathscr{F}$ are coherent subsheaves of $\mathscr{G}$, then*

$$\mathscr{E} \cap \mathscr{F} = (\mathrm{E}(\mathcal{U}) \cap \mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}, \quad \mathscr{E} + \mathscr{F} = (\mathrm{E}(\mathcal{U}) + \mathrm{F}(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$$

are coherent sheaves.

(vi) *if $\mathscr{E}$ and $\mathscr{F}$ are coherent, then $\mathscr{E} \oplus \mathscr{F}$ is coherent.*

Proof (i) Let $\mathscr{E}' = (E'(\mathcal{U}))_{\mathcal{U}\in\mathscr{O}}$ be a finitely generated subsheaf of $\mathscr{E}$, let $\mathcal{U} \in \mathscr{O}$ be open, and let $s'_1, \dots, s'_k \in E'(\mathcal{U})$. Let $\iota_{\mathcal{U}}$ be the inclusion of $E'(\mathcal{U})$ in $E(\mathcal{U})$. Then, since $s'_1, \dots, s'_k \in E'(\mathcal{U}) \subseteq E(\mathcal{U})$, $\ker(\varrho(\iota_{\mathcal{U}}(s'_1), \dots, \iota_{\mathcal{U}}(s'_k)))$ is a locally finitely generated subsheaf of $(\mathscr{R}|\mathcal{U})^k$. Since

$$\varrho(\iota_{\mathcal{U}}(s'_1), \dots, \iota_{\mathcal{U}}(s'_k))_x([f^1]_x, \dots, [f^k]_x) = \sum_{j=1}^{k}[f^j]_x[\iota_{\mathcal{U}}(s'_j)]_x = \iota_{\mathcal{U}}\Big(\sum_{j=1}^{k}[f^j]_x[s'_j]_x\Big)$$

and since $\iota_{\mathcal{U}}$ is injective,

$$\ker(\varrho(\iota_{\mathcal{U}}(s'_1), \dots, \iota_{\mathcal{U}}(s'_k))) = \ker(\varrho(s'_1, \dots, s'_k)),$$

and the result follows.

(iii) (for $\ker(\Phi)$ and $\mathrm{image}(\Phi)$) We shall first prove that $\ker(\Phi)$ and $\mathrm{image}(\Phi)$ are coherent. Note that $\mathrm{image}(\Phi)$ is locally finitely generated, being the image of a locally finitely generated sheaf. By part (i) it follows that $\mathrm{image}(\Phi)$ is coherent. Let $x_0 \in \mathcal{S}$. Since $\mathscr{E}$ is locally finitely generated, let $\mathcal{U}$ be a neighbourhood of x_0 and $s_1, \dots, s_k \in E(\mathcal{U})$ be such that $[s_1]_x, \dots, [s_k]_x$ generate $\mathscr{E}_x$ for every $x \in \mathcal{U}$. Since $\mathscr{F}$ is coherent, there exists a neighbourhood $\mathcal{U}' \subseteq \mathcal{U}$ of x_0 and $\alpha_1, \dots, \alpha_m \in (\mathscr{R}|\mathcal{U}')^k$ be such that $[\alpha_1]_x, \dots, [\alpha_m]_x$ generate $\ker(\varrho(\Phi(s_1), \dots, \Phi(s_k)))_x$ for every $x \in \mathcal{U}'$. Define $t_1, \dots, t_m \in E(\mathcal{U}')$ by

$$t_l = \sum_{j=1}^{k}\alpha_l^j s_j, \qquad l \in \{1, \dots, m\}.$$

We claim that, for every $x \in \mathcal{U}'$, $[t_1]_x, \dots, [t_m]_x$ generate $\ker(\Phi)_x$. Let $x \in \mathcal{U}'$. First of all, since

$$\Phi_x([t_l]_x) = \sum_{j=1}^{k}[\alpha_l^j]_x\Phi_x([s_j]_x),$$

it follows that $[t_l]_x \in \ker(\Phi_x)$ for every $l \in \{1, \dots, m\}$. Moreover, if $[s]_x \in \ker(\Phi_x)$ then

$$[s]_x = \sum_{j=1}^{k}[f^j]_x[s_j]_x$$

for some $[f^1]_x, \dots, [f^k]_x \in \mathscr{R}_x$, and we additionally have

$$0 = \Phi_x([s]_x) = \sum_{j=1}^{k}[f^j]_x\Phi_x([s_j]_x).$$

Thus, taking $[\alpha]_x = ([f^1]_x, \dots, [f^k]_x)$, $[\alpha_x] \in \ker(\varrho(\Phi(s_1), \dots, \Phi(s_k)))_x$ and so

$$[\alpha]_x = \sum_{l=1}^{m} [g^l]_x[\alpha_l]_x \quad \Longleftrightarrow \quad [f^j]_x = \sum_{l=1} [g^l]_x[\alpha_l^j]_x.$$

Thus

$$[s]_x = \sum_{j=1}^{k}\sum_{l=1}^{m} [g^l]_x[\alpha_l^j]_x[s_j]_x = \sum_{l=1}^{m} [g^l]_x[t_l]_x,$$

as desired.

(iv) Suppose that $\mathscr{F}$ and $\mathscr{G}$ are coherent. Since $\mathscr{E}$ corresponds is isomorphic to image$(\Phi) = \ker(\Psi)$, it is coherent by part (ii) above.

Suppose that $\mathscr{E}$ and $\mathscr{F}$ are coherent. Then $\mathscr{G}$ is locally finitely generated since it is the image of a locally finitely generated sheaf. To show coherence of $\mathscr{G}$, let $\mathcal{U} \in \mathscr{O}$, let $t_1, \dots, t_k \in G(\mathcal{U})$, and let $x_0 \in \mathcal{U}$. By Proposition 7.1.51 there exist $[s_1]_{x_0}, \dots, [s_k]_{x_0} \in \mathscr{F}_{x_0}$ such that $\Psi_{x_0}([s_j]_{x_0}) = [t_j]_{x_0}$ for each $j \in \{1, \dots, k\}$. Thus there exists a neighbourhood $\mathcal{U}'$ of x_0 such that $\Psi_x([s_j]_x) = [t_j]_x$ for each $x \in \mathcal{U}'$ and $j \in \{1, \dots, k\}$. Since $\mathscr{E}$ is finitely generated, we can (possibly after shrinking $\mathcal{U}'$) find $r_1, \dots, r_m \in E(\mathcal{U}')$ such that $[r_1]_x, \dots, [r_m]_x$ generate $\mathscr{E}_x$ for every $x \in \mathcal{U}'$. Let $\mathrm{pr}\colon (\mathscr{R}|\mathcal{U}')^{m+k} \to (\mathscr{R}|\mathcal{U}')^k$ be the projection onto the last k components. We claim that

$$\ker(\varrho(t_1, \dots, t_k))_x = \mathrm{pr}_x(\ker(\varrho(\Phi(r_1), \dots, \Phi(r_m), s_1, \dots, s_k))_x)$$

for all $x \in \mathcal{U}'$. Let $x \in \mathcal{U}'$. First suppose that $[\alpha]_x \in \ker(\varrho(t_1, \dots, t_k))_x$. Thus

$$0 = \sum_{j=1}^{k} [\alpha^j]_x[t_j]_x = \Psi_x\Big(\sum_{j=1}^{k} [\alpha^j]_x[s_j]_x\Big),$$

which, by exactness, implies that

$$\sum_{j=1}^{k} [\alpha^j]_x[s_j]_x = \sum_{l=1}^{m} [\beta^l]_x[\Phi(r_l)]_x$$

for some $[\beta]_x \in \mathscr{R}_x^m$. Thus

$$([\beta^1]_x, \dots, [\beta^m]_x, [\alpha^1]_x, \dots, [\alpha^k]_x) \in \ker(\varrho(\Phi(r_1), \dots, \Phi(r_m), s_1, \dots, s_k))_x$$

and so

$$[\alpha]_x \in \mathrm{pr}_x(\ker(\varrho(\Phi(r_1), \dots, \Phi(r_m), s_1, \dots, s_k))_x).$$

For the converse inclusion, suppose that

$$[\alpha]_x \in \mathrm{pr}_x(\ker(\varrho(\Phi(r_1), \dots, \Phi(r_m), s_1, \dots, s_k))_x)$$

so that there exists $[\beta]_x \in \mathscr{R}_x^m$ for which

$$\sum_{j=1}^{k} [\alpha^j]_x[s_j]_x = \sum_{l=1}^{m} [\beta^l]_x[\Phi(r_l)]_x.$$

By exactness this implies that

$$0 = \sum_{j=1}^{k} [\alpha^j]_x \Psi_x([s_j]_x) = \sum_{j=1}^{k} [\alpha^j]_x [t_j]_x,$$

giving the desired conclusion. This also shows that $\ker(\varrho(t_1, \dots, t_k))_x$ is locally finitely generated and so $\mathscr{G}$ is coherent.

Finally, suppose that $\mathscr{E}$ and $\mathscr{G}$ are coherent. For $x_0 \in \mathcal{S}$ let $\mathcal{U}$ be a neighbourhood of x_0 for which there are $r_1, \dots, r_m \in E(\mathcal{U})$ and $t_1, \dots, t_k \in G(\mathcal{U})$ having the property that $[r_1]_x, \dots, [r_m]_x$ and $[t_1]_x, \dots, [t_k]_x$ generate $\mathscr{E}_x$ and $\mathscr{G}_x$, respectively, for each $x \in \mathcal{U}$. As in the previous part of the proof, let $\mathcal{U}' \subseteq \mathcal{U}$ be a neighbourhood of x_0 such that there exist $s_1, \dots, s_k \in F(\mathcal{U}')$ for which $\Psi_x([s_j]_x) = [t_j]_x$ for each $j \in \{1, \dots, k\}$ and $x \in \mathcal{U}'$. We claim that $[\Phi(r_1)]_x, \dots, [\Phi(r_m)]_x, [s_1]_x, \dots, [s_k]_x$ generate $\mathscr{F}_x$ for every $x \in \mathcal{U}'$. Indeed, let $[s]_x \in \mathscr{F}_x$. suppose first that $\Psi_x([s]_x) = 0$. Exactness then gives $[s]_x$ in the span of $[\Phi(r_1)]_x, \dots, [\Phi(r_m)]_x$. On the other hand, if $\Psi_x([s]_x) \neq 0$ then $\Psi_x([s]_x)$ is in the span of $[t_1]_x, \dots, [t_k]_x$, i.e.,

$$\Psi_x\Big([s]_x - \sum_{j=1}^{k} [f^j]_x [s_j]_x\Big) = 0.$$

By exactness we have $[s]_x$ in the span of $[\Phi(r_1)]_x, \dots, [\Phi(r_m)]_x, [s_1]_x, \dots, [s_k]_x$, as desired.

Now let $\mathcal{V} \in \mathscr{O}$ be open and let $\sigma_1, \dots, \sigma_d \in F(\mathcal{V})$. Let $\tau_c = \Psi_{\mathcal{V}}(\sigma_c)$, $c \in \{1, \dots, d\}$. Let $x_0 \in \mathcal{V}$. Coherence of $\mathscr{G}$ implies that there exists a neighbourhood $\mathcal{V}' \subseteq \mathcal{V}$ of x_0 and $\alpha_1, \dots, \alpha_p \in (\mathscr{R}|\mathcal{V}')^r$ such that $[\alpha_1]_x, \dots, [\alpha_p]_x$ generate $\ker(\varrho(\tau_1, \dots, \tau_d))_x$ for every $x \in \mathcal{V}'$. Define

$$F(\mathcal{V}') \ni \beta_q = \sum_{c=1}^{d} \alpha_q^c \sigma_c, \qquad q \in \{1, \dots, p\},$$

and note that

$$\Psi_x([\beta_q]_x) = \sum_{c=1}^{d} [\alpha_q^c]_x \Psi_x([\sigma_c]_x) = \sum_{c=1}^{d} [\alpha_q^c]_x [\tau_c]_x = 0,$$

and so by exactness $[\beta_q]_x \in \operatorname{image}(\Phi_x)$ for each $q \in \{1, \dots, p\}$. By coherence of $\mathscr{E}$ it follows that there exists neighbourhood $\mathcal{V}'' \subseteq \mathcal{V}'$ of x_0 and $\gamma_1, \dots, \gamma_b \in (\mathscr{R}|\mathcal{V}'')^p$ such that $[\gamma_1]_x, \dots, [\gamma_b]_x$ generate $\ker(\varrho(\beta_1, \dots, \beta_p))_x$ for each $x \in \mathcal{V}''$. Define

$$(\mathscr{R}|\mathcal{V}'')^d \ni \kappa_a = \sum_{q=1}^{p} \gamma_a^q \alpha_q, \qquad a \in \{1, \dots, b\}.$$

We claim that $[\kappa_1]_x, \dots, [\kappa_b]_x$ generate $\ker(\varrho(\sigma_1, \dots, \sigma_d))_x$ for each $x \in \mathcal{V}''$. First of all, note that

$$\sum_{c=1}^{d} [\kappa_a^c]_x [\sigma_c]_x = \sum_{c=1}^{d} \sum_{q=1}^{p} [\gamma_a^q]_x [\alpha_q^c]_x [\sigma_c]_x = \sum_{q=1}^{p} [\gamma_a^q]_x [\beta_q]_x = 0,$$

and so $[\kappa_a]_x \in \ker(\varrho(\sigma_1, \dots, \sigma_d))_x$ for each $a \in \{1, \dots, b\}$ and $x \in \mathcal{V}''$. Now let $x \in \mathcal{V}''$. If $[\kappa]_x \in \ker(\varrho(\sigma_1, \dots, \sigma_d))_x$ then

$$\sum_{c=1}^{d} [\kappa^c]_x [\sigma_c]_x = 0 \quad \Longrightarrow \quad \sum_{c=1}^{d} [\kappa^c]_x [\tau_c]_x = 0.$$

Thus $[\kappa]_x \in \ker(\varrho(\tau_1, \dots, \tau_d))_x$ and so

$$[\kappa]_x = \sum_{q=1}^{p} [f^q]_x [\alpha_q]_x$$

for some $[f^1]_1, \dots, [f^p]_x \in \mathscr{R}_x$. Thus

$$\sum_{c=1}^{d} \sum_{q=1}^{p} [f^q]_x [\alpha_q^c]_x [\sigma_c]_x = 0,$$

which shows that $([f^1]_x, \dots, [f^p]_x) \in \ker(\varrho(\beta_1, \dots, \beta_p))_x$. Thus

$$[f^q]_x = \sum_{a=1}^{b} [g^a]_x [\gamma_a^q]_x$$

and so

$$[\kappa]_x = \sum_{q=1}^{p} \sum_{a=1}^{b} [g^a]_x [\gamma_a^q]_x [\alpha_q]_x,$$

as desired.

(ii) We have the exact sequence

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{E} \longrightarrow 0$$

and so the conclusion follows from part (iv).

(iii) (for coker(Φ)) This follows from part (ii).

(v) Here, one verifies that $\mathscr{E} \cap \mathscr{F}$ is the kernel of the composition of the inclusion of $\mathscr{E}$ in $\mathscr{G}$ and the projection of $\mathscr{G}$ onto the quotient $\mathscr{G}/\mathscr{F}$. Since $\mathscr{G}/\mathscr{F}$ is coherent by part (ii), the result follows from part (iii). For $\mathscr{E} + \mathscr{F}$ we note that this is a locally finitely generated subsheaf of $\mathscr{G}$ and so by part (i) is coherent.

(vi) In this case we have the exact sequence

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E} \oplus \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow 0$$

and from part (iv) it follows that the middle term in the sequence is coherent since the left and right terms are. ■

Let us now see how coherence allows stalkwise properties to be extended to local properties. The first such result we consider is the following.

7.3.7 Proposition (For coherent sheaves, stalkwise exactness implies local exactness) *Let* $(\mathsf{S}, \mathscr{O})$ *be a topological space, let* $\mathscr{R} = (\mathsf{R}(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$ *be a sheaf of rings over* S*, let* $\mathscr{E} = (\mathsf{E}(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$*,* $\mathscr{F} = (\mathsf{F}(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$*, and* $\mathscr{G} = (\mathsf{G}(\mathcal{U}))_{\mathcal{U} \in \mathscr{O}}$ *be coherent sheaves of* $\mathscr{R}$*-modules over* S*, and let* $\mathsf{x} \in \mathsf{S}$*. If the sequence*

$$\mathscr{E}_{\mathsf{x}} \xrightarrow{\Phi_{\mathsf{x}}} \mathscr{F}_{\mathsf{x}} \xrightarrow{\Psi_{\mathsf{x}}} \mathscr{G}_{\mathsf{x}}$$

of sheaves of $\mathscr{R}$*-modules is exact, then, for each* $x \in \mathcal{S}$*, there exists a neighbourhood* $\mathcal{U}$ *of* x *such that the sequence*

$$\mathscr{E}|\mathcal{U} \xrightarrow{\Phi_{\mathcal{U}}} \mathscr{F}|\mathcal{U} \xrightarrow{\Psi_{\mathcal{U}}} \mathscr{G}|\mathcal{U}$$

is exact.

Proof Since $\Psi_x \circ \Phi_x = 0$, $\mathscr{E}_x / \ker(\Psi_x \circ \Phi_x) = 0$. Note that the sheaf $\mathscr{E}/\ker(\Psi \circ \Phi)$ is coherent by Proposition 7.3.6(iii). Thus it is finitely generated, and so being zero at x it is zero in a neighbourhood $\mathcal{U}$ of x by Lemma 7.2.6. This implies that $\mathrm{image}(\Phi_{\mathcal{U}}) \subseteq \ker(\Psi_{\mathcal{U}})$. By parts (ii) and (ii) of Proposition 7.3.6, it follows that $\ker(\Psi_{\mathcal{U}})/\mathrm{image}(\Phi_{\mathcal{U}})$ is a coherent sheaf over $\mathcal{U}$. Moreover, as we just argued for $\mathscr{E}/\ker(\Psi \circ \Phi)$, since it is zero at x, it is zero in a neighbourhood of x, giving the desired conclusion. ■

Before we get to specific examples of coherent sheaves, let us present an important example of a sheaf that is *not* coherent.

7.3.8 Proposition (The sheaf of smooth functions is not coherent) *If* M *is a smooth manifold whose connected components have positive dimension, then* $\mathscr{C}^{\infty}_{\mathsf{M}}$ *is not coherent.*

Proof It is convenient to first make some general algebraic constructions. First let us give a simple version of a more general result known as Nakayama's Lemma, referring to [Eisenbud 1995, §4.1] for further discussion.

1 Lemma *If* R *is a local ring with maximal ideal* $\mathfrak{m}$ *and if* $\mathsf{I} \subseteq \mathsf{R}$ *is a finitely generated ideal for which* $\mathsf{I} = \mathfrak{m}\mathsf{I}$*, then* $\mathsf{I} = \{0\}$*.*

Proof Let $\mathscr{I}_k$ be the set of ideals of R that admit k generators. We prove by induction on k that the lemma holds for ideals in $\mathscr{I}_k$. If $\mathsf{I} \in \mathscr{I}_0$ then the lemma holds trivially. Suppose that the lemma holds for all ideals in $\mathscr{I}_k$ for $k \in \{0, 1, \dots, m-1\}$ and let $\mathsf{I} \in \mathscr{I}_m$. Let $r_1, \dots, r_m$ be generators for I and note that, by hypothesis, $r_s \in \mathsf{I} = \mathfrak{m}\mathsf{I}$ so that

$$r_m = a_1 r_1 + \dots + a_m r_m$$

for $a_1, \dots, a_m \in \mathfrak{m}$. Note that $1 - a_m \notin \mathfrak{m}$ since otherwise we would have $(1 - a_m) + a_m = 1 \in \mathfrak{m}$. Since $\mathsf{R} \setminus \mathfrak{m}$ is comprised of units (we showed this in the proof of Corollary 2.2.19), we thus have

$$r_m = (1 - a_m)^{-1}(a_1 r_1 + \dots + a_{m-1} r_{m-1}).$$

Thus $\mathsf{I} \in \mathscr{I}_{m-1}$ and so $\mathsf{I} = \{0\}$ by the induction hypothesis. ▼

Let us say that a commutative unit ring R is ***coherent*** if every finitely generated ideal of I is finitely presented. The following elementary observation is then useful.

2 Lemma *If* $\mathscr{R}$ *is a coherent sheaf of rings over a topological space* $(\mathcal{S}, \mathscr{O})$ *then the stalks* $\mathscr{R}_x$ *are coherent rings.*

Proof Suppose that $\mathsf{I} \subseteq \mathscr{R}_x$ is a finitely generated ideal. By [Hungerford 1980, Corollary IV.2.2] we have an epimorphism

$$\mathscr{R}_x^k \xrightarrow{\Phi} \mathsf{I} \longrightarrow 0$$

of rings. Coherence of $\mathscr{R}$ ensures that $\ker(\Phi)$ is finitely generated, and so $\mathscr{R}_x$ is coherent, as desired. ▼

With these simple facts behind us, let us prove the proposition. By Lemma 2 the result is local so we can take $\mathsf{M} = \mathbb{R}^n$ and show that the ring of germs $\mathscr{C}^\infty_{\mathbf{0},\mathbb{R}^n}$ is not coherent.

Let us first consider the case $n = 1$. First recall from Proposition 2.3.5 that $\mathscr{C}^\infty_{0,\mathbb{R}}$ is a local ring with maximal ideal

$$\mathfrak{m} = \{[f]_0 \mid f(x) = 0\}.$$

Let $f \in \mathrm{C}^\infty(\mathbb{R})$ satisfy $f(x) = 0$ for $x \in \mathbb{R}_{\le 0}$ and $f(x) > 0$ for $x \in \mathbb{R}_{>0}$. Let $m_f : \mathscr{C}^\infty_{0,\mathbb{R}} \to \mathscr{C}^\infty_{0,\mathbb{R}}$ be defined by $m_f([g]_0) = [f]_0[g]_0$ and let $\mathsf{I}_f = \ker(m_f)$. Note that

$$\mathsf{I}_f = \{[g]_0 \in \mathscr{C}^\infty_{0,\mathbb{R}} \mid g(x) = 0 \text{ for } x \in \mathbb{R}_{\ge 0}\}.$$

To show that $\mathscr{C}^\infty_{0,\mathbb{R}}$ is not coherent, it suffices to show that I_f is not finitely generated, for then the finitely generated ideal image(m_f) is not finitely presented. Suppose that I_f is finitely generated. By Lemma 1 from the proof of Proposition 4.3.4 we have $\mathsf{I}_f = \mathfrak{m}\mathsf{I}_f$. By Lemma 1 above, this means that $\mathsf{I}_f = \{0\}$, which is a contradiction.

Finally, let us prove the result for general n. Given $f \in \mathrm{C}^\infty(\mathbb{R})$ define $\hat{f} \in \mathrm{C}^\infty(\mathbb{R}^n)$ by

$$\hat{f}(x_1, \dots, x_n) = f(x_1).$$

Now define $\psi : \mathscr{C}^\infty_{0,\mathbb{R}} \to \mathscr{C}^\infty_{\mathbf{0},\mathbb{R}^n}$ by $\psi([f]_0) = [\hat{f}]_{\mathbf{0}}$. It is immediate from the definition that ψ is an injective ring homomorphism. Thus ψ maps ideals of $\mathscr{C}^\infty_{0,\mathbb{R}}$ isomorphically to ideals of $\mathscr{C}^\infty_{\mathbf{0},\mathbb{R}^n}$. In particular, since $\mathscr{C}^\infty_{0,\mathbb{R}}$ contains a finitely generated ideal that is not finitely presented, so too does $\mathscr{C}^\infty_{\mathbf{0},\mathbb{R}^n}$. ■

The preceding result effectively removes sheaves of $\mathscr{C}^\infty_{\mathsf{M}}$-modules from consideration as coherent sheaves. This leaves us with the holomorphic and real analytic case.

7.3.9 Definition (Holomorphic or real analytic sheaf) Let M be a holomorphic or real analytic manifold. A ***holomorphic sheaf*** (resp. ***real analytic sheaf***) over M is a sheaf of $\mathscr{C}^{\mathrm{hol}}_{\mathsf{M}}$-modules (resp. $\mathscr{C}^{\omega}_{\mathsf{M}}$-modules). We shall sometimes use the expression ***analytic sheaf*** to stand for either "holomorphic sheaf" or "real analytic sheaf." •

In the next section we shall show that there are important classes of holomorphic and real analytic sheaves. Here we show that there are holomorphic and real analytic sheaves that are not coherent.

7.3.10 Example (An analytic sheaf that is not coherent) We continue Example 7.3.2 from above. Thus we let $r \in \{\omega, \mathrm{hol}\}$ and take $\mathbb{F} = \mathbb{R}$ if $r = \omega$ and $\mathbb{F} = \mathbb{C}$ if $r = \mathrm{hol}$. We consider $\mathsf{M} = \mathbb{F}$ and let

$$S = \{\tfrac{1}{j} \mid j \in \mathbb{Z}_{>0}\} \cup \{0\}.$$

Consider the presheaf $\mathscr{I}_S = (I_S(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ given by

$$I_S(\mathcal{U}) = \{f \in \mathrm{C}^r(\mathcal{U}) \mid f(x) = 0 \text{ for } x \in \mathcal{U} \cap S\}.$$

One can easily verify that $\mathscr{I}_S$ is a sheaf. In Example 7.3.2 we showed that $\mathscr{I}_S$ is not locally finitely generated. We also claim that $\mathscr{C}^r_{\mathbb{F}}/\mathscr{I}_S$ is not coherent. First of all, for an open set $\mathcal{U} \subseteq \mathbb{F}$, we have the exact sequence

$$\mathscr{C}^r_{\mathbb{F}}(\mathcal{U}) \longrightarrow \mathscr{C}^r_{\mathbb{F}}(\mathcal{U})/\mathscr{I}_S(\mathcal{U}) \longrightarrow 0$$

By Corollary 7.3.5, if $\mathscr{C}^r_{\mathbb{F}}/\mathscr{I}_S$ is coherent there exists $k \in \mathbb{Z}_{>0}$ and a morphism Ψ such that the sequence

$$(\mathscr{C}^r_{\mathbb{F}}(\mathcal{U}))^k \xrightarrow{\Psi} \mathscr{C}^r_{\mathbb{F}}(\mathcal{U}) \longrightarrow \mathscr{C}^r_{\mathbb{F}}(\mathcal{U})/\mathscr{I}_S(\mathcal{U}) \longrightarrow 0$$

is exact, possibly after shrinking $\mathcal{U}$. However, by exactness this implies that $\mathscr{I}_S(\mathcal{U}) = \text{image}(\Psi)$ is finitely generated, which we just showed above is generally not true, e.g., if $0 \in \mathcal{U}$. •

what We shall comprehensively develop the phenomenon of the example in , and we shall see there that there are some interesting differences between the holomorphic and real analytic cases.

For coherent real analytic sheaves, the ascending chain property of Noetherian modules holds locally, not just for germs.

7.3.11 Theorem (Noetherian-like property for coherent real analytic sheaves) *Let* M *be a real analytic manifold, let* $\mathscr{E} = (\mathrm{E}(\mathcal{U}))_{\mathcal{U}\ open}$ *be a coherent real analytic sheaf over* M*, and let* $\mathscr{E}_{\mathrm{j}} = (\mathrm{E}_{\mathrm{j}}(\mathcal{U}))_{\mathcal{U}\ open}$, $\mathrm{j} \in \mathbb{Z}_{>0}$*, be a sequence of coherent subsheaves of* $\mathscr{E}$ *satisfying* $\mathscr{E}_{\mathrm{j}} \subseteq \mathscr{E}_{\mathrm{j}+1}$ *for every* $\mathrm{j} \in \mathbb{Z}_{>0}$*. Then, for every compact set* $\mathrm{K} \subseteq \mathsf{M}$ *there exists* $\mathrm{N} \in \mathbb{Z}_{>0}$ *such that* $\mathscr{E}_{\mathrm{j,x}} = \mathscr{E}_{\mathrm{N,x}}$ *for every* $\mathrm{x} \in \mathrm{K}$ *and for every* $\mathrm{j} \geq \mathrm{N}$*.*

Proof Since any compact set will intersect only finitely many connected components of M, we can without loss of generality suppose that M is connected.

We first prove that if, for every $x \in \mathsf{M}$, there exists a neighbourhood $\mathcal{U}_x$ of x and $N_x \in \mathbb{Z}_{>0}$ such that $\mathscr{E}_{j,y} = \mathscr{E}_{N_x,y}$ for every $y \in \mathcal{U}_x$ and $j \geq N_x$, then the conclusions of the theorem follow if $\mathscr{E}$ and $(\mathscr{E}_j)_{j\in\mathbb{Z}_{>0}}$ satisfy the hypotheses of the theorem. Indeed, let $K \subseteq \mathsf{M}$ be compact. Let $x_1, \dots, x_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{U}_{x_j}$ and take $N = \{N_{x_1}, \dots, N_{x_k}\}$. Then we clearly have $\mathscr{E}_{j,x} = \mathscr{E}_{N,x}$ for every $x \in K$ and $j \geq N$.

We next prove that if the theorem holds for $\mathscr{E} = (\mathscr{C}^\omega_{\mathsf{M}})^k$ for every $k \in \mathbb{Z}_{>0}$, then it holds for a general sheaf $\mathscr{E}$. Let $x \in K$ and let $\mathcal{U}_x$ be a neighbourhood of x such that there exists an epimorphism $\kappa_x \colon (\mathscr{C}^\omega_{\mathcal{U}_x})^{k_x} \to \mathscr{E}|\mathcal{U}_x$ for some $k_x \in \mathbb{Z}_{>0}$ (this being possible by Proposition 7.3.4). If $(\mathscr{E}_j)_{j\in\mathbb{Z}_{>0}}$ is as in the statement of the theorem, then the sequence $(\kappa_x^{-1}(\mathscr{E}_j))_{j\in\mathbb{Z}_{>0}}$ also satisfies the hypotheses of the theorem. Let $\mathcal{V}_x \subseteq \mathcal{U}_x$ be a relatively compact neighbourhood such that $\mathrm{cl}(\mathcal{V}_x) \subseteq \mathcal{U}_x$. Our assumption that the theorem holds for $(\mathscr{C}^\omega_{\mathcal{U}_x})^{k_x}$ implies that there exists $N_x \in \mathbb{Z}_{>0}$ such that $\kappa_x^{-1}(\mathscr{E}_j)_y = \kappa_x^{-1}(\mathscr{E}_{N_x})_y$ for every $y \in \mathcal{V}_x$ and $j \geq N_x$. This directly implies that $\mathscr{E}_{j,y} = \mathscr{E}_{N,y}$ for every $y \in \mathcal{V}_x$ and $j \geq N_x$. By the first part of the proof, the theorem holds for $(\mathscr{E}_j)_{j\in\mathbb{Z}_{>0}}$.

Let us next prove by induction that if the theorem holds for $\mathscr{E} = \mathscr{C}^\omega_{\mathsf{M}}$ then it holds for $\mathscr{E} = (\mathscr{C}^\omega_{\mathsf{M}})^k$ for any $k \in \mathbb{Z}_{>0}$. Let $K \subseteq \mathsf{M}$ be compact. Consider the exact sequence

$$0 \longrightarrow \mathscr{C}^\omega_{\mathsf{M}} \xrightarrow{\iota} (\mathscr{C}^\omega_{\mathsf{M}})^k \xrightarrow{\mathrm{pr}} (\mathscr{C}^\omega_{\mathsf{M}})^{k-1} \longrightarrow 0$$

where ι is the inclusion of the first factor and pr is the projection onto the last $k-1$ factors. Let $(\mathscr{E}_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of subsheaves of $(\mathscr{C}^\omega_{\mathsf{M}})^k$ as in the statement of the theorem. Then $(\iota^{-1}(\mathscr{E}_j))_{j\in\mathbb{Z}_{>0}}$ and $(\mathrm{pr}(\mathscr{E}_j))_{j\in\mathbb{Z}_{>0}}$ are sequences of subsheaves of $\mathscr{C}^\omega_{\mathsf{M}}$ and $(\mathscr{C}^\omega_{\mathsf{M}})^{k-1}$, respectively, that satisfy the hypotheses of the theorem. By the induction hypothesis, there exists

$N \in \mathbb{Z}_{>0}$ such that $\iota^{-1}(\mathscr{E}_j)_x = \iota^{-1}(\mathscr{E}_N)_x$ and $\mathrm{pr}(\mathscr{E}_j)_x = \mathrm{pr}(\mathscr{E}_N)_x$ for every $x \in K$ and $j \geq N$. We then directly verify that $\mathscr{E}_{j,x} = \mathscr{E}_{N,x}$ for every $x \in K$ and $j \geq N$.

Finally we prove by induction on $\dim(\mathsf{M})$ that the theorem holds when $\mathscr{E} = \mathscr{C}^\omega_{\mathsf{M}}$. In this case, $E_j(\mathcal{U})$ is an ideal of $E(\mathcal{U})$ for every $j \in \mathbb{Z}_{>0}$ and open $\mathcal{U} \subseteq \mathsf{M}$. Let us also suppose (since otherwise the theorem in this case is trivial) that $\mathscr{E}_k$ is not the zero sheaf for some $k \in \mathbb{Z}_{>0}$. By the identity principle, Theorem 4.1.5, it follows that $\mathscr{E}_{k,x} \neq \{0\}$ for every $x \in \mathsf{M}$.

We consider first the case when $\dim(\mathsf{M}) = 1$. Let $x_0 \in \mathsf{M}$ and let $(\mathcal{U}, \phi)$ be a chart about x such that $\phi(x_0) = 0$. As we saw in the proof of Theorem 2.3.4, $\mathscr{E}_{k,x_0}$ is either equal to $\mathscr{C}^\omega_{x_0,\mathsf{M}}$ or is generated by $[x^m]$, where $x\colon \mathcal{U} \to \mathbb{F}$ is the coordinate function and for some $m \in \mathbb{Z}_{>0}$. Now let ■ finish

7.3.2 The Oka Coherence Theorem

Having in the previous section introduced coherent sheaves and their basic properties, in this section we introduce a large class of coherent real analytic sheaves. The result here was proved by Oka [1950], and our proof follows that of Hörmander [1973].

7.3.12 Theorem (Oka Coherence Theorem) *Let $\mathrm{r} \in \{\omega, \mathrm{hol}\}$ and let $\mathbb{F} = \mathbb{R}$ if $\mathrm{r} = \omega$ and let $\mathbb{F} = \mathbb{C}$ if $\mathrm{r} = \mathrm{hol}$. If $\pi\colon \mathsf{E} \to \mathsf{M}$ is a vector bundle of class C^r then $\mathscr{G}^\mathrm{r}_{\mathsf{E}}$ is a coherent sheaf of $\mathscr{C}^\mathrm{r}_{\mathsf{M}}$-modules.*

Proof We can without loss of generality assume that E is a local vector bundle: $\mathsf{E} = \mathcal{U} \times \mathbb{F}^m$. The base space is $\mathcal{U}$ and the vector bundle projection is $\pi(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{x}$. By considering only principal parts, sections $\boldsymbol{\xi}\colon \mathcal{U} \to \mathcal{U} \times \mathbb{F}^m$ become mappings $\boldsymbol{f}\colon \mathcal{U} \to \mathbb{F}^m$. Thus we suppose that we have $\boldsymbol{f}_1, \dots, \boldsymbol{f}_k \in \mathrm{C}^\omega(\mathcal{U}; \mathbb{F}^m)$, and we define

$$\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{f}_1, \dots, \boldsymbol{f}_k) = \Big\{[\Phi]_{\boldsymbol{x}} \in \mathscr{C}^\omega_{\boldsymbol{x}}(\mathcal{U}; \mathbb{F}^k) \;\Big|\; \sum_{j=1}^k [\Phi^j]_{\boldsymbol{x}} [\boldsymbol{f}_j]_{\boldsymbol{x}} = \boldsymbol{0}\Big\}.$$

To prove the theorem, we must show that, for any $\boldsymbol{x}_0 \in \mathcal{U}$, there exists a neighbourhood $\mathcal{U}_0 \subseteq \mathcal{U}$ of $\boldsymbol{x}_0$ and $\Phi_1, \dots, \Phi_r \in \mathrm{C}^\omega(\mathcal{U}_0; \mathbb{F}^k)$ such that $\mathsf{R}_{\boldsymbol{x}}(\boldsymbol{f}_1, \dots, \boldsymbol{f}_k)$ is generated by $[\Phi_1]_{\boldsymbol{x}}, \dots, [\Phi_r]_{\boldsymbol{x}}$ for each $\boldsymbol{x} \in \mathcal{U}_0$. For notational simplicity, we will sometimes denote germs by $\mathscr{C}^\omega_{\boldsymbol{x},\mathbb{F}^n}$ (and similarly for germs of mappings) in order to not have to keep track of the name of specific neighbourhoods. Also for notational simplicity we suppose that $\boldsymbol{x}_0 = \boldsymbol{0}$.

By Proposition 7.3.6(vi) we observe that if, for fixed n, the theorem holds for $m = 1$ it holds for arbitrary m. We observe that the theorem holds vacuously when $n = 0$. So we assume that the theorem has been proved for arbitrary m and for $n \in \{0, 1, \dots, n_0\}$, and take $m = 1$ and $n = n_0 + 1$. In the usual way, we denote a point in $\mathbb{F}^{n_0+1}$ by $(\boldsymbol{x}, y)$, and we suppose that the neighbourhood of $\mathbb{F}^{n_0+1}$ in which we work is of the form $\mathcal{U} \times \mathcal{V}$ for a neighbourhood $\mathcal{U} \subseteq \mathbb{F}^{n_0}$ of $\boldsymbol{0}$ and a neighbourhood $\mathcal{V} \subseteq \mathbb{F}$ of 0. We first suppose that all components of $f_1, \dots, f_k$ are Weierstrass polynomials, and denote $f_j = W_j$, $j \in \{1, \dots, k\}$. Thus $W_j \in \mathrm{C}^\omega(\mathcal{U})[\eta]$, possible after shrinking $\mathcal{U}$ and $\mathcal{V}$. We let d be the maximum of the degrees of $W_1, \dots, W_k$.

We wish to understand the character of $\mathsf{R}_{(\boldsymbol{x},y)}(W_1, \dots, W_k)$ for $(\boldsymbol{x}, y) \in \mathcal{U} \times \mathcal{V}$. To do this, if $[f]_{(\boldsymbol{x},y)} \in \mathscr{C}^\omega_{(\boldsymbol{x},y),\mathbb{F}^{n_0}\times\mathbb{F}}$ we define $[f']_{(\boldsymbol{0},0)} \in \mathscr{C}^\omega_{(\boldsymbol{0},0),\mathbb{F}^{n_0}\times\mathbb{F}}$ by taking $f'(\boldsymbol{u}, v) = f(\boldsymbol{x} + \boldsymbol{u}, y + v)$ for $(\boldsymbol{u}, v)$ in a neighbourhood of $(\boldsymbol{0}, 0)$. We can do this for germs of mappings taking values in

Euclidean spaces as well, and, up to the end of the proof of the lemma we are about to state and prove, we will use without comment the $'$ to denote a germ at $(\mathbf{0}, 0)$ corresponding to a germ at (x, y). In particular, we let $W'_1, \dots, W'_k \in \mathscr{C}^{\omega}_{(\mathbf{0},0),\mathbb{F}^{n_0}\times\mathbb{F}}$ correspond to $W_1, \dots, W_k$. Note, then, that

$$\mathsf{R}_{(\mathbf{0},0)}(W'_1, \dots, W'_k) = \{[\Phi']_{(\mathbf{0},0)} \mid [\Phi]_{(x,y)} \in \mathsf{R}_{(x,y)}(W_1, \dots, W_k)\}.$$

Note also that $W'_1, \dots, W'_k$ are polynomials in v of degree the same as $W_1, \dots, W_k$ are in y, but are not necessarily Weierstrass polynomials.

The following lemma is now useful.

1 **Lemma** *For each* $(\mathbf{x}, y) \in \mathcal{U} \times \mathcal{V}$ *the* $\mathscr{C}^{\omega}_{(\mathbf{x},y),\mathbb{F}^{n_0}\times\mathbb{F}}$*-module* $\mathsf{R}_{(\mathbf{x},y)}(W_1, \dots, W_k)$ *is generated by its elements of the form* $([P_1]_{(\mathbf{x},y)}, \dots, [P_k]_{(\mathbf{x},y)})$ *where* $[P_j]_{(\mathbf{x},y)} \in \mathscr{C}^{\omega}_{\mathbf{x},\mathbb{F}^{n_0}}[\eta]$, $j \in \{1, \dots, k\}$, *are polynomial functions of* y *of degree at most* d.

Proof Without loss of generality, suppose that $\deg(W'_k) = d$. By the Weierstrass Preparation Theorem, thinking of $[W'_k]_{(\mathbf{0},0)}$ as an element of $\mathscr{C}^{\omega}_{(\mathbf{0},0),\mathbb{F}^{n_0}\times\mathbb{F}'}$ write $[W'_k]_{(\mathbf{0},0)} = [E']_{(\mathbf{0},0)}[W']_{(\mathbf{0},0)}$ for a unit $[E']_{(\mathbf{0},0)} \in \mathscr{C}^{\omega}_{(\mathbf{0},0),\mathbb{F}^{n_0}\times\mathbb{F}}$ and a Weierstrass polynomial W'. Note that W'_k is a polynomial in v of degree d and so, by Lemma 2.1.4(i), E is a polynomial in v, and its highest degree coefficient must therefore be 1. Let $d_{E'}$ and $d_{W'}$ be the degrees of E' and W', respectively, noting that $d_{E'} + d_{W'} = d$.

For $[\Phi']_{(\mathbf{0},0)} \in \mathsf{R}_{(\mathbf{0},0)}(W'_1, \dots, W'_k)$, by the Weierstrass Preparation Theorem, write

$$[\Phi'^{j}]_{(\mathbf{0},0)} = [Q'^{j}]_{(\mathbf{0},0)}[W'_k]_{(\mathbf{0},0)} + [R'^{j}]_{(\mathbf{0},0)}, \qquad j \in \{1, \dots, k-1\},$$

where R'^{j} is a polynomial of degree less than $d_{W'}$. Define

$$[R'^{k}]_{(\mathbf{0},0)} = [\Phi'^{k}]_{(\mathbf{0},0)} + \sum_{j=1}^{k-1} [Q'^{j}]_{(\mathbf{0},0)}[W'_j]_{(\mathbf{0},0)}.$$

We claim that $([R'^{1}]_{(\mathbf{0},0)}, \dots, [R'^{k}]_{(\mathbf{0},0)}) \in \mathsf{R}_{(\mathbf{0},0)}(W'_1, \dots, W'_k)$. Indeed,

$$\begin{aligned}\sum_{j=1}^{k} [R'^{j}]_{(\mathbf{0},0)}[W'_j]_{(\mathbf{0},0)} &= \sum_{j=1}^{k-1} \big([\Phi'^{j}]_{(\mathbf{0},0)} - [Q'^{j}]_{(\mathbf{0},0)}[W'_k]_{(\mathbf{0},0)}\big)[W'_j]_{(\mathbf{0},0)} \\ &\quad + [\Phi'^{k}]_{(\mathbf{0},0)}[W'_k]_{(\mathbf{0},0)} + \sum_{j=1}^{k-1} [Q'^{j}]_{(\mathbf{0},0)}[W'_j]_{(\mathbf{0},0)}[W'_k]_{(\mathbf{0},0)} \\ &= \sum_{j=1}^{k} [\Phi'^{j}]_{(\mathbf{0},0)}[W'_j]_{(\mathbf{0},0)} = 0.\end{aligned}$$

Thus we have

$$\sum_{j=1}^{k-1} [R'^{j}]_{(\mathbf{0},0)}[W'_j]_{(\mathbf{0},0)} + [R'^{k}]_{(\mathbf{0},0)}[E']_{(\mathbf{0},0)}[W']_{(\mathbf{0},0)} = 0.$$

The sum on the left is one whose terms are polynomial in v of degree less than $d_{W'} + d$ which implies that $[R'^{k}]_{(\mathbf{0},0)}[E']_{(\mathbf{0},0)}[W']_{(\mathbf{0},0)}$ is a polynomial in v of degree less than $d_{W'} + d$.

Since $[W']_{(\mathbf{0},0)}$ is a Weierstrass polynomial of degree $d_{W'}$, from Lemma 2.1.4(i) we have that $[R'^k]_{(\mathbf{0},0)}[E']_{(\mathbf{0},0)}$ must be a polynomial, and so have degree less than d. Recalling that $[E']_{(\mathbf{0},0)}$ is a unit, we can write

$$[R'^k]_{(\mathbf{0},0)} = [E']^{-1}_{(\mathbf{0},0)}[E']_{(\mathbf{0},0)}[R'^k]_{(\mathbf{0},0)},$$

showing that the polynomial degree of $[R'^k]_{(\mathbf{0},0)}$ has degree less than d, as do $[R'^1]_{(\mathbf{0},0)}, \dots, [R'^{(k-1)}]_{(\mathbf{0},0)}$. Now we write

$$\begin{aligned}([\Phi'^1]_{(\mathbf{0},0)}, \dots, [\Phi'^k]_{(\mathbf{0},0)}) &= ([W'_k]_{(\mathbf{0},0)}, \dots, 0, -[W'_1]_{(\mathbf{0},0)})[Q'^1]_{(\mathbf{0},0)} \\ &+ \dots + ([0, \dots, [W'_k]_{(\mathbf{0},0)}, [W'_{k-1}]_{(\mathbf{0},0)}[Q'^{(k-1)}]_{(\mathbf{0},0)} + ([R'^1]_{(\mathbf{0},0)}, \dots, [R'^k]_{(\mathbf{0},0)}).\end{aligned}$$

Thus shows that elements of $\mathsf{R}_{(\mathbf{0},0)}(W'_1, \dots, W'_k)$ are linear combinations of vectors whose components are polynomials in v with degree at most d. By "unpriming" this equation we get

$$\begin{aligned}([\Phi^1]_{(x,y)}, \dots, [\Phi^k]_{(x,y)}) &= ([W_k]_{(x,y)}, \dots, 0, -[W_1]_{(x,y)})[Q^1]_{(x,y)} \\ &+ \dots + ([0, \dots, [W_k]_{(x,y)}, [W_{k-1}]_{(x,y)}[Q^{k-1}]_{(x,y)} + ([R^1]_{(x,y)}, \dots, [R^k]_{(x,y)}),\end{aligned}$$

and the lemma follows. ▼

To complete the proof of the theorem in the case when $f_1, \dots, f_k$ are Weierstrass polynomials, let $[\Psi]_{(x,y)} \in \mathsf{R}_{(x,y)}(W_1, \dots, W_k)$ have the form of the lemma. Thus

$$[\Psi^j]_{(x,y)} = \sum_{a=0}^{d} [\psi^{ja}]_x [y^a]_{(x,0)}, \qquad j \in \{1, \dots, k\},$$

for $[\psi^{ja}]_x \in \mathscr{C}^{\omega}_{\mathbf{0},\mathbb{F}^{n_0}}[\eta]$, $j \in \{1, \dots, k\}$, $a \in \{0, 1, \dots, d\}$. Note that we must have

$$\sum_{j=1}^{k} \sum_{a=0}^{d} [\psi^{ja}]_{(x,y)} [y^a]_{(x,0)} [W_j]_{(x,y)} = 0, \qquad j \in \{1, \dots, k\}.$$

Since $W_1, \dots, W_k$ are polynomials with degree at most d, this preceding equation is a polynomial equation in η of degree at most $2d$. Let $[C^b]_x \in \mathscr{C}^{\omega}_{x,\mathbb{F}^{n_0}}$, $b \in \{0, 1, \dots, 2d\}$, be the coefficient of y^a for the expression on the left, noting that this will be a linear function of the coefficients $[\psi^{ja}]_x$, $j \in \{1, \dots, k\}$, $a \in \{1, \dots, d\}$. Thus we can write

$$[C^b]_x = \sum_{j=1}^{k} \sum_{a=1}^{d} [c^b_{ja}]_x [\psi^{ja}]_x = 0, \qquad b \in \{0, 1, \dots, 2d\}, \tag{7.5}$$

where $c^b_{ja} \in \mathrm{C}^{\omega}(\mathcal{U})$, $b \in \{0, 1, \dots, 2d\}$, $a \in \{1, \dots, d\}$, $j \in \{1, \dots, k\}$, are real analytic functions on $\mathcal{U}$ determined from the coefficients of the polynomials $W_1, \dots, W_k$. Thus the induction hypotheses give a neighbourhood $\mathcal{U}_0$ of $(\mathbf{0}, 0)$ and functions $\phi^{ja}_s \in \mathrm{C}^{\omega}(\mathcal{U}_0)$, $s \in \{1, \dots, r\}$, $a \in \{1, \dots, d\}$, $j \in \{1, \dots, k\}$, such that every solution $[\psi^{ja}]_x$ to (7.5) is given by

$$[\psi^{ja}]_x = \sum_{s=1}^{r} [\alpha^s]_x [\phi^{ja}_s]_x, \qquad j \in \{1, \dots, k\},\ a \in \{0, 1, \dots, d\},$$

for some $[\alpha^s]_{\boldsymbol{x}} \in \mathscr{C}^{\omega}_{\boldsymbol{x},\mathbb{F}^{n_0}}$. Then, by the lemma above, if we define $\Phi_s \in \mathrm{C}^{\omega}(\mathcal{U}_0 \times \mathcal{V}; \mathbb{F}^k)$, $s \in \{1, \dots, r\}$, by

$$\Phi_s^j(\boldsymbol{x}, y) = \sum_{a=0}^{d} \phi_s^{ja}(\boldsymbol{x}) y^a, \qquad j \in \{1, \dots, k\},\ s \in \{1, \dots, r\},$$

then $[\Phi_1]_{(\boldsymbol{x},y)}, \dots, [\Phi_r]_{(\boldsymbol{x},y)}$ generate $\mathsf{R}_{(\boldsymbol{x},y)}(W_1, \dots, W_k)$ for every $(\boldsymbol{x}, y) \in \mathcal{U}_0 \times \mathcal{V}$. This proves the theorem in the case that $f_1, \dots, f_k$ are Weierstrass polynomials.

Now let us suppose that this is not necessarily the case. By Lemma 2.1.3 let $\psi\colon \mathbb{F}^{n_0+1} \to \mathbb{F}^{n_0+1}$ be an orthogonal transformation such that $\psi^* f_1, \dots, \psi^* f_k$ are normalised. Then, by the Weierstrass Preparation Theorem, write $\psi^* f_j = E_j W_j$ for $(\boldsymbol{x}, y)$ in some neighbourhood $\mathcal{U}' \times \mathcal{V}'$ of $(\boldsymbol{0}, 0)$, and where $W_1, \dots, W_k$ are Weierstrass polynomials and $E_1, \dots, E_k \in \mathrm{C}^{\omega}(\mathcal{U}' \times \mathcal{V}')$ are nonzero at $(\boldsymbol{0}, 0)$. Let us suppose that $\mathcal{U}' \times \mathcal{V}'$ is sufficiently small that E is nowhere zero. By the proof for Weierstrass polynomials above, there exists a neighbourhood $\mathcal{U}_0 \times \mathcal{V}_0 \subseteq \mathcal{U}' \times \mathcal{V}'$ of $(\boldsymbol{0}, 0)$ and $\Phi_1, \dots, \Phi_r \in \mathrm{C}^{\omega}(\mathcal{U}_0 \times \mathcal{V}_0; \mathbb{F}^k)$ such that $[\Phi_1]_{(\boldsymbol{x},y)}, \dots, [\Phi_r]_{(\boldsymbol{x},y)}$ generate $\mathsf{R}_{(\boldsymbol{x},y)}(W_1, \dots, W_k)$ for each $(\boldsymbol{x}, y) \in \mathcal{U}_0 \times \mathcal{V}_0$. One then directly sees that $[\psi_*(E^{-1}\Phi_1)]_{(\boldsymbol{x},y)}, \dots, [\psi_*(E^{-1}\Phi_r)]_{(\boldsymbol{x},y)}$ generate $\mathsf{R}_{(\boldsymbol{x},y)}(f_1, \dots, f_k)$ for each $(\boldsymbol{x}, y) \in \psi(\mathcal{U}_0 \times \mathcal{V}_0)$, where $\psi_* = (\psi^{-1})^*$. Since $\psi(\mathcal{U}_0 \times \mathcal{V}_0)$ is a neighbourhood of $(\boldsymbol{0}, 0)$, the theorem follows. ■

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