Coherent Algebraic Sheaves

Jean-Pierre Serre

Translated by Piotr Achinger and Łukasz Krupa

Introduction

We know that the cohomological methods, in particular sheaf theory, play an increasing role not only in the theory of several complex variables ([5]), but also in classical algebraic geometry (let me recall the recent works of Kodaira-Spencer on the Riemann-Roch theorem). The algebraic character of these methods suggested that it is possible to apply them also to abstract algebraic geometry; the aim of this paper is to demonstrate that this is indeed the case.

The content of the particular chapters is as follows:

Chapter I is dedicated to general sheaf theory. It contains proofs of the results of this theory needed for the two other chapters. Various algebraic operations one might perform on sheaves are described in §1; we follow quite exactly the exposition of Cartan ([2], [5]). In §2 we study coherent sheaves of modules; these generalize analytic coherent sheaves (cf. [3], [5]), admitting almost the same properties. §3 contains the definition of cohomology groups of a space X with values in a sheaf $\mathscr{F}$. In subsequent applications, X is an algebraic variety, equipped with the Zariski topology, so it is not topologically separated [1]. and the methods used by Leray [10] and Cartan [3] (basing on "partitions of unity" or "fine" sheaves) do not apply; so one is led to follow the method of Čech and define the cohomology groups $H^q(X, \mathscr{F})$ by passing to the limit with finer and finer open coverings. Another difficulty arising from the non-separatedness of X regards the "cohomology exact sequence" (cf. n^{os} 24 and 25): we could construct this exact sequence only for particular cases, yet sufficient for the purposes we had in mind (cf. n^{os} 24 and 47).

Chapter II starts with the definition of an algebraic variety, analogous to that of Weil ([17], Chapter VII), but including the case of reducible varieties (note that, contrary to Weil's usage, we reserved the word *variety* only for irreducible ones); we define the structure of an algebraic variety using the data consisting of the topology (Zariski topology) and a sub-sheaf of the sheaf of germs of functions (a sheaf of local rings). An algebraic coherent sheaf on an algebraic variety V is simply a coherent sheaf of $\mathscr{O}_V$-modules, $\mathscr{O}_V$ being the sheaf of local rings on V; we give various examples in §2. The results obtained are in fact similar to related facts concerning Stein manifolds (cf. [3], [5]): if $\mathscr{F}$ is a coherent algebraic sheaf on an affine variety V, then $H^q(V, \mathscr{F}) = 0$ for all $q > 0$ and $\mathscr{F}_x$ is generated by $H^0(V, \mathscr{F})$ for all $x \in V$. Moreover (§4), $\mathscr{F}$ is determined by $H^0(V, \mathscr{F})$ considered as a module over the ring of coordinates on V.

Chapter III, concerning projective varieties, contains the results which are essential for this paper. We start with establishing a correspondence between coherent algebraic sheaves $\mathscr{F}$ on a projective space $X = \mathbb{P}_r(K)$ and graded S-modules satisfying the condition (TF) of n° 56 (S denotes the polynomial

[1] i.e. Hausdorff

algebra $K[t_0, \ldots, t_r]$); this correspondence is bijective if one identifies two S-modules whose homogeneous components differ only in low degrees (for precise statements, see n^{os} 57, 59 and 65). In consequence, every question concerning $\mathscr{F}$ could be translated into a question concerning the associated S-module M. This way we obtain a method allowing an algebraic determination of $H^q(X, \mathscr{F})$ starting from M, which in particular lets us study the properties of $H^q(X, \mathscr{F}(n))$ for n going to $+\infty$ (for the definition of $\mathscr{F}(n)$, see n° 54); the results obtained are stated in n^{os} 65 and 66. In §4, we relate the groups $H^q(X, \mathscr{F})$ to the functors Ext^q_S introduced by Cartan-Eilenberg [6]; this allows us, in §5, to study the behavior of $H^q(X, \mathscr{F}(n))$ for n tending to $-\infty$ and give a homological characterization of varieties k *times of the first kind*. §6 exposes certain properties of the Euler-Poincaré characteristic of a projective variety with values in a coherent algebraic sheaf.

Moreover, we demonstrate how one can apply the general results of this paper in diverse particular problems, and notably extend to the abstract case the "duality theorem" of [15], thus a part of the results of Kodaira-Spencer on the Riemann-Roch theorem; in these applications, the theorems of n^{os} 66, 75 and 76 play an essential role. We also show that, if the base field is the field of complex numbers, the theory of coherent algebraic sheaves is essentially identical to that of coherent analytic sheaves (cf. [4]).

Contents

Chapter I

Sheaves

§1 Operations on sheaves

1 Definition of a sheaf

Let X be a topological space. A *sheaf of abelian groups* on X (or simply a *sheaf*) consists of:

(a) *A function $x \to \mathscr{F}_x$, giving for all $x \in X$ an abelian group $\mathscr{F}_x$,*

(b) *A topology on the set $\mathscr{F}$, the sum of the sets $\mathscr{F}_x$.*

If f is an element of $\mathscr{F}_x$, we put $\pi(f) = x$; we call the mapping of π the *projection* of $\mathscr{F}$ onto X; the family in $\mathscr{F} \times \mathscr{F}$ consisting of pairs (f, g) such that $\pi(f) = \pi(g)$ is denoted by $\mathscr{F} + \mathscr{F}$.

Having stated the above definitions, we impose two axioms on the data (a) and (b):

(I) *For all $f \in \mathscr{F}$ there exist open neighborhoods V of f and U of $\pi(f)$ such that the restriction of π to V is a homeomorphism of V and U.*

(In other words, π is a local homeomorphism).

(II) *The mapping $f \mapsto -f$ is a continuous mapping from $\mathscr{F}$ to $\mathscr{F}$, and the mapping $(f, g) \mapsto f + g$ is a continuous mapping from $\mathscr{F} + \mathscr{F}$ to $\mathscr{F}$.*

We shall see that, even when X is separated (which we do not assume), $\mathscr{F}$ is not necessarily separated, which is shown by the example of the sheaf of germs of functions (cf. n° 3).

Example of a sheaf. For G an abelian group, set $\mathscr{F}_x = G$ for all $x \in X$; the set $\mathscr{F}$ can be identified with the product $X \times G$ and, if it is equipped with the product topology of the topology of X by the discrete topology on G, one obtains a sheaf, called the *constant sheaf* isomorphic with G, often identified with G.

2 Sections of a sheaf

Let $\mathscr{F}$ be a sheaf on a space X, and let U be a subset of X. By a *section* of $\mathscr{F}$ over U we mean a continuous mapping $s : U \to \mathscr{F}$ for which $\pi \circ s$ coincides with the identity on U. We therefore have $s(x) \in \mathscr{F}_x$ for all $x \in U$. The set of sections of $\mathscr{F}$ over U is denoted by $\Gamma(U, \mathscr{F})$; axiom (II) implies that $\Gamma(U, \mathscr{F})$ is an abelian group. If $U \subset V$, and if s is a section over V, the restriction of s to U is a section over U; hence we have a homomorphism $\rho_U^V : \Gamma(V, \mathscr{F}) \to \Gamma(U, \mathscr{F})$.

If U is open in X, $s(U)$ is open in $\mathscr{F}$, and if U runs over a base of the topology of X, then $s(U)$ runs over a base of the topology of $\mathscr{F}$; this is only another wording of axiom (I).

Note also one more consequence of axiom (I): for all $f \in \mathscr{F}_x$, there exists a section s over an open neighborhood of x for which $s(x) = f$, and two sections with this property coincide on an open neighborhood of x. In other words, $\mathscr{F}_x$ is an *inductive limit* of $\Gamma(U, \mathscr{F})$ for U running over the filtering order of all open neighborhoods of x.

3 Construction of sheaves

Given for all open $U \subset X$ an abelian group $\mathscr{F}_U$ and for all pairs of open $U \subset V$ a homomorphism $\phi_U^V : \mathscr{F}_V \to \mathscr{F}_U$, satisfying the transitivity condition $\phi_U^V \circ \phi_V^W = \phi_U^W$ whenever $U \subset V \subset W$.

The collection $(\mathscr{F}_U, \phi_U^V)$ allows us to define a sheaf $\mathscr{F}$ in the following way:

(a) Put $\mathscr{F}_x = \lim \mathscr{F}_U$ (inductive limit of the system of open neighborhoods of x). If x belongs to an open subset U, we have a canonical morphism ϕ_x^U : $\mathscr{F}_U \to \mathscr{F}_x$.

(b) Let $t \in \mathscr{F}_U$ and denote by $[t, U]$ the set of $\phi_x^U(t)$ for x running over U ; we have $[t, U] \subset \mathscr{F}$ and we give $\mathscr{F}$ the topology generated by $[t, U]$. Moreover, an element $f \in \mathscr{F}_x$ has a base of neighborhoods consisting of the sets $[t, U]$ for $x \in U$ and $\phi_x^U(t) = f$.

One verifies immediately that the data (a) and (b) satisfy the axioms (I) and (II), in other words, that $\mathscr{F}$ is a sheaf. We say that this is the sheaf *defined by the system* $(\mathscr{F}_U, \phi_U^V)$.

If $f \in \mathscr{F}_U$, the mapping $x \mapsto \phi_x^U(t)$ is a section of $\mathscr{F}$ over U ; hence we have a canonical morphism $\iota : \mathscr{F}_U \to \Gamma(U, \mathscr{F})$.

Proposition 1. *$\iota : \mathscr{F}_U \to \Gamma(U, \mathscr{F})$ is injective* [1] *if and only if the following condition holds:*

If an element $t \in \mathscr{F}_U$ is such that there exists an open covering $\{U_i\}$ of U with $\phi_{U_i}^U(t) = 0$ for all i, then $t = 0$.

If $t \in \mathscr{F}_U$ satisfies the condition above, we have

$$\phi_x^U(t) = \phi_x^{U_i} \circ \psi_{U_i}^U(t) = 0 \quad \text{if } x \in U_i,$$

which means that $\iota(t) = 0$. Conversely, suppose that $\iota(t) = 0$ with $t \in \mathscr{F}_U$; since $\phi_x^U(t) = 0$ for $x \in U$, there exists an open neighborhood $U(x)$ of x such that $\phi_{U(x)}^U(t) = 0$, by the definition of an inductive limit. The sets $U(x)$ form therefore an open covering of U satisfying the condition stated above.

Proposition 2. *Let U be an open subset of X, and let $\iota : \mathscr{F}_V \to \Gamma(V, \mathscr{F})$ be injective for all open $V \subset U$. Then $\iota : \mathscr{F}_U \to \Gamma(U, \mathscr{F})$ is surjective*[1] *(and therefore bijective) if and only if the following condition is satisfied:*

For all open coverings $\{U_i\}$ of U, and all systems $\{t_i\}$, $t_i \in \mathscr{F}_{U_i}$ such that $\phi_{U_i \cap U_j}^{U_i}(t_i) = \phi_{U_i \cap U_j}^{U_j}(t_j)$ for all pairs (i, j), there exists a $t \in \mathscr{F}_U$ with $\phi_{U_i}^U(t) = t_i$ for all i.

The condition is necessary: every t_i defines a section $s_i = \iota(t_i)$ over U_i, and we have $s_i = s_j$ over $U_i \cap U_j$; so there exists a section s over U which coincides with s_i over U_i for all i; if $\iota : \mathscr{F}_U \to \Gamma(U, \mathscr{F})$ is surjective, there exists $t \in \mathscr{F}_U$

[1] Recall (cf. [1]) that a function $f : E \to E'$ is **injective** if $f(e_1) = f(e_2)$ implies $e_1 = e_2$, **surjective** if $f(E) = E'$, **bijective** when it is both injective and surjective. An injective (resp. surjective, bijective) mapping is called an **injection** (resp. a **surjection**, a **bijection**).

such that $\iota(t) = s$. If we put $t'_i = \phi^U_{U_i}(t)$, the section defined by t'_i over U_i does not differ from s_i; since $\iota(t_i - t'_i) = 0$, which implies $t_i = t'_i$ for ι was supposed injective.

The condition is sufficient: if s is a section of $\mathscr{F}$ over U, there exists an open covering $\{U_i\}$ of U and elements $t_i \in \mathscr{F}_{U_i}$ such that $\iota(t_i)$ coincides with the restriction of s to U_i; it follows that the elements $\phi^{U_i}_{U_i \cap U_j}(t_i)$ and $\phi^{U_j}_{U_i \cap U_j}(t_j)$ define the same section over $U_i \cap U_j$, so, by the assumption made on ι, they are equal. If $t \in \mathscr{F}_U$ satisfies $\phi^U_{U_i}(t) = t_i$, $\iota(t)$ coincides with s over each U_i, so also over S, q.e.d.

Proposition 3. *If $\mathscr{F}$ is a sheaf of abelian groups on X, the sheaf defined by the system $(\Gamma(U, \mathscr{F}), \rho^V_U)$ is canonically isomorphic with $\mathscr{F}$.*

This is an immediate result of properties of sections stated in n° 2.

Proposition 3 shows that every sheaf can be defined by an appropriate system $(\mathscr{F}_U, \phi^V_U)$. We will see that different systems can define the same sheaf $\mathscr{F}$; however, if we impose on $(\mathscr{F}_U, \phi^V_U)$ the conditions of Propositions 1 and 2, we shall have only one (up to isomorphism) possible system: the one given by $(\Gamma(U, \mathscr{F}), \rho^V_U)$.

Example. Let G be an abelian group and denote by $\mathscr{F}_U$ the set of functions on U with values in G; define $\phi^V_U : \mathscr{F}_V \to \mathscr{F}_U$ by *restriction* of such functions. We thus obtain a system $(\mathscr{F}_U, \phi^V_U)$, and hence a sheaf $\mathscr{F}$, called the *sheaf of germs of functions* with values in G. One checks immediately that the system $(\mathscr{F}_U, \phi^V_U)$ satisfies the conditions of Propositions 1 and 2; we thus can identify sections of $\mathscr{F}$ over an open U with the elements of $\mathscr{F}_U$.

4 Glueing sheaves

Let $\mathscr{F}$ be a sheaf on X, and let U be a subset of X; the set $\pi^{-1}(U) \subset \mathscr{F}$, with the topology induced from $\mathscr{F}$, forms a sheaf over U, called a sheaf *induced* by $\mathscr{F}$ on U, end denoted by $\mathscr{F}(U)$ (or just $\mathscr{F}$, when it does not cause confusion).

We see that conversely, we can define a sheaf on X by means of sheaves on open subsets covering X:

Proposition 4. *Let $\mathsf{U} = \{U_i\}_{i \in I}$ be an open covering of X and, for all $i \in I$, let $\mathscr{F}_i$ be a sheaf over U_i; for all pairs (i, j) let θ_{ij} be an isomorphism from $\mathscr{F}_j(U_i \cap U_j)$ to $\mathscr{F}_i(U_i \cap U_j)$; suppose that we have $\theta_{ij} \circ \theta_{jk} = \theta_{ik}$ at each point of $U_i \cap U_j \cap U_k$ for all triples (i, j, k).*

Then there exists a sheaf $\mathscr{F}$ and for all i an isomorphism η_i from $\mathscr{F}(U_i)$ to $\mathscr{F}_i$, such that $\theta_{ij} = \eta_i \circ \eta_j^{-1}$ at each point of $U_i \cap U_j$. Moreover, $\mathscr{F}$ and η_i are determined up to isomorphism by the preceding conditions.

The uniqueness of $\{\mathscr{F}, \eta_i\}$ is evident; for the proof of existence, we could define $\mathscr{F}$ as a quotient space of the sum of $\mathscr{F}_i$, but we will rather use the methods of n° 3: if U is an open subset of X, let $\mathscr{F}_U$ be the group whose elements are systems $\{s_k\}_{k\in I}$ with $s_k \in \Gamma(U \cap U_k, \mathscr{F}_k)$ and $s_k = \theta_{kj}(s_j)$ on $U \cap U_j \cap U_k$; if $U \subset V$, we define ϕ_U^V in an obvious way. The sheaf defined by the system $(\mathscr{F}_U, \phi_U^V)$ is the sheaf $\mathscr{F}$ we look for; moreover, if $U \in U_i$, the mapping sending a system $\{s_k\} \in \mathscr{F}_U$ to the element $s_i \in \Gamma(U_i, \mathscr{F}_i)$ is an isomorphism from $\mathscr{F}_U$ to $\Gamma(U, \mathscr{F}_i)$, because of the transitivity condition; we so obtain an isomorphism $\eta_i : \mathscr{F}(U_i) \to \mathscr{F}_i$, which obviously satisfies the stated condition.

We say that the sheaf $\mathscr{F}$ is obtained by *glueing* the sheaves $\mathscr{F}_i$ by means of the isomorphisms θ_{ij}.

5 Extension and restriction of a sheaf

Let X be a topological space, Y its closed subspace and $\mathscr{F}$ a sheaf on X. We say that $\mathscr{F}$ is *concentrated on* Y, or that it is *zero outside of* Y if we have $\mathscr{F}_x = 0$ for all $x \in X - Y$.

Proposition 5. *If a sheaf $\mathscr{F}$ is concentrated on Y, the homomorphism*

$$\rho_Y^X : \Gamma(X, \mathscr{F}) \to \Gamma(Y, \mathscr{F}(Y))$$

is bijective.

If a section of $\mathscr{F}$ over X is zero over Y, it is zero everywhere since $\mathscr{F}_x = 0$ if $x \notin Y$, which shows that ρ_Y^X is injective. Conversely, let s be a section of $\mathscr{F}(Y)$ over Y, and extend s onto X by putting $s(x) = 0$ for $x \notin Y$; the mapping $x \mapsto s(x)$ is obviously continuous on $X - Y$; on the other hand, if $x \in Y$, there exists a section s' of $\mathscr{F}$ over an open neighborhood U of x for which $s'(x) = s(x)$; since s is continuous on Y by assumption, there exists an open neighborhood V of x, contained in U and such that $s'(y) = s(y)$ for all $y \in V \cap Y$; since $\mathscr{F}_y = 0$ if $y \notin Y$, we also have that $s'(y) = s(y)$ for $y \in V - (V \cap Y)$; hence s and s' coincide on V, which proves that s is continuous in a neighborhood of Y, so it is continuous everywhere. This shows that ρ_Y^X is surjective, which ends the proof.

We shall now prove that the sheaf $\mathscr{F}(Y)$ determines the sheaf $\mathscr{F}$ uniquely:

Proposition 6. *Let Y be a closed subspace of X, and let $\mathscr{G}$ be a sheaf on Y. Put $\mathscr{F}_x = \mathscr{G}_x$ if $x \in Y$, $\mathscr{F}_x = 0$ if $x \notin Y$, and let $\mathscr{F}$ be the sum of the sets $\mathscr{F}_x$. Then $\mathscr{F}$ admits a unique structure of a sheaf over X such that $\mathscr{F}(Y) = \mathscr{G}$.*

Let U be an open subset of X; if s is a section of $\mathscr{G}$ on $U \cap Y$, extend s by 0 on $U - (U \cap Y)$; when s runs over $\Gamma(U \cap Y, \mathscr{G})$, we obtain this way a group $\mathscr{F}_U$ of mappings from U to $\mathscr{F}$. Proposition 5 then shows that if $\mathscr{F}$ is equipped a structure of a sheaf such that $\mathscr{F}(Y) = \mathscr{G}$, we have $\mathscr{F}_U = \Gamma(U, \mathscr{F})$, which proves the uniqueness of the structure in question. The existence is proved using the methods of n° 3 applied to $\mathscr{F}_U$ and the restriction homomorphisms $\phi_U^V : \mathscr{F}_U \to \mathscr{F}_V$.

We say that a sheaf $\mathscr{F}$ is obtained by *extension of the sheaf $\mathscr{G}$ by 0 outside* Y ; we denote this sheaf by $\mathscr{G}^X$, or simply $\mathscr{G}$ if it does not cause confusion.

6 Sheaves of rings and sheaves of modules

The notion of a sheaf defined in n° 1 is that of a sheaf of *abelian groups*. It is clear that there exist analogous definitions for all algebraic structures (we could even define "sheaves of rings", where $\mathscr{F}_x$ would not admit an algebraic structure, and we only require axiom **(I)**). From now on, we will encounter mainly sheaves of *rings* and sheaves of *modules*:

A sheaf of rings $\mathscr{A}$ is a sheaf of abelian groups $\mathscr{A}_x$, $x \in X$, where each $\mathscr{A}_x$ has a structure of a ring such that the mapping $(f, g) \mapsto f \cdot g$ is a continuous mapping from $\mathscr{A} + \mathscr{A}$ to $\mathscr{A}$ (the notation being that of n° 1). We shall always assume that $\mathscr{A}_x$ has a unity element, varying continuously with x.

If $\mathscr{A}$ is a sheaf of rings satisfying the preceding condition, $\Gamma(U, \mathscr{A})$ is a ring with unity, and $\rho_U^V : \Gamma(V, \mathscr{A}) \to \Gamma(U, \mathscr{A})$ is a homomorphism of rings preserving unity if $U \subset V$. Conversely, given rings $\mathscr{A}_U$ with unity and homomorphisms $\phi_U^V : \mathscr{A}_V \to \mathscr{A}_U$ preserving unity and satisfying $\phi_U^V \circ \phi_V^W = \phi_U^W$, the sheaf $\mathscr{A}$ defined by the system $(\mathscr{A}_U, \phi_U^V)$ is a sheaf of rings. For example, if G is a ring with unity, the ring of germs of functions with values in G (defined in n° 3) is a sheaf of rings.

Let $\mathscr{A}$ be a sheaf of rings. A sheaf $\mathscr{F}$ is called a *sheaf of $\mathscr{A}$-modules* if every $\mathscr{F}_x$ carries a structure of a left unitary[2] $\mathscr{A}_x$-module, varying "continuously" with x, in the following sense: if $\mathscr{A} + \mathscr{F}$ is the subspace of $\mathscr{A} \times \mathscr{F}$ consisting of the pairs (a, f) with $\pi(a) = \pi(f)$, the mapping $(a, f) \mapsto a \cdot f$ is a continuous mapping from $\mathscr{A} + \mathscr{F}$ to $\mathscr{F}$.

If $\mathscr{F}$ is a sheaf of $\mathscr{A}$-modules, $\Gamma(U, \mathscr{F})$ is a unitary module over $\Gamma(U, \mathscr{A})$. Conversely, if $\mathscr{A}$ is defined by the system $(\mathscr{A}_U, \phi_U^V)$ as above, and let $\mathscr{F}$ be a sheaf defined by the system $(\mathscr{F}, \psi_U^V)$, where every $\mathscr{F}_U$ is a unitary $\mathscr{A}_U$-module, with $\psi_U^V(a \cdot f) = \phi_U^V(a) \cdot \psi_U^V(f)$; then $\mathscr{F}$ is a sheaf of $\mathscr{A}$-modules.

Every sheaf of abelian groups can be considered a sheaf of $\mathbb{Z}$-modules, $\mathbb{Z}$ being the constant sheaf isomorphic to the ring of integers. This will allow us to narrow our study to sheaves of modules from now on.

7 Subsheaf and quotient sheaf

Let $\mathscr{A}$ be a sheaf of rings, $\mathscr{F}$ a sheaf of $\mathscr{A}$-modules. For all $x \in X$, let $\mathscr{G}_x$ be a subset of $\mathscr{F}_x$. We say that $\mathscr{G} = \overrightarrow{\quad} \mathscr{G}_x$ is a *subsheaf* of $\mathscr{F}$ if:

(a) *$\mathscr{G}_x$ is a sub-$\mathscr{A}_x$-module of $\mathscr{F}_x$ for all $x \in X$,*

(b) *$\mathscr{G}$ is an open subset of $\mathscr{F}$.*

Condition (b) can be also expressed as:

(b') *If x is a point of X, and if s is a section of $\mathscr{F}$ over a neighborhood of x such that $s(x) \in \mathscr{G}_x$, we have $s(y) \in \mathscr{G}_y$ for all y close enough to x.*

[2] i.e. with the unity acting as identity

It is clear that, if these conditions are satisfied, $\mathscr{G}$ is a sheaf of $\mathscr{A}$-modules.

Let $\mathscr{G}$ be a subsheaf of $\mathscr{F}$ and put $\mathscr{K}_x = \mathscr{F}_x/\mathscr{G}_x$ for all $x \in X$. Give $\mathscr{K} = \quad \mathscr{K}_x$ the quotient topology of $\mathscr{F}$; we see easily that we also obtain a sheaf of $\mathscr{A}$-modules, called the *quotient sheaf* of $\mathscr{F}$ by $\mathscr{G}$, and denoted by $\mathscr{F}/\mathscr{G}$. We can give another definition, using the methods of n° 3: if U is an open subset of X, set $\mathscr{K}_U = \Gamma(U,\mathscr{F})/\Gamma(U,\mathscr{G})$ and let ϕ_U^V a homomorphism obtained by passing to the quotient with $\rho_U^V : \Gamma(V,\mathscr{F}) \to \Gamma(U,\mathscr{F})$; the sheaf defined by the system $(\mathscr{K}_U, \phi_U^V)$ coincides with $\mathscr{K}$.

The second definition of $\mathscr{K}$ shows that, if s is a section of $\mathscr{K}$ over a neighborhood of x, there exists a section t of $\mathscr{F}$ over a neighborhood of x such that the class of $t(y) \mod \mathscr{G}_y$ is equal to $s(y)$ for all y close enough to x. Of course, this does not hold globally in general: if U is an open subset of X we only have an exact sequence

$$0 \to \Gamma(U,\mathscr{G}) \to \Gamma(U,\mathscr{F}) \to \Gamma(U,\mathscr{K}),$$

the homomorphism $\Gamma(U,\mathscr{F}) \to \Gamma(U,\mathscr{K})$ not being surjective in general (cf. n° 24).

8 Homomorphisms

Let $\mathscr{A}$ be a sheaf of rings, $\mathscr{F}$ and $\mathscr{G}$ two sheaves of $\mathscr{A}$-modules. An $\mathscr{A}$-*homomorphism* (or an $\mathscr{A}$-linear homomorphism, or simply a homomorphism) from $\mathscr{F}$ to $\mathscr{G}$ is given by, for all $x \in X$, an $\mathscr{A}_x$-homomorphism $\phi_x : \mathscr{F}_x \to \mathscr{G}_x$, such that the mapping $\phi : \mathscr{F} \to \mathscr{G}$ defined by the ϕ_x is continuous. This condition can also be expressed by saying that, if s is a section of $\mathscr{F}$ over U, $x \mapsto \phi_x(s(x))$ is a section of $\mathscr{G}$ over U (we denote this section by $\phi(s)$, or $\phi \circ s$). For example, if $\mathscr{G}$ is a subsheaf of $\mathscr{F}$, the injection $\mathscr{G} \to \mathscr{F}$ and the projection $\mathscr{F} \to \mathscr{F}/\mathscr{G}$ both are homomorphisms.

Proposition 7. *Let ϕ be a homomorphisms from $\mathscr{F}$ to $\mathscr{G}$. For all $x \in X$, let $\mathscr{N}_x$ be the kernel of ϕ_x and let $\mathscr{I}_x$ be the image of ϕ_x. Then $\mathscr{N} = \quad \mathscr{N}_x$ is a subsheaf of $\mathscr{F}$, $\mathscr{I} = \quad \mathscr{I}_x$ is a subsheaf of $\mathscr{G}$ and ϕ defines an isomorphism of $\mathscr{F}/\mathscr{N}$ and $\mathscr{I}$.*

Since ϕ_x is an $\mathscr{A}_x$-homomorphism, $\mathscr{N}_x$ and $\mathscr{I}_x$ are submodules of $\mathscr{F}$ and $\mathscr{G}$ respectively, and ϕ_x defines an isomorphism of $\mathscr{F}_x/\mathscr{N}_x$ with $\mathscr{I}_x$. If on the other hand s is a local section of $\mathscr{F}$, such that $s(x) \in \mathscr{N}_x$, we have $\phi \circ s(x) = 0$, hence $\phi \circ s(y) = 0$ for y close enough to x, so $s(y) \in \mathscr{N}_y$, which shows that $\mathscr{N}$ is a subsheaf of $\mathscr{F}$. If t is a local section of $\mathscr{G}$, such that $t(x) \in \mathscr{I}_x$, there exists a local section $s \in \mathscr{F}$, such that $\phi \circ s(x) = t(x)$, hence $\phi \circ s = t$ in the neighborhood of x, showing that $\mathscr{I}$ is a subsheaf of $\mathscr{G}$, isomorphic with $\mathscr{F}/\mathscr{N}$.

The sheaf $\mathscr{N}$ is called the *kernel* of ϕ and denoted by $\mathrm{Ker}(\phi)$; the sheaf $\mathscr{I}$ is called the *image* of ϕ and denoted by $\mathrm{Im}(\phi)$; the sheaf $\mathscr{G}/\mathscr{I}$ is called the *cokernel* of ϕ and denoted by $\mathrm{Coker}(\phi)$. A homomorphism ϕ is called *injective*, or one-to-one, if each ϕ_x is injective, or equivalently if $\mathrm{Ker}(\phi) = 0$; it is called *surjective* if each ϕ_x is surjective, or equivalently if $\mathrm{Coker}(\phi) = 0$; it is called *bijective* if it is both injective and surjective, and Proposition 7 shows that it is an isomorphism of $\mathscr{F}$ and $\mathscr{G}$ and that ϕ^{-1} is a homomorphism. All the definitions related to homomorphisms of modules translate naturally to sheaves of modules; for example, a sequence of homomorphisms is called *exact* if the image of each homomorphisms coincides with the kernel of the homomorphism following it. If $\phi : \mathscr{F} \to \mathscr{G}$ is a homomorphism, the sequences:

$$0 \to \mathrm{Ker}(\phi) \to \mathscr{F} \to \mathrm{Im}(\phi) \to 0$$

$$0 \to Im(\phi) \to \mathscr{G} \to \mathrm{Coker}(\phi) \to 0$$

are exact.

If ϕ is a homomorphism from $\mathscr{F}$ to $\mathscr{G}$, the mapping $s \mapsto \phi \circ s$ is a $\Gamma(U, \mathscr{A})$-homomorphism from $\Gamma(U, \mathscr{F})$ to $\Gamma(U, \mathscr{G})$. Conversely, if $\mathscr{A}, \mathscr{F}, \mathscr{G}$ are defined by the systems $(\mathscr{A}_U, \phi_U^V)$, $(\mathscr{F}_U, \psi_U^V)$, $(\mathscr{G}_U, \chi_U^V)$ as in n° 6, and take for every open $U \subset X$ an $\mathscr{A}_U$-homomorphism $\phi_U : \mathscr{F}_U \to \mathscr{G}_U$ such that $\chi_U^V \circ \phi_V = \phi_U \circ \psi_U^V$; by passing to the inductive limit, the ϕ_U define a homomorphism $\phi : \mathscr{F} \to \mathscr{G}$.

9 The direct sum of two sheaves

Let $\mathscr{A}$ be a sheaf of rings, $\mathscr{F}$ and $\mathscr{G}$ two sheaves of $\mathscr{A}$-modules; for all $x \in X$, form the module $\mathscr{F}_x \oplus \mathscr{G}_x$, the *direct sum* of $\mathscr{F}_x$ and $\mathscr{G}_X$; an element of $\mathscr{F}_x \oplus \mathscr{G}_x$ is a pair (f, g) with $f \in \mathscr{F}_x$ and $g \in \mathscr{G}_x$. Let $\mathscr{K}$ be the sum of the sets $\mathscr{F}_x \oplus \mathscr{G}_x$ for $x \in X$; we can identify $\mathscr{K}$ with the subset of $\mathscr{F} \times \mathscr{G}$ consisting of the pairs (f, g) with $\pi(f) = \pi(g)$. We give $\mathscr{K}$ the topology induced from $\mathscr{F} \times \mathscr{G}$ and verify immediately that $\mathscr{K}$ is a sheaf of $\mathscr{A}$-modules; we call this sheaf the *direct sum* of $\mathscr{F}$ and $\mathscr{G}$, and denote it by $\mathscr{F} \oplus \mathscr{G}$. A section of $\mathscr{F} \oplus \mathscr{G}$ is of the form $x \mapsto (s(x), t(x))$, where s and t are sections of $\mathscr{F}$ and $\mathscr{G}$ over U; in other words, $\Gamma(U, \mathscr{F} \oplus \mathscr{G})$ is isomorphic to the direct sum $\Gamma(U, \mathscr{F}) \oplus \Gamma(U, \mathscr{G})$.

The definition of the direct sum extends by recurrence to a finite number of $\mathscr{A}$-modules. In particular, a direct sum of p sheaves isomorphic to one sheaf $\mathscr{F}$ is denoted by $\mathscr{F}^p$.

10 The tensor product of two sheaves

Let $\mathscr{A}$ be a sheaf of rings, $\mathscr{F}$ a sheaf right of $\mathscr{A}$-modules, $\mathscr{G}$ a sheaf of left $\mathscr{A}$-modules. For all $x \in X$ we set $\mathscr{K}_x = \mathscr{F}_x \otimes \mathscr{G}_x$, the tensor product being taken over the ring $\mathscr{A}_x$ (cf. for example [6], Chapter II, §2); let $\mathscr{K}$ be the sum of the sets $\mathscr{K}_x$.

Proposition 8. *There exists a structure of a sheaf on $\mathscr{K}$, unique with the property that if s and t are sections of $\mathscr{F}$ and $\mathscr{G}$ over an open subset U, the mapping $x \mapsto s(x) \otimes t(x) \in \mathscr{K}_x$ gives a section of $\mathscr{K}$ over U.*

The sheaf $\mathscr{K}$ thus defined is called the tensor product (over $\mathscr{A}$) of $\mathscr{F}$ and $\mathscr{G}$, and denoted by $\mathscr{F} \otimes_{\mathsf{A}} \mathscr{G}$; if the rings $\mathscr{A}_x$ are commutative, it is a sheaf of $\mathscr{A}$-modules.

If $\mathscr{K}$ has a structure of a sheaf satisfying the above condition, and if f_i and g_i are sections of $\mathscr{F}$ and $\mathscr{G}$ over an open $U \subset X$, the mapping $x \mapsto \quad s_i(x) \otimes t_i(x)$ is a section of $\mathscr{K}$ on U. In fact, all $h \in \mathscr{K}_x$ can be expressed in the form $h = \quad f_i \otimes g_i$, $f_i \in \mathscr{F}_X$, $g_i \in \mathscr{G}_x$, therefore also the form $\quad s_i(x) \otimes t_i(x)$, where s_i and t_i are defined in an open neighborhood U of x; in result, every section of $\mathscr{K}$ can be locally expressed in the preceding form, which shows the uniqueness of the structure of a sheaf on $\mathscr{K}$.

Now we show the existence. We might assume that $\mathscr{A}$, $\mathscr{F}$, $\mathscr{G}$ are defined by the systems $(\mathscr{A}_U, \phi_U^V)$, $(\mathscr{F}_U, \psi_U^V)$, $(\mathscr{G}_U, \chi_U^V)$ as in n° 6. Now set $\mathscr{K}_U = \mathscr{F}_U \otimes \mathscr{G}_U$, the tensor product being taken over $\mathscr{A}_U$; the homomorphisms ψ_U^V and χ_U^V define, by passing to the tensor product, a homomorphism $\eta_U^V : \mathscr{K}_V \to \mathscr{K}_U$; besides, we have $\lim_{x \in U} \mathscr{K}_U = \lim_{x \in U} \mathscr{F}_U \otimes \lim_{x \in U} \mathscr{G}_U = \mathscr{K}_x$, the tensor product being taken over $\mathscr{A}_x$ (for the commutativity of the tensor product with inductive limits, see for example [6], Chapter VI, Exercise 18). The sheaf defined by the system $(\mathscr{K}_U, \eta_U^V)$ can be identified with $\mathscr{K}$, and $\mathscr{K}$ is thus given a structure of a sheaf obviously satisfying the imposed condition. Finally, if the $\mathscr{A}_x$ are commutative, we can suppose that the $\mathscr{A}_U$ are also commutative (it suffices to take for $\mathscr{A}_U$ the ring $\Gamma(U, \mathscr{A})$), so $\mathscr{K}_U$ is a $\mathscr{A}_U$-module, and $\mathscr{K}$ is a sheaf of $\mathscr{A} - modules$.

Now let ϕ be an $\mathscr{A}$-homomorphism from $\mathscr{F}$ to $\mathscr{F}'$ and let ψ be an $\mathscr{A}$-homomorphism form $\mathscr{G}$ to $\mathscr{G}'$; in that case $\phi_x \otimes \psi_x$ is a homomorphism (of abelian groups in general – of $\mathscr{A}_x$-modules, if $\mathscr{A}_x$ is commutative) and the definition of $\mathscr{F} \otimes_{\mathsf{A}} \mathscr{G}$ shows that the collection of $\phi_x \otimes \psi_x$ is a homomorphism from $\mathscr{F} \otimes_{\mathsf{A}} \mathscr{G}$ to $\mathscr{F}' \otimes_{\mathsf{A}} \mathscr{G}'$; this homomorphism is denoted by $\phi \otimes \psi$; if ψ is the identity, we write ϕ instead of $\phi \otimes 1$.

All of the usual properties of the tensor product of two modules translate to the tensor product of two sheaves of modules. For example, all exact sequences:

$$\mathscr{F} \to \mathscr{F}' \to \mathscr{F}'' \to 0$$

give rise to an exact sequence:

$$\mathscr{F} \otimes_{\mathsf{A}} \mathscr{G} \to \mathscr{F}' \otimes_{\mathsf{A}} \mathscr{G} \to \mathscr{F}'' \otimes_{\mathsf{A}} \mathscr{G} \to 0.$$

We have the canonical isomorphisms:

$$\mathscr{F} \otimes_{\mathsf{A}} (\mathscr{G}_1 \oplus \mathscr{G}_2) \approx \mathscr{F} \otimes_{\mathsf{A}} \mathscr{G}_1 \oplus \mathscr{F} \otimes_{\mathsf{A}} \mathscr{G}_2, \quad \mathscr{F} \otimes_{\mathsf{A}} \mathscr{A} \approx \mathscr{F},$$

and (supposing that $\mathscr{A}_x$ are commutative, to simplify the notation):

$$\mathscr{F} \otimes_{\mathsf{A}} \mathscr{G} \approx \mathscr{G} \otimes_{\mathsf{A}} \mathscr{F}, \quad \mathscr{F} \otimes_{\mathsf{A}} (\mathscr{G} \otimes_{\mathsf{A}} \mathscr{K}) \approx (\mathscr{F} \otimes_{\mathsf{A}} \mathscr{G}) \otimes_{\mathsf{A}} \mathscr{K}.$$

11 The sheaf of germs of homomorphisms from one sheaf to another

Let $\mathscr{A}$ be a sheaf of rings, $\mathscr{F}$ and $\mathscr{G}$ two sheaves of $\mathscr{A}$-modules. If U is an open subset of X, let $\mathscr{K}_U$ be the group of homomorphisms from $\mathscr{F}(U)$ to $\mathscr{G}(U)$ (we also write "homomorphism from $\mathscr{F}$ to $\mathscr{G}$ over U" in place of "homomorphism from $\mathscr{F}(U)$ to $\mathscr{G}(U)$"). The operation of restricting a homomorphism defines $\phi_U^V : \mathscr{K}_V \to \mathscr{K}_U$; the sheaf defined by $(\mathscr{K}_U, \phi_U^V)$ is called the *sheaf of germs of homomorphisms from* $\mathscr{F}$ *to* $\mathscr{G}$ and denoted by $\mathrm{Hom}_{\mathsf{A}}(\mathscr{F}, \mathscr{G})$. If $\mathscr{A}_x$ are commutative, $\mathrm{Hom}_{\mathsf{A}}(\mathscr{F}, \mathscr{G})$ is a sheaf of $\mathscr{A}$-modules.

An element of $\mathrm{Hom}_{\mathsf{A}}(\mathscr{F}, \mathscr{G})$, being a germ of a homomorphism from $\mathscr{F}$ to $\mathscr{G}$ in a neighborhood of x, defines an $\mathscr{A}_x$-homomorphism from $\mathscr{F}_x$ to $\mathscr{G}_x$; hence a canonical homomorphism

$$\rho : \mathrm{Hom}_{\mathsf{A}}(\mathscr{F}, \mathscr{G})_x \to \mathrm{Hom}_{\mathsf{A}_x}(\mathscr{F}_x, \mathscr{G}_x).$$

But, contrary to what happened with the operations studied up to now, the homomorphism ρ is not a bijection in general ; we will give in n° 14 a sufficient condition for that.

If $\phi : \mathscr{F}' \to \mathscr{F}$ and $\psi : \mathscr{G} \to \mathscr{G}'$ are homomorphisms, we define in an obvious way a homomorphism

$$\mathrm{Hom}_{\mathsf{A}}(\phi, \psi) : \mathrm{Hom}_{\mathsf{A}}(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{\mathsf{A}}(\mathscr{F}', \mathscr{G}').$$

Every exact sequence $0 \to \mathscr{G} \to \mathscr{G}' \to \mathscr{G}''$ gives rise to an exact sequence:

$$0 \to \mathrm{Hom}_{\mathsf{A}}(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{\mathsf{A}}(\mathscr{F}, \mathscr{G}') \to \mathrm{Hom}_{\mathsf{A}}(\mathscr{F}, \mathscr{G}'').$$

We also have the canonical isomorphisms: $\mathrm{Hom}_{\mathsf{A}}(\mathscr{A}, \mathscr{G}) \approx \mathscr{G}$,

$$\mathrm{Hom}_{\mathsf{A}}(\mathscr{F}, \mathscr{G}_1 \oplus \mathscr{G}_2) \approx \mathrm{Hom}_{\mathsf{A}}(\mathscr{F}, \mathscr{G}_1) \oplus \mathrm{Hom}_{\mathsf{A}}(\mathscr{F}, \mathscr{G}_2)$$

$$\mathrm{Hom}_{\mathsf{A}}(\mathscr{F}_1 \oplus \mathscr{F}_2, \mathscr{G}) \approx \mathrm{Hom}_{\mathsf{A}}(\mathscr{F}_1, \mathscr{G}) \oplus \mathrm{Hom}_{\mathsf{A}}(\mathscr{F}_2, \mathscr{G}).$$

§2 Coherent sheaves of modules

In this paragraph, X denotes a topological space and $\mathscr{A}$ a sheaf of rings on X. We suppose that all the rings $\mathscr{A}_x$, $x \in X$ are commutative and have a unity element varying continuously with x. All sheaves considered until n° 16 are sheaves of $\mathscr{A}$-modules and all homomorphisms are $\mathscr{A}$-homomorphisms.

12 Definitions

Let $\mathscr{F}$ be a sheaf of $\mathscr{A}$-modules, and let $s_1, \ldots, s_p$ be sections of $\mathscr{F}$ over an open $U \subset X$. If we assign to any family $f_1, \ldots, f_p$ of elements of $\mathscr{A}_x$ the element $\sum_{i=1}^{i=p} f_i \cdot s_i(x)$ of $\mathscr{F}_x$, we obtain a homomorphism $\phi : \mathscr{A}^p \to \mathscr{F}$, defined over an open subset U (being precise, ϕ is a homomorphism from $\mathscr{A}^p(U)$ to $\mathscr{F}(U)$, with the notations from n° 4). The kernel $\mathscr{R}(s_1, \ldots, s_p)$ of the homomorphism ϕ is a subsheaf of $\mathscr{A}^p$, called the *sheaf of relations* between the s_i; the image of ϕ is a subsheaf of $\mathscr{F}$ generated by s_i. Conversely, any homomorphism $\phi : \mathscr{A}^p \to \mathscr{F}$ defines the sections $s_1, \ldots, s_p$ by the formulas

$$s_1(x) = \phi_x(1, 0, \ldots, 0), \quad \ldots, \quad s_p(x) = \phi_x(0, \ldots, 0, 1).$$

Definition 1. *A sheaf of $\mathscr{A}$-modules $\mathscr{F}$ is said to be* of finite type *if it is locally generated by a finite number of its sections.*

In another words, for every point $x \in X$, there exists an open neighborhood U of x and a finite number of sections $s_1, \ldots, s_p$ of $\mathscr{F}$ over U such that every element of $\mathscr{F}_y$, $y \in U$ is a linear combination, with coefficients in $\mathscr{A}_y$, of $s_i(y)$. According to the preceding statements, it is another way of saying that the restriction of $\mathscr{F}$ to U is isomorphic to a quotient sheaf of $\mathscr{A}^p$.

Proposition 1. *Let $\mathscr{F}$ be a sheaf of finite type. If $s_1, \ldots, s_p$ are sections of $\mathscr{F}$, defined over a neighborhood of a point $x \in X$ and generating $\mathscr{F}_x$, then they also generate $\mathscr{F}_y$ for all y close enough to x.*

Because $\mathscr{F}$ is of finite type, there is a finite number of sections of $\mathscr{F}$ in a neighborhood of x, say $t_1, \ldots, t_q$, which generate $\mathscr{F}_y$ for y close enough to x. Since $s_j(x)$ generate $\mathscr{F}_x$, there exist sections f_{ij} of $\mathscr{A}$ in a neighborhood of x such that $t_i(x) = \sum_{j=1}^{j=p} f_{ij}(x) \cdot s_j(x)$; it follows that, for y close enough to x, we have:

$$t_i(y) = \sum_{j=1}^{j=p} f_{ij}(y) \cdot s_j(y),$$

which implies that $s_j(y)$ generate $\mathscr{F}_y$, q.e.d.

Definition 2. *A sheaf of $\mathscr{A}$-modules $\mathscr{F}$ is said to be* coherent *if:*

(a) *$\mathscr{F}$ is of finite type,*

(b) *If $s_1, \ldots, s_p$ are sections of $\mathscr{F}$ over an open $U \subset X$, the sheaf of relations between the s_i is of finite type (over the open set U).*

We will observe the *local* character of definitions 1 and 2.

Proposition 2. *Locally, every coherent sheaf is isomorphic to the cokernel of a homomorphism* $\phi : \mathscr{A}^q \to \mathscr{A}^p$*.*

This is an immediate result of the definitions and the remarks preceding definition 1.

Proposition 3. *Any subsheaf of finite type of a coherent sheaf is coherent.*

Indeed, if a sheaf $\mathscr{F}$ satisfies condition (b) of definition 2, then any subsheaf of $\mathscr{F}$ satisfies it also.

13 Main properties of coherent sheaves

Theorem 1. *Let* $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{K} \to 0$ *be an exact sequence of homomorphisms. If two of the sheaves* $\mathscr{F}$, $\mathscr{G}$, $\mathscr{K}$ *are coherent, so is the third.*

Suppose that $\mathscr{G}$ and $\mathscr{K}$ are coherent. Locally, there exists a homomorphism $\gamma : \mathscr{A}^p \to \mathscr{G}$; let $\mathscr{I}$ the kernel of $\beta \circ \gamma$; since $\mathscr{K}$ is coherent, $\mathscr{I}$ is a sheaf of finite type (condition **(b)**); thus $\gamma(\mathscr{I})$ is a sheaf of finite type, thus coherent by Proposition 3; since α is an isomorphism from $\mathscr{F}$ to $\gamma(\mathscr{I})$, it follows that $\mathscr{F}$ is also coherent.

Suppose that $\mathscr{F}$ and $\mathscr{G}$ are coherent. Because $\mathscr{G}$ is of finite type, $\mathscr{K}$ is also of finite type, so it remains to prove that $\mathscr{K}$ satisfies the condition **(b)** of definition 2. Let $s_1, \ldots, s_p$ be a finite number of sections of $\mathscr{K}$ in a neighborhood of a point $x \in X$. The question being local, we can assume that there exist sections $s'_1, \ldots, s'_p$ of $\mathscr{G}$ such that $s_i = \beta(s'_i)$. Let $n_1, \ldots, n_q$ be a finite number of sections of $\mathscr{F}$ in a neighborhood of x, generating $\mathscr{F}_y$ for y close enough to x. A family $f_1, \ldots, f_p$ of elements of $\mathscr{A}_y$ belongs to $\mathscr{R}(s_1, \ldots, s_p)_y$ if and only if one can find $g_1, \ldots, g_q \in \mathscr{A}_y$ such that

$$\sum_{i=1}^{i=p} f_i \cdot s'_i = \sum_{j=1}^{j=q} g_j \cdot \alpha(n_j) \quad \text{in } y.$$

Now the sheaf of relations between the s'_i and the $\alpha(n_j)$ is of finite type, because $\mathscr{G}$ is coherent. The sheaf $\mathscr{R}(s_1, \ldots, s_p)$, the image of the preceding by the canonical projection from $\mathscr{A}^{p+q}$ to $\mathscr{A}^p$ is thus of finite type, which shows that $\mathscr{K}$ is coherent.

Suppose that $\mathscr{F}$ and $\mathscr{K}$ are coherent. The question being local, we might assume that $\mathscr{F}$ (resp. $\mathscr{K}$) is generated by a finite number of sections $n_1, \ldots, n_q$ (resp. $s_1, \ldots, s_p$); furthermore we might assume that there exist sections s'_i of $\mathscr{G}$ such that $s_i = \beta(s'_i)$. It is clear that the sections s'_i and $\alpha(n_j)$ generate $\mathscr{G}$, which proves that $\mathscr{G}$ is a sheaf of finite type. Now let $t_1, \ldots, t_r$ be a finite number of sections of $\mathscr{G}$ in a neighborhood of a point x; since $\mathscr{K}$ is coherent, there exist sections f^i_j or $\mathscr{A}^r$ $(1 \leq i \leq r,\ 1 \leq j \leq s)$, defined in the neighborhood of x, which generate the sheaf of relations between the $\beta(t_i)$. Put $u_j = \sum_{i=1}^{i=r} f_j^i \cdot t_i$; since $\sum_{i=1}^{i=r} f_j^i \cdot \beta(t_i) = 0$, the u_j are contained in $\alpha(\mathscr{F})$ and, since $\mathscr{F}$ is coherent, the sheaf of relations between the u_j is generated, in a neighborhood of x, by a finite number of sections, say g_k^j $(1 \leq j \leq s,\ 1 \leq k \leq t)$. I say that the $\sum_{j=1}^{j=s} g_k^j \cdot f_j^i$ generate the sheaf $\mathscr{R}(t_1, \dots, t_r)$ in a neighborhood of x; indeed, if $\sum_{i=1}^{i=r} f_i \cdot t_i = 0$ on y, with $f_i \in \mathscr{A}_y$, we have $\sum_{i=1}^{i=r} f_i \cdot \beta(t_i) = 0$ and there exist $g_j \in \mathscr{A}_y$ with $f_i = \sum_{j=1}^{j=s} g_j f_j^i$; noting that $\sum_{i=1}^{i=r} f_i \cdot t_i = 0$, one obtains $\sum_{j=1}^{j=s} g_j \cdot u_j = 0$, thus making the system g_j a linear combination of the systems g_k^j and showing our assertion. It follows that $\mathscr{G}$ satisfies condition **(b)**, which ends the proof.

Corollary. *A direct sum of a finite family of coherent sheaves is coherent.*

Theorem 2. *Let ϕ be a homomorphism from a coherent sheaf $\mathscr{F}$ to a coherent sheaf $\mathscr{G}$. The kernel, the cokernel and the image of ϕ are also coherent sheaves.*

Because $\mathscr{F}$ is coherent, $\Im(\phi)$ is of finite type, thus coherent by Proposition 3. We apply Theorem 1 to the exact sequences

$$0 \to \mathrm{Ker}(\phi) \to \mathscr{F} \to \mathrm{Im}(\phi) \to 0$$

$$0 \to Im(\phi) \to \mathscr{G} \to \mathrm{Coker}(\phi) \to 0$$

seeing that $\mathrm{Ker}(\phi)$ and $\mathrm{Coker}(\phi)$ are also coherent.

Corollary. *Let $\mathscr{F}$ and $\mathscr{G}$ be two coherent subsheaves of a coherent sheaf $\mathscr{K}$. The sheaves $\mathscr{F} + \mathscr{G}$ and $\mathscr{F} \cap \mathscr{G}$ are coherent.*

For $\mathscr{F} + \mathscr{G}$, this follows from Proposition 3; and for $\mathscr{F} \cap \mathscr{G}$, this is the kernel of $\mathscr{F} \to \mathscr{K}/\mathscr{G}$.

14 Operations on coherent sheaves

We have just seen that a direct sum of a finite number coherent sheaves is a coherent sheaf. We will now show analogous results for the functors $\otimes$ and Hom.

Proposition 4. *If $\mathscr{F}$ and $\mathscr{G}$ are two coherent sheaves, $\mathscr{F} \otimes_{\mathsf{A}} \mathscr{G}$ is a coherent sheaf.*

By Proposition 2, $\mathscr{F}$ is locally isomorphic to the cokernel of a homomorphism $\phi : \mathscr{A}^q \to \mathscr{A}^p$; thus $\mathscr{F} \otimes_{\mathsf{A}} \mathscr{G}$ is locally isomorphic to the cokernel of $\phi : \mathscr{A}^q \otimes_{\mathsf{A}} \mathscr{G} \to \mathscr{A}^p \otimes_{\mathsf{A}} \mathscr{G}$. But $\mathscr{A}^q \otimes_{\mathsf{A}} \mathscr{G}$ and $\mathscr{A}^p \otimes_{\mathsf{A}} \mathscr{G}$ are isomorphic to $\mathscr{G}^q$ and $\mathscr{G}^p$ respectively, which are coherent (Corollary of Theorem 1). Thus $\mathscr{F} \otimes_{\mathsf{A}} \mathscr{G}$ is coherent (Theorem 2).

Proposition 5. *Let $\mathscr{F}$ and $\mathscr{G}$ be two sheaves, $\mathscr{F}$ being coherent. For all $x \in X$, the module $\mathrm{Hom}_{\mathsf{A}}(\mathscr{F}, \mathscr{G})_x$ is isomorphic to $\mathrm{Hom}_{\mathsf{A}_x}(\mathscr{F}_x, \mathscr{G}_x)$.*

Precisely, we prove that the homomorphism

$$\rho : \mathrm{Hom}_{\mathsf{A}}\,(\mathscr{F}, \mathscr{G})_x \to \mathrm{Hom}_{\mathsf{A}}\,(\mathscr{F}, \mathscr{G})_x,$$

defined in n° 11, is bijective. First of all, let $\psi : \mathscr{F} \to \mathscr{G}$ be a homomorphism defined in a neighborhood of x, being zero in $\mathscr{F}_x$; since $\mathscr{F}$ is of finite type, we conclude immediately that ψ is zero in a neighborhood of x, which proves that ρ is injective. We will show that ρ is surjective, or in other words, that if ϕ is a $\mathscr{A}_x$-homomorphism from $\mathscr{F}_x$ to $\mathscr{G}_x$, there exists a homomorphism $\psi : \mathscr{F} \to \mathscr{G}$, defined in a neighborhood of x and such that $\psi_x = \phi$. Let $m_1, \ldots, m_p$ be a finite number of sections of $\mathscr{F}$ in a neighborhood of x, generating $\mathscr{F}_y$ for all y close enough to x, and let f^i_j $(1 \leq i \leq p,\ 1 \leq j \leq q)$ be sections of $\mathscr{A}^p$ generating $\mathscr{R}(m_1, \ldots, m_p)$ in a neighborhood of x. There exist local sections of $\mathscr{G}$, say $n_1, \ldots, n_p$, such that $n_i(x) = \phi(m_i(x))$. Put $p_j = \sum_{i=1}^{i=p} f^i_j \cdot n_i$, $1 \leq j \leq q$; the p_j are local sections of $\mathscr{G}$ being zero in x, so in every point of a neighborhood U of x. It follows that for $y \in U$, the formula $\sum f_i \cdot m_i(y) = 0$ with $f_i \in \mathscr{A}_y$, implies $\sum f_i \cdot n_i(y) = 0$; for any element $m = \sum f_i \cdot m_i(y) \in \mathscr{F}_y$, we thus can put:

$$\psi_y(m) = \sum_{i=1}^{i=p} f_i \cdot n_i(y) \in \mathscr{G}_y.$$

The collection of ψ_y, $y \in U$ constitutes a homomorphism $\psi : \mathscr{F} \to \mathscr{G}$, defined over U and such that $\psi_x = \phi$, which ends the proof.

Proposition 6. *If $\mathscr{F}$ and $\mathscr{G}$ are two coherent sheaves, then* $\mathrm{Hom}_{\mathsf{A}}\,(\mathscr{F}, \mathscr{G})$ *is a coherent sheaf.*

The question being local, we might assume, by Proposition 2, that we have an exact sequence $\mathscr{A}^q \to \mathscr{A}^p \to \mathscr{F} \to 0$. From the preceding Proposition it follows that the sequence:

$$0 \to \mathrm{Hom}_{\mathsf{A}}\,(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{\mathsf{A}}\,(\mathscr{A}^p, \mathscr{G}) \to \mathrm{Hom}_{\mathsf{A}}\,(\mathscr{A}^q, \mathscr{G})$$

is exact. Now the sheaf $\mathrm{Hom}_{\mathsf{A}}\,(\mathscr{A}^p, \mathscr{G})$ is isomorphic to $\mathscr{G}^p$, thus is coherent, the same for $\mathrm{Hom}_{\mathsf{A}}\,(\mathscr{A}^q, \mathscr{G})$. Theorem 2 then shows that $\mathrm{Hom}_{\mathsf{A}}\,(\mathscr{F}, \mathscr{G})$ is coherent.

15 Coherent sheaves of rings

A sheaf of rings $\mathscr{A}$ can be regarded as a sheaf of $\mathscr{A}$-modules; if this sheaf of $\mathscr{A}$-modules is coherent, we say that $\mathscr{A}$ is a *coherent sheaf of rings*. Since $\mathscr{A}$ is clearly of finite type, this means that $\mathscr{A}$ satisfies condition **(b)** of Proposition 2. In other words:

Definition 3. *A sheaf $\mathscr{A}$ is a* coherent sheaf of rings *if the sheaf of relations between a finite number of sections of $\mathscr{A}$ over an open subset U is a sheaf of finite type on U.*

Examples. **(1)** If X is a complex analytic variety, the sheaf of germs of holomorphic functions on X is a coherent sheaf of rings, by a theorem of K. Oka (cf. [3], statement XV, or [5], §5).

(2) If X is an algebraic variety, the sheaf of local rings of X is a coherent sheaf of rings (cf. n° 37, Proposition 1).

When $\mathscr{A}$ is a coherent sheaf of rings, we have the following results:

Proposition 7. *For a sheaf of $\mathscr{A}$-modules, being coherent is equivalent to being locally isomorphic to the cokernel of a homomorphism $\phi : \mathscr{A}^q \to \mathscr{A}^p$.*

The necessity part is Proposition 2; the sufficiency follows from the coherence of $\mathscr{A}^p$ and $\mathscr{A}^q$ and from Theorem 2.

Proposition 8. *A subsheaf of $\mathscr{A}$ is coherent if and only if it is of finite type.*

This is a special case of Proposition 3.

Corollary. *The sheaf of relations between a finite number of sections of a coherent sheaf is coherent.*

In fact, this sheaf is of finite type, from the definition of a coherent sheaf.

Proposition 9. *Let $\mathscr{F}$ be a coherent sheaf of $\mathscr{A}$-modules. For all $x \in X$, let $\mathscr{I}_x$ be an ideal of $\mathscr{A}_x$ consisting of those $a \in \mathscr{A}_x$ for which $a \cdot f = 0$ for all $f \in \mathscr{F}_x$. Then the $\mathscr{I}_x$ form a coherent sheaf of ideals (called the annihilator of $\mathscr{F}$).*

In fact, $\mathscr{I}_x$ is the kernel of the homomorphism $\mathscr{A}_x \to \mathrm{Hom}_{\mathsf{A_x}}(\mathscr{F}_x, \mathscr{G}_x)$; we then apply Propositions 5 and 6 and Theorem 2.

More generally, the *conductor* $\mathscr{F} : \mathscr{G}$ of a coherent sheaf $\mathscr{G}$ into its coherent subsheaf $\mathscr{F}$ is a coherent sheaf of ideals (being the annihilator of $\mathscr{G}/\mathscr{F}$).

16 Change of ring

The notions of a sheaf of finite type, and of a coherent sheaf, are dependent on the fixed sheaf of rings $\mathscr{A}$. When we will consider multiple sheaves of rings, we will say "of finite type over $\mathscr{A}$", "$\mathscr{A}$-coherent" to point out that we mean sheaves of $\mathscr{A}$-modules.

Theorem 3. *Let $\mathscr{A}$ be a coherent sheaf of rings, $\mathscr{I}$ a coherent sheaf of ideals of $\mathscr{A}$. Let $\mathscr{F}$ be a sheaf of $\mathscr{A}/\mathscr{I}$-modules. Then $\mathscr{F}$ is $\mathscr{A}/\mathscr{I}$-coherent if and only if it is $\mathscr{A}$-coherent. In particular, $\mathscr{A}/\mathscr{I}$ is a coherent sheaf of rings.*

It is clear that "of finite type over $\mathscr{A}$" is the same as "of finite type over $\mathscr{A}/\mathscr{I}$". For the other part, if $\mathscr{F}$ is $\mathscr{A}$-coherent, and if $s_1, \dots, s_p$ are sections of $\mathscr{F}$ over an open U, the sheaf of relations between the s_i with coefficients in $\mathscr{A}$, is of finite type over $\mathscr{A}$. It follows immediately that the sheaf of relations between the s_i with coefficients in $\mathscr{A}/\mathscr{I}$, is of finite type over $\mathscr{A}/\mathscr{I}$, since it is the image of the preceding by the canonical mapping $\mathscr{A}^p \to (\mathscr{A}/\mathscr{I})^p$. Thus

$\mathscr{F}$ is $\mathscr{A}/\mathscr{I}$-coherent. In particular, since $\mathscr{A}/\mathscr{I}$ is $\mathscr{A}$-coherent, it is also $\mathscr{A}/\mathscr{I}$-coherent, in other words, $\mathscr{A}/\mathscr{I}$ is a coherent sheaf of rings. Conversely, if $\mathscr{F}$ is $\mathscr{A}/\mathscr{I}$-coherent, it is locally isomorphic to the cokernel of a homomorphism $\phi : (\mathscr{A}/\mathscr{I})^q \to (\mathscr{A}/\mathscr{I})^p$ and since $\mathscr{A}/\mathscr{I}$ is $\mathscr{A}$-coherent, $\mathscr{F}$ is coherent by Theorem 2.

17 Extension and restriction of a coherent sheaf

Let Y be a closed subspace of a space X. If $\mathscr{G}$ is a sheaf over Y, we denote by $\mathscr{G}^X$ the a sheaf obtained by extending $\mathscr{G}$ by 0 outside Y; it is a sheaf over X (cf. n° 5). If $\mathscr{A}$ is a sheaf of rings over Y, $\mathscr{A}^X$ is a sheaf of rings over X, and if $\mathscr{F}$ is a sheaf of $\mathscr{A}$-modules, then $\mathscr{F}^X$ is a sheaf of $\mathscr{A}^X$-modules.

Proposition 10. *$\mathscr{F}$ is of finite type over $\mathscr{A}$ if and only if $\mathscr{F}^X$ is of finite type over $\mathscr{A}^X$.*

Let U be an open subset of X, and let $V = U \cap Y$. Any homomorphism $\phi : \mathscr{A}^p \to \mathscr{F}$ over V defines a homomorphism $\phi^X : (\mathscr{A}^X)^p \to \mathscr{F}^X$ over U, and conversely; so ϕ is surjective if and only if ϕ^X is. The proposition follows immediately from this.

We therefore show:

Proposition 11. *$\mathscr{F}$ is $\mathscr{A}$-coherent if and only if $\mathscr{F}^X$ is $\mathscr{A}^X$-coherent.*

Hence, by putting $\mathscr{F} = \mathscr{A}$:

Corollary. *$\mathscr{A}$ is a coherent sheaf of rings if and only if $\mathscr{A}^X$ is a coherent sheaf of rings.*

§3 Cohomology of a space with values in a sheaf

In this paragraph, X is a topological space, separated or not. By a *covering* of X we will always mean an open covering.

18 Cochains of a covering

Let $\mathsf{U} = \{U_i\}_{i \in I}$ be a covering of X. If $s = (i_0, \ldots, i_p)$ is a finite sequence of elements of I, we put

$$U_s = U_{i_0 \ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}.$$

Let $\mathscr{F}$ be a sheaf of abelian groups on the space X. If p is an integer ≥ 0, we call a *p-cochain of U with values in $\mathscr{F}$* a function f assigning to every $s = (i_0, \ldots, i_p)$ of $p+1$ elements of I a section $f_s = f_{i_0 \ldots i_p}$ of $\mathscr{F}$ over $U_{i_0 \ldots i_p}$. The p-cochains form an abelian group, denoted by $C^p(\mathsf{U}, \mathscr{F})$; it is the product group $\prod \Gamma(U_s, \mathscr{F})$, the product being over all sequences s of $p+1$ elements of I. The family of $C^p(\mathsf{U}, \mathscr{F})$, $p = 0, 1, \ldots$ is denoted by $C(\mathsf{U}, \mathscr{F})$. A p-cochain is also called a cochain of degree p.

A p-cochain is said to be *alternating* if:

(a) $f_{i_0 \ldots i_p} = 0$ whenever any two of the indices $i_0, \ldots, i_p$ are equal,

(b) $f_{i_{\sigma 0} \ldots i_{\sigma p}} = \varepsilon_\sigma f_{i_0 \ldots i_p}$ if σ is a permutation of the set $\{0, \ldots, p\}$ (ε_σ denotes the sign of σ).

The alternating cochains form a subgroup $C'^p(\mathsf{U}, \mathscr{F})$ of the group $C^p(\mathsf{U}, \mathscr{F})$; the family of the $C'^p(\mathsf{U}, \mathscr{F})$ is denoted by $C'(\mathsf{U}, \mathscr{F})$.

19 Simplicial operations

Let $S(I)$ be the simplex with the set I as its set of vertices; an (ordered) simplex of $S(I)$ is a sequence $s = (i_0, \ldots, i_p)$ of elements of I; p is called the dimension of s. Let $K(I) = \bigoplus_{p=0}^{\infty} K_p(I)$ be the complex defined by $S(I)$; by definition, $K_p(I)$ is a free group with the set of simplexes of dimension p of $S(I)$ as its base.

If s is a simplex of $S(I)$, we denote by $|s|$ the set of vertices of s.

A mapping $h : K_p(I) \to K_q(I)$ is called a *simplicial endomorphism* if

(i) h is a homomorphism,

(ii) For any simplex s of dimension p of $S(I)$ we have

$$h(s) = \sum_{s'} c_s^{s'} \cdot s', \quad \text{with } c_s^{s'} \in \mathbb{Z},$$

the sum being over all simplexes s' of dimension q such that $|s'| \subset |s|$.

Let h be a simplicial endomorphism, and let $f \in C^q(\mathsf{U}, \mathscr{F})$ be a cochain of degree q. For any simplex s of dimension p put:

$$({}^t h f)_s = \sum_{s'} c_s^{s'} \cdot \rho_s^{s'}(f_{s'}),$$

$\rho^{s'}$ denoting the restriction homomorphism: $\Gamma(U_{s'}, \mathscr{F}) \to \Gamma(U_s, \mathscr{F})$, which makes sense because $|s'| \subset |s|$. The mapping $s \mapsto ({}^t h f)_s$ is a p-cochain, denoted by ${}^t h f$. The mapping $f \mapsto {}^t h f$ is a homomorphism

$${}^t h : C^q(\mathsf{U}, \mathscr{F}) \to C^p(\mathsf{U}, \mathscr{F}),$$

and one verifies immediately the formulas:

$${}^t(h_1 + h_2) = {}^t h_1 + {}^t h_2, \quad {}^t(h_1 \circ h_2) = {}^t h_2 \circ {}^t h_1, \quad {}^t 1 = 1.$$

Note. In practice, we often do not write the restriction homomorphism $\rho_s^{s'}$.

20 Complexes of cochains

We apply the above to the simplicial endomorphism

$$\partial : K_{p+1}(I) \to K_p(I),$$

defined by the usual formula:

$$\partial(i_0, \ldots, i_{p+1}) = \sum_{j=0}^{j=p+1} (-1)^j (i_0, \ldots, \hat{i}_j, \ldots, i_{p+1}),$$

the sign ˆ meaning, as always, that the symbol below it should be omitted.

We thus obtain a homomorphism ${}^t\partial : C^p(\mathsf{U}, \mathscr{F}) \to C^{p+1}(\mathsf{U}, \mathscr{F})$, which we denote by d; from definition, we have that

$$(df)_{i_0 \ldots i_{p+1}} = \sum_{j=0}^{j=p+1} (-1)^j \rho_j (f_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}),$$

where ρ_j denotes the restriction homomorphism

$$\rho_j : \Gamma(U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}, \mathscr{F}) \to \Gamma(U_{i_0 \ldots i_{p+1}}, \mathscr{F}).$$

Since $\partial \circ \partial = 0$, we have $d \circ d = 0$. Thus we find that $C(\mathsf{U}, \mathscr{F})$ is equipped with a coboundary operator making it a complex. The q-th cohomology group of the complex $C(\mathsf{U}, \mathscr{F})$ will be denoted by $H^q(\mathsf{U}, \mathscr{F})$. We have:

Proposition 1. $H^0(\mathsf{U}, \mathscr{F}) = \Gamma(X, \mathscr{F})$.

A 0-cochain is a system $(f_i)_{i\in I}$ with every f_i being a section of $\mathscr{F}$ over U_i. It is a cocycle if and only if it satisfies $f_i - f_j = 0$ over $U_i \cap U_j$, or in other words, if there is a section f of $\mathscr{F}$ on X coinciding with f_i on U_i for all $i \in I$. Hence the Proposition.

(Thus $H^0(\mathsf{U}, \mathscr{F})$ is independent of U; of course this is not true for $H^q(\mathsf{U}, \mathscr{F})$ in general).

We see immediately that df is alternating if f is alternating; in other words, d preserves $C'(\mathsf{U}, \mathscr{F})$ which forms a subcomplex of $C(\mathsf{U}, \mathscr{F})$. The cohomology groups of $C'(\mathsf{U}, \mathscr{F})$ are denoted by $H'^q(\mathsf{U}, \mathscr{F})$.

Proposition 2. *The inclusion of $C'(\mathsf{U}, \mathscr{F})$ in $C(\mathsf{U}, \mathscr{F})$ induces an isomorphism of $H^q(\mathsf{U}, \mathscr{F})$ and $H^q(\mathsf{U}, \mathscr{F})$, for every $q \geq 0$.*

We equip the set I with a structure of a total order, and let h be a simplicial endomorphism of $K(I)$ defined in the following way:

$h((i_0, \ldots, i_q)) = 0$ if any two indices $i_0, \ldots, i_q$ are equal,

$h((i_0, \ldots, i_q)) = \varepsilon_\sigma(i_{\sigma 0} \ldots i_{\sigma q})$ if all indices $i_0, \ldots, i_q$ are distinct and σ is a permutation of $\{0, \ldots, q\}$ for which $i_{\sigma 0} < i_{\sigma 1} < \ldots < i_{\sigma q}$.

We verify right away that h commutes with ∂ and that $h(s) = s$ if $\dim(s) = 0$; in result (cf. [7], Chapter VI, §5) there exists a simplicial endomorphism k, raising the dimension by one, such that $1 - h = \partial \circ k + k \circ \partial$. Hence, by passing to $C(\mathsf{U}, \mathscr{F})$,

$$1 - {}^t h = {}^t k \circ d + d \circ {}^t k.$$

But we check immediately that ${}^t h$ is a *projection* from $C(\mathsf{U}, \mathscr{F})$ onto $C'(\mathsf{U}, \mathscr{F})$; since the preceding formula shows that it is a homotopy operator, the Proposition is proved. (Compare with [7], Chapter VI, theorem 6.10).

Corollary. $H^q(\mathsf{U}, \mathscr{F}) = 0$ *for* $q > \dim(\mathsf{U})$.

By the definition of $\dim(\mathsf{U})$, we have $U_{i_0 \ldots i_q} = \emptyset$ for $q > \dim(\mathsf{U})$, if the indices $i_0, \ldots, i_q$ are distinct; hence $C'^q(\mathsf{U}, \mathscr{F}) = 0$, which shows that

$$H^q(\mathsf{U}, \mathscr{F}) = H'^q(\mathsf{U}, \mathscr{F}) = 0.$$

21 Passing to a finer covering

A covering $\mathsf{U} = \{U_i\}_{i\in I}$ is said to be *finer* than the covering $\mathsf{V} = \{V_j\}_{j\in J}$ if there exists a mapping $\tau : I \to J$ such that $U_i \subset V_{\tau i}$ for all $i \in I$. If $f \in C^q(\mathsf{V}, \mathscr{F})$, put

$$(\tau f)_{i_0,\ldots,i_q} = \rho_U^V(f_{\tau i_0 \ldots \tau i_q}),$$

ρ_U^V denoting the restriction homomorphism defined by the inclusion of $U_{i_0 \ldots i_q}$ in $V_{\tau i_0 \ldots \tau i_q}$. The mapping $f \mapsto \tau f$ is a homomorphism from $C^q(\mathsf{V}, \mathscr{F})$ to $C^q(\mathsf{U}, \mathscr{F})$, defined for all $q \geq 0$ and commuting with d, thus it defines homomorphisms

$$\tau^* : H^q(\mathsf{V}, \mathscr{F}) \to H^q(\mathsf{U}, \mathscr{F}).$$

Proposition 3. *The homomorphisms $\tau^* : H^q(\mathsf{V}, \mathscr{F}) \to H^q(\mathsf{U}, \mathscr{F})$ depend only on* U *and* V *and not on the chosen mapping τ.*

Let τ and τ' be two mappings from I to J such that $U_i \subset V_{\tau i}$ and $U_i \subset V_{\tau'_i}$; we have to show that $\tau^* = \tau'*$.

Let $f \in C^q(\mathsf{V}, \mathscr{F})$; set

$$(kf)_{i_0 \ldots i_{q-1}} = \sum_{h=0}^{h=q-1} (-1)^h \rho_h (f_{\tau i_0 \ldots \tau i_j \tau' i_h \ldots \tau' i_{q-1}}),$$

where ρ_h denotes the restriction homomorphism defined by the inclusion of $U_{i_0 \ldots i_{q-1}}$ in $V_{\tau i_0 \ldots \tau i_j \tau' i_h \ldots \tau' i_{q-1}}$.

We verify by direct computation (cf. [7], Chapter VI, §3) that we have

$$dkf + k\,df = \tau' f - \tau f,$$

which ends the proof of the Proposition.

Thus, if U is finer than V, there exists for every integer $q \geq 0$ a canonical homomorphism from $H^q(\mathsf{V}, \mathscr{F})$ to $H^q(\mathsf{U}, \mathscr{F})$. From now on, this homomorphism will be denoted by $\sigma(\mathsf{U}, \mathsf{V})$.

22 Cohomology groups of X with values in a sheaf $\mathscr{F}$

The relation "U is finer than V" (which we denote henceforth by $\mathsf{U} \prec \mathsf{V}$) is a relation of a *preorder*[3] between coverings of X; moreover, this relation is *filtered*[4], since if $\mathsf{U} = \{U_i\}_{i \in I}$ and $\mathsf{V} = \{V_j\}_{j \in J}$ are two coverings, $\mathsf{W} = \{U_i \cap V_j\}_{(i,j) \in I \times J}$ is a covering finer than U and than V.

We say that two coverings U and V are equivalent if we have $\mathsf{U} \prec \mathsf{V}$ and $\mathsf{V} \prec \mathsf{U}$. Any covering U is equivalent to a covering U′ whose set of indices is a subset of $\mathsf{P}(X)$; in fact, we can take for U′ the *set* of open subsets of X belonging to the *family* U. We can thus speak of the set of classes of coverings with respect to this equivalence relation; this is an ordered filtered set. [5]

If $\mathsf{U} \prec \mathsf{V}$, we have defined at the end of the preceding n° a well defined homomorphism $\sigma(\mathsf{U}, \mathsf{V}) : H^q(\mathsf{V}, \mathscr{F}) \to H^q(\mathsf{U}, \mathscr{F})$, defined for every integer $q \geq 0$ and every sheaf $\mathscr{F}$ on X. It is clear that $\sigma(\mathsf{U}, \mathsf{U})$ is the identity and that $\sigma(\mathsf{U}, \mathsf{V}) \circ \sigma(\mathsf{V}, \mathsf{W}) = \sigma(\mathsf{U}, \mathsf{W})$ if $\mathsf{U} \prec \mathsf{V} \prec \mathsf{W}$. It follows that, if U is equivalent to V, then $\sigma(\mathsf{U}, \mathsf{V})$ and $\sigma(\mathsf{V}, \mathsf{U})$ are inverse isomorphisms; in other words, $H^q(\mathscr{F}, \mathsf{U})$ depends only on the class of the covering U.

Definition. We call the q-th cohomology group of X with values in a sheaf $\mathscr{F}$, and denote by $H^q(X, \mathscr{F})$, the inductive limit of groups $H^q(\mathsf{U}, \mathscr{F})$, where U

[3] i.e. quasiorder

[4] i.e. directed

[5] To the contrary, we cannot speak about the "set" of coverings, because a covering is a family whose set of indices is arbitrary.

runs over the filtered ordering of classes of coverings of X, with respect to the homomorphisms $\sigma(\mathsf{U},\mathsf{V})$.

In other words, an element of $H^q(X,\mathscr{F})$ is just a pair (U,x) with $x \in H^q(\mathsf{U},\mathscr{F})$, and we identify two such pairs (U,x) and (V,y) whenever there exists a W with $\mathsf{W} \prec \mathsf{U}$, $\mathsf{W} \prec \mathsf{V}$ and $\sigma(\mathsf{W},\mathsf{U})(x) = \sigma(\mathsf{W},\mathsf{V})(y)$ in $H^q(\mathsf{W},\mathscr{F})$. Any covering U in X is thus associated a canonical homomorphism $\sigma(U) : H^q(\mathsf{U},\mathscr{F}) \to H^q(X,\mathscr{F})$.

We will see that $H^q(X,\mathscr{F})$ can also be defined by an inductive limit of $H^q(\mathsf{U},\mathscr{F})$ where U runs over a **cofinal** family of coverings. Thus, if X is quasi-compact (resp. quasi-paracompact), we can consider only finite (resp. locally finite) coverings.

When $q = 0$, by Proposition 1 we have:

Proposition 4. $H^0(X,\mathscr{F}) = \Gamma(X,\mathscr{F})$.

23 Homomorphisms of sheaves

Let ϕ be a homomorphism from a sheaf $\mathscr{F}$ to a sheaf $\mathscr{G}$. If U is a covering of X, we can assign to any $f \in C^q(\mathsf{U},\mathscr{F})$ an element $\phi f \in C^q(\mathsf{U},\mathscr{G})$ defined by the formula $(\phi f)_s = \phi(f_s)$. The mapping $f \mapsto \phi f$ is a homomorphism from $C(\mathsf{U},\mathscr{F})$ to $C(\mathsf{U},\mathscr{G})$ commuting with the coboundary, thus it defines homomorphisms $\phi^* : H^q(\mathsf{U},\mathscr{F}) \to H^q(\mathsf{U},\mathscr{G})$. We have $\phi^* \circ \sigma(\mathsf{U},\mathsf{V}) = \sigma(\mathsf{U},\mathsf{V}) \circ \psi^*$, hence, by passing to the limit, the homomorphisms

$$\phi^* : H^q(X,\mathscr{F}) \to H^q(X,\mathscr{G}).$$

When $q = 0$, ϕ^* coincides with the homomorphism from $\Gamma(X,\mathscr{F})$ to $\Gamma(X,\mathscr{G})$ induced in the natural way by ϕ.

In general, the homomorphisms ϕ^* satisfy usual formal properties:

$$(\phi+\psi)^* = \phi^* + \psi^*, \quad (\phi\circ\psi)^*, \quad 1^* = 1.$$

In other words, for all $q \geq 0$, $H^q(X,\mathscr{F})$ is a covariant additive functor of $\mathscr{F}$. Hence we gather that if $\mathscr{F}$ is the direct sum of two sheaves $\mathscr{G}_1$ and $\mathscr{G}_2$, then $H^q(X,\mathscr{F})$ is the direct sum of $H^q(X,\mathscr{G}_1)$ and $H^q(X,\mathscr{G}_2)$.

Suppose that $\mathscr{F}$ is a sheaf of $\mathscr{A}$-modules. Any section of $\mathscr{A}$ on X defines an endomorphism of $\mathscr{F}$, therefore of $H^q(X,\mathscr{F})$. It follows that $H^q(X,\mathscr{F})$ are modules over the ring $\Gamma(X,\mathscr{A})$.

24 Exact sequence of sheaves: the general case

Let $0 \to \mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C} \to 0$ be an exact sequence of sheaves. If U is a covering of X, the sequence

$$0 \to C(\mathsf{U}, \mathscr{A}) \xrightarrow{\alpha} C(\mathsf{U}, \mathscr{B}) \xrightarrow{\beta} C(\mathsf{U}, \mathscr{C})$$

is obviously exact, but the homomorphism β need not be surjective in general. Denote by $C_0(\mathsf{U}, \mathscr{C})$ the image of this homomorphism; it is a subcomplex of $C(\mathsf{U}, \mathscr{C})$ whose cohomology groups will be denoted by $H_0^q(\mathsf{U}, \mathscr{C})$. The exact sequence of complexes:

$$0 \to C(\mathsf{U}, \mathscr{A}) \to C(\mathsf{U}, \mathscr{B}) \to C_0(\mathsf{U}, \mathscr{C}) \to 0$$

giving rise to an exact sequence of cohomology:

$$\ldots \to H^q(\mathsf{U}, \mathscr{B}) \to H_0^q(\mathsf{U}, \mathscr{C}) \xrightarrow{d} H^{q+1}(\mathsf{U}, \mathscr{A}) \to H^{q+1}(\mathsf{U}, \mathscr{B}) \to \ldots,$$

where the coboundary operator d is defined as usual.

Now let $\mathsf{U} = \{U_i\}_{i \in I}$ and $\mathsf{V} = \{V_j\}_{j \in J}$ be two coverings and let $\tau : I \to J$ be such that $U_i \subset V_{\tau i}$; we thus have $\mathsf{U} \prec \mathsf{V}$. The commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & C(\mathsf{V}, \mathscr{A}) & \longrightarrow & C(\mathsf{V}, \mathscr{B}) & \longrightarrow & C(\mathsf{V}, \mathscr{C}) \\
& \downarrow \tau & & \downarrow \tau & & \downarrow \tau \\
0 \longrightarrow & C(\mathsf{U}, \mathscr{A}) & \longrightarrow & C(\mathsf{U}, \mathscr{B}) & \longrightarrow & C(\mathsf{U}, \mathscr{C})
\end{array}$$

shows that τ maps $C_0(\mathsf{V}, \mathscr{C})$ into $C_0(\mathsf{U}, \mathscr{C})$, thus defining the homomorphisms $\tau^* : H_0^q(\mathsf{V}, \mathscr{C}) \to H_0^q(\mathsf{U}, \mathscr{C})$. Moreover, the homomorphisms τ^* are independent of the choice of the mapping τ: this follows from the fact that, if $f \in C_0^q(\mathsf{V}, \mathscr{C})$, we have $kf \in C_0^{q-1}(\mathsf{U}, \mathscr{C})$, with the notations of the proof of Proposition 3. We have thus obtained canonical homomorphisms $\sigma(\mathsf{U}, \mathsf{V}) : H_0^q(\mathsf{V}, \mathscr{C}) \to H_0^q(\mathsf{U}, \mathscr{C})$; we might then define $H_0^q(X, \mathscr{C})$ as the inductive limit of the groups $H_0^q(\mathsf{U}, \mathscr{C})$.

Because an inductive limit of exact sequences is an exact sequence (cf. [7], Chapter VIII, theorem 5.4), we obtain:

Proposition 5. *The sequence*

$$\ldots \to H^q(X, \mathscr{B}) \xrightarrow{\beta^*} H_0^q(X, \mathscr{C}) \xrightarrow{d} H^{q+1}(X, \mathscr{A}) \xrightarrow{\alpha^*} H^{q+1}(X, \mathscr{B}) \to \ldots$$

is exact.

(d denotes the homomorphism obtained by passing to the limit with the homomorphisms $d : H_0^q(\mathsf{U}, \mathscr{C}) \to H^{q+1}(\mathsf{U}, \mathscr{A})$).

To apply the preceding Proposition, it is convenient to compare the groups $H_0^q(X, \mathscr{C})$ and $H^q(X, \mathscr{C})$. The inclusion of $C_0(\mathsf{U}, \mathscr{C})$ in $C(\mathsf{U}, \mathscr{C})$ defines the homomorphisms $H_0^q(\mathsf{U},\mathscr{C}) \to H^q(\mathsf{U},\mathscr{C})$, hence, by passing to the limit with U, the homomorphisms:

$$H_0^q(X,\mathscr{C}) \to H^q(X,\mathscr{C}).$$

Proposition 6. *The canonical homomorphism $H_0^q(X,\mathscr{C}) \to H^q(X,\mathscr{C})$ is bijective for $q = 0$ and injective for $q = 1$.*

We will prove the following lemma:

Lemma 1. *Let $\mathsf{V} = \{V_j\}_{j\in J}$ be a covering and let $f = (f_j)$ be an element of $C^0(\mathsf{V},\mathscr{C})$. There exists a covering $\mathsf{U} = \{U_i\}_{i\in I}$ and a mapping $\tau : I \to J$ such that $U_i \subset V_{\tau i}$ and $\tau f \in C_0^0(\mathsf{U},\mathscr{C})$.*

For any $x \in X$, take a $\tau x \in J$ such that $x \in V_{\tau x}$. Since $f_{\tau x}$ is a section of $\mathscr{C}$ over $V_{\tau x}$, there exists an open neighborhood U_x of x, contained in $V_{\tau x}$ and a section b_x of $\mathscr{B}$ over U_x such that $\beta(b_x) = f_{\tau x}$ on U_x. The $\{U_x\}_{x\in X}$ form a covering U of X, and the b_x form a 0-chain b of U with values in V; since $\tau f = \beta(b)$, we have that $\tau f \in C_0^0(\mathsf{U},\mathscr{C})$.

We will now show that $H_0^1(X,\mathscr{C}) \to H^1(X,\mathscr{C})$ is injective. An element of the kernel of this mapping may be represented by a 1-cocycle $z = (z_{j_0 j_1}) \in C_0'(\mathsf{V},\mathscr{C})$ such that there exists an $f = (f_j) \in C^0(\mathsf{V},\mathscr{C})$ with $df = z$; applying Lemma 1 to f yields a covering U such that $\tau f \in C_0^0(\mathsf{U},\mathscr{C})$, which shows that τz is cohomologous to 0 in $C_0(\mathsf{U},\mathscr{C})$, thus its image in $H_0^1(X,\mathscr{C})$ is 0. This shows that $H_0^0(X,\mathscr{C}) \to H^0(X,\mathscr{C})$ is bijective.

Corollary 1. *We have an exact sequence:*

$$0 \to H^0(X,\mathscr{A}) \to H^0(X,\mathscr{B}) \to H^0(X,\mathscr{C}) \to H^1(X,\mathscr{A}) \to H^1(X,\mathscr{B}) \to H^1(X,\mathscr{C}).$$

This is an immediate consequence of Propositions 5 and 6.

Corollary 2. *If $H^1(X,\mathscr{A}) = 0$, then $\Gamma(X,\mathscr{B}) \to \Gamma(X,\mathscr{C})$ is surjective.*

25 Exact sequence of sheaves: the case of X paracompact

Recall that a space X is said to be paracompact if it is separated and if any covering of X admits a locally finite finer covering. On paracompact spaces, we can extend Proposition 6 for all values of q (I do not know whether that extension is possible for nonseparated spaces):

Proposition 7. *If X is paracompact, the canonical homomorphism*

$$H_0^q(X,\mathscr{C}) \to H^q(X,\mathscr{C})$$

is bijective for all $q \geq 0$.

This Proposition is an immediate consequence of the following lemma, analogous to Lemma 1:

Lemma 2. *Let* $\mathsf{V} = \{V_j\}_{j\in J}$ *be a covering, and let* $f = (f_{j_0\dots j_q})$ *be an element of* $C^q(\mathsf{V}, \mathscr{C})$*. There exists a covering* $\mathsf{U} = \{U_i\}_{i\in I}$ *and a mapping* $\tau : I \to J$ *such that* $U_i \subset V_{\tau i}$ *and* $\tau f \in C^q_0(\mathsf{U}, \mathscr{C})$*.*

Since X is paracompact, we might assume that V is locally finite. Then there exists a covering $\{W_j\}_{j\in J}$ such that $W_j \subset V_j$. For every $x \in X$, choose an open neighborhood U_x of x such that

(a) If $x \in V_j$ (resp. $x \in W_j$), then $U_x \subset V_j$ (resp. $U_x \subset W_j$),

(b) If $U_x \cap W_j \neq \emptyset$, then $U_x \subset W_j$,

(c) If $x \in V_{j_0\dots j_q}$, there exists a section b of $\mathscr{B}$ over U_x such that $\beta(b) = f_{j_0\dots j_q}$ over U_x.

The condition (c) can be satisfied due to the definition of the quotient sheaf and to the fact that x belongs to a finite number of sets $V_{j_0\dots j_q}$. Having (c) satisfied, it suffices to restrict U_x appropriately to satisfy (a) and (b).

The family $\{U_x\}_{x\in X}$ forms a covering U; for any $x \in X$, choose $\tau x \in J$ such that $x \in W_{\tau x}$. We now check that τf belongs to $C^q_0(U, \mathscr{C})$, in other words, that $f_{\tau x_0\dots\tau x_q}$ is the image by β of a section of $\mathscr{B}$ over $U_{x_0} \cap \dots \cap U_{x_q}$. If $U_{x_0} \cap \dots \cap U_{x_q}$ is empty, this is obvious; if not, we have $U_{x_0} \cap U_{x_k} \neq \emptyset$ for $0 \leq k \leq q$, and since $U_{x_k} \subset U_{\tau x_k}$, we have $U_{x_0} \cap W_{\tau x_k} \neq \emptyset$, which implies by (b) that $U_{x_0} \subset V_{\tau x_k}$, hence $x_0 \in V_{\tau x_0\dots\tau x_q}$; we then apply (c), seeing that there exists a section b of $\mathscr{B}$ over U_{x_0} such that $\beta(b)_x = f_{\tau x_0\dots\tau x_q}$ on U_{x_0}, so also on $U_{x_0} \cap \dots \cap U_{x_q}$, which ends the proof.

Corollary. *If* X *is paracompact, we have an exact sequence:*

$$\dots \to H^q(X, \mathscr{B}) \xrightarrow{\beta^*} H^q(X, \mathscr{C}) \xrightarrow{d} H^{q+1}(X, \mathscr{A}) \xrightarrow{\alpha^*} H^{q+1}(X, \mathscr{B}) \to \dots$$

(the map d being defined as the composition of the inverse of the isomorphism $H^q_0(X, \mathscr{C}) \to H^q(X, \mathscr{C})$ with $d : H^q_0(X, \mathscr{C}) \to H^{q+1}(X, \mathscr{A})$).

The exact sequence mentioned above is called the *exact sequence of cohomology* defined by a given exact sequence of sheaves $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$. More generally, it exists whenever we can show that $H^q_0(X, \mathscr{C}) \to H^q(X, \mathscr{C})$ is bijective (we will see in n° 47 that this is the case when X is an algebraic variety and when $\mathscr{A}$ is an algebraic coherent sheaf).

26 Cohomology of a closed subspace

Let $\mathscr{F}$ be a sheaf over a space X, and let Y be a subspace of Y. Let $\mathscr{F}(Y)$ be the sheaf induced by $\mathscr{F}$ on Y, in the sense of n° 4. If $\mathsf{U} = \{U_i\}_{i\in I}$ is a covering of X, the sets $U'_i = Y \cap U_i$ form a covering U' of Y; if $f_{i_0\dots i_Q}$ is a section of $\mathscr{F}$ over $U_{i_0\dots i_q}$, the restriction of $f_{i_0\dots i_q}$ to $U'_{i_0\dots i_q} = Y \cap U_{i_0\dots i_q}$ is a section of $\mathscr{F}(Y)$. The operation of restriction is a homomorphism $\rho : C(\mathsf{U}, \mathscr{F}) \to C(\mathsf{U}', \mathscr{F}(Y))$, commuting with d, thus defining $\rho^* : H^q(\mathsf{U}, \mathscr{F}) \to H^q(\mathsf{U}', \mathscr{F}(Y))$. If $\mathsf{U} \prec \mathsf{V}$, we have $\mathsf{U}' \prec \mathsf{V}'$, and $\rho^* \circ \sigma(\mathsf{U}, \mathsf{V}) = \sigma(\mathsf{U}', \mathsf{V}') \circ \rho^*$; thus the homomorphisms

ρ^* define, by passing to the limit with U, homomorphisms $\rho^* : H^q(X, \mathscr{F}) \to H^q(Y, \mathscr{F}(Y))$.

Proposition 8. *Assume that Y is closed in X and that $\mathscr{F}$ is zero outside Y. Then $\rho^* : H^q(X, \mathscr{F}) \to H^q(Y, \mathscr{F}(Y))$ is bijective for all $q \geq 0$.*

The Proposition is implied by the following facts:

(a) Any covering $\mathsf{W} = \{W_i\}_{i \in I}$ of Y is of the form U' for some covering U of X.

Indeed, it suffices to put $U_i = W_i \cup (X - Y)$, since Y is closed in X.

(b) For any covering U of X, $\rho : C(\mathsf{U}, \mathscr{F}) \to C(\mathsf{U}', \mathscr{F}(Y))$ is bijective. Indeed, the result follows from Proposition 5 of n° 5, applied to $U_{i_0 \ldots i_q}$ and the sheaf $\mathscr{F}$.

We can also express Proposition 8 in the following manner: If $\mathscr{G}$ is a sheaf on Y, and if $\mathscr{G}^X$ is the sheaf obtained by extending $\mathscr{G}$ by 0 outside Y, we have $H^q(Y, \mathscr{G}) = H^q(X, \mathscr{G}^X)$ for all $q \geq 0$; in other words, the identification of $\mathscr{G}$ with $\mathscr{G}^X$ is compatible with passing to cohomology groups.

§4 Comparison of cohomology groups of different coverings

In this paragraph, X denotes a topological space and $\mathscr{F}$ is a sheaf on X. We pose conditions on a covering U of X, under which we have $H^n(\mathsf{U}, \mathscr{F}) = H^n(X, \mathscr{F})$ for all $n \geq 0$.

27 Double complexes

A double complex (cf. [6], Chapter VI, §4) is a bigraded abelian group

$$K = \bigoplus_{p,q} K^{p,q}, \quad p \geq 0, q \geq 0,$$

equipped with two endomorphisms d' and d'' satisfying the following properties:
— d' maps $K^{p,q}$ to $K^{p+1,q}$ and d'' maps $K^{p,q}$ to $K^{p,q+1}$,
— $d' \circ d' = 0$, $d' \circ d'' + d'' \circ d' = 0$, $d'' \circ d'' = 0$.

An element of $K^{p,q}$ is said to be bihomogenous of bidegree (p, q), and of total degree $p + q$. The endomorphism $d = d' + d''$ satisfies $d \circ d = 0$, and the cohomology groups of K with respect to this coboundary operator are denoted by $H^n(K)$, where n means the total degree.

We can treat d' as a coboundary operator on K; since d' is compatible with the bigrading of K, we also obtain cohomology groups, denoted by $H_I^{p,q}(K)$; for d'', we have the groups $H_{II}^{p,q}(K)$.

We denote by K_{II}^q the subgroup of $K^{0,q}$ consisting of elements x such that $d'(x) = 0$, and by K_{II} the direct sum of K_{II}^q ($q = 0, 1, \ldots$). We have an analogous definition of $K_I = \bigoplus_{p=0}^{\infty} K_I^p$. We note that

$$K_{II}^q = H_I^{0,q}(K) \quad \text{and} \quad K_I^p = H_{II}^{p,0}(K).$$

K_{II} is a subcomplex of K, and the operator d coincides on K_{II} with the operator d''.

Proposition 1. *If $H_I^{p,q}(K) = 0$ for $p > 0$ and $q \geq 0$, the inclusion $K_{II} \to K$ defines a bijection from $H^n(K_{II})$ to $H^n(K)$, for all $n \geq 0$.*

(Cf. [4], statement XVII-6, whose proof we shall repeat here).

By replacing K by K/K_{II}, we are led to prove that if $H_I^{p,q}(K) = 0$ for $p \geq 0$ and $q \geq 0$, then $H^n(K) = 0$ for all $n \geq 0$. Put

$$K_h = \bigoplus_{q \geq h} K^{p,q}.$$

The groups K_h ($h = 0, 1, \ldots$) are subcomplexes embedded in K, and K_h/K_{h+1} is isomorphic to $\bigoplus_{p=0}^{\infty} K^{p,h}$, equipped with the coboundary operator d'. We thus have $H^n(K_h/K_{h+1}) = H_I^{h,n-h}(K) = 0$ for any n and h, therefore $H^n(K_h) = H^n(K_{h+1})$. Since $H^n(K_h) = 0$ if $h > n$, we deduce, by descending recursion on h, that $H^n(K_h) = 0$ for all n and h, and since K_0 is equal to K, the Proposition follows.

28 The double complex defined by two coverings

Let $\mathsf{U} = \{U_i\}_{i \in I}$ and $\mathsf{V} = \{V_j\}_{j \in J}$ be two coverings of X. If s is a p-simplex of $S(I)$ and s' a q-simplex of $S(J)$, we denote by U_s the intersection of U_i, $i \in s$ (cf. n° 18), the intersection of V_j, $j \in s'$, by V_s the covering of U_s formed by $\{U_s \cap V_j\}_{j \in J}$ and by $\mathsf{U}_{s'}$ the covering of $V_{s'}$ formed by $\{V_{s'} \cap U_i\}_{i \in I}$.

We define a double complex $C(\mathsf{U}, \mathsf{V}; \mathscr{F}) = \bigoplus_{p,q} C^{p,q}(\mathsf{U}, \mathsf{V}; \mathscr{F})$ as follows:

$C^{p,q}(\mathsf{U}, \mathsf{V}; \mathscr{F}) = \prod \Gamma(U_s \cap V_{s'}, \mathscr{F})$, the product taken over all pairs (s, s') where s is a simplex of dimension p of $S(I)$ and s' is a simplex of dimension q of $S(J)$.

An element $f \in C^{p,q}(\mathsf{U}, \mathsf{V}; \mathscr{F})$ is thus a system $(f_{s,s'})$ of sections of $\mathscr{F}$ on $U_s \cap V_{s'}$ or, with the notations of n° 18, it is a system

$$f_{i_0 \dots i_p, j_0 \dots j_q} \in \Gamma(U_{i_0 \dots i_p} \cap V_{j_0 \dots j_q}, \mathscr{F}).$$

We can also identify $C^{p,q}(\mathsf{U}, \mathsf{V}; \mathscr{F})$ with $\prod_{s'} C^p(\mathsf{U}_{s'}, \mathscr{F})$; thus, for all s', we have a coboundary operator $d : C^p(\mathsf{U}_{s'}, \mathscr{F} \to C^{p+1}(\mathsf{U}_{s'}, \mathscr{F})$, giving a homomorphism

$$d_{\mathsf{U}} : C^{p,q}(\mathsf{U}, \mathsf{V}; \mathscr{F}) \to C^{p+1,q}(\mathsf{U}, \mathsf{V}; \mathscr{F}).$$

Making the definition of d_{U} explicit, we obtain:

$$(d_{\mathsf{U}} f)_{i_0 \dots i_{p+1}, j_0 \dots j_q} = \sum_{k=0}^{k=p+1} (-1)^k \rho_k (f_{i_0 \dots \hat{i}_k \dots i_{p+1}, j_0 \dots j_q}),$$

ρ_k being the restriction homomorphism defined by the inclusion of

$$U_{i_0 \dots i_p} \cap V_{j_0 \dots j_q} \quad \text{in} \quad U_{i_0 \dots i_k \dots i_{p+1}} \cap V_{j_0 \dots j_q}.$$

We define $d_{\mathsf{V}} : C^{p,q}(\mathsf{U}, \mathsf{V}; \mathscr{F}) \to C^{p,q+1}(\mathsf{U}, \mathsf{V}; \mathscr{F})$ the same way and we have

$$(d_{\mathsf{V}} f)_{i_0 \dots i_p, j_0 \dots j_{q+1}} = \sum_{h=0}^{h=q+1} (-1)^h \rho_h (f_{i_0 \dots i_p, j_0 \dots \hat{j}_h \dots j_{q+1}}).$$

It is clear that $d_{\mathsf{U}} \circ d_{\mathsf{U}} = 0$, $d_{\mathsf{U}} \circ d\mathsf{V} = d_{\mathsf{V}} \circ d_{\mathsf{U}}$, $d_{\mathsf{V}} \circ d_{\mathsf{V}} = 0$. We thus put $d' = d_{\mathsf{U}}$, $d'' = (-1)^p d_{\mathsf{V}}$, equipping $C(\mathsf{U}, \mathsf{V}; \mathscr{F})$ with a structure of a double complex. We now apply to $K = C(\mathsf{U}, \mathsf{V}; \mathscr{F})$ the definitions from the preceding n° ; the groups or complexes denoted in the general case by $H^n(K)$, $H_I^{p,q}(K)$,

$H_I^{p,q}(K)$, $H_{II}^{p,q}(K)$, K_I, K_{II} will be denoted by $H^n(\mathsf{U},\mathsf{V};\mathscr{F})$, $H_I^{p,q}(\mathsf{U},\mathsf{V};\mathscr{F})$, $H_{II}^{p,q}(\mathsf{U},\mathsf{V};\mathscr{F})$, $C_I(\mathsf{U},\mathsf{V};\mathscr{F})$ and $C_{II}(\mathsf{U},\mathsf{V};\mathscr{F})$, respectively.

From the definitions of d' and d'', we immediately obtain:

Proposition 2. *$H_I^{p,q}(\mathsf{U},\mathsf{V};\mathscr{F})$ is isomorphic to $\prod_{s^\delta} H^p(\mathsf{U}_{s^\delta},\mathscr{F})$, the product being taken over all simplexes of dimension q of $S(J)$. In particular,*

$$C_{II}^q(\mathsf{U},\mathsf{V};\mathscr{F}) = H_I^{0,q}(\mathsf{U},\mathsf{V};\mathscr{F})$$

is isomorphic to $\prod_{s^\delta} H^0(\mathsf{U}_{s^\delta},\mathscr{F}) = C^q(\mathsf{V},\mathscr{F})$.

We denote by ι'' the canonical isomorphism: $C(\mathsf{V},\mathscr{F}) \to C_{II}(\mathsf{U},\mathsf{V};\mathscr{F})$. If $(f_{j_0\ldots j_q})$ is an element of $C^q(\mathsf{V},\mathscr{F})$, we thus have

$$(\iota'' f)_{i_0,j_0\ldots j_q} = \rho_{i_0}(f_{j_0\ldots j_q}),$$

where ρ_{i_0} denotes the restriction homomorphism defined by the inclusion of

$$U_{i_0} \cap V_{j_0\ldots j_q} \quad \text{in} \quad V_{j_0\ldots j_q}.$$

Obviously, a statement analogous to Proposition 2 holds for $H_I^{p,q}I(\mathsf{U},\mathsf{V};\mathscr{F})$, and we have an isomorphism $\iota' : C(\mathsf{U},\mathscr{F}) \to C_I(\mathsf{U},\mathsf{V};\mathscr{F})$.

29 Applications

Proposition 3. *Assume that $H^p(\mathsf{U}_{s^\delta},\mathscr{F}) = 0$ for every s' and all $p > 0$. Then the homomorphism $H^n(\mathsf{V},\mathscr{F}) \to H^n(\mathsf{U},\mathsf{V};\mathscr{F})$, defined by ι'', is bijective for all $n \geq 0$.*

This is an immediate consequence of Propositions 1 and 2.

Before stating Proposition 4, we prove a lemma:

Lemma 1. *Let $\mathsf{W} = \{W_i\}_{i\in I}$ be a covering of a space Y and let $\mathscr{F}$ be a sheaf on Y. If there exists an $i \in I$ such that $W_i = Y$, then $H^p(\mathsf{W},\mathscr{F}) = 0$ for all $p > 0$.*

Let W' be a covering of Y consisting of a single open set Y; we obviously have $\mathsf{W} \prec \mathsf{W}'$, and the assumption made on W means that $\mathsf{W}' \prec \mathsf{W}$. In result (n° 22) we have $H^p(\mathsf{W},\mathscr{F}) = H^p(\mathsf{W}',\mathscr{F}) = 0$ if $p > 0$.

Proposition 4. *Suppose that the covering V is finer than the covering U. Then $\iota'' : H^n(\mathsf{V},\mathscr{F}) \to H^n(\mathsf{U},\mathsf{V};\mathscr{F})$ is bijective for all $n \geq 0$. Moreover, the homomorphism $\iota' \circ \iota''^{-1} : H^n(\mathsf{U},\mathscr{F}) \to H^n(\mathsf{V},\mathscr{F})$ coincides with the homomorphism $\sigma(\mathsf{V},\mathsf{U})$ defined in n° 21.*

We apply Lemma 1 to $\mathsf{W} = \mathsf{U}_{s^\delta}$ and $Y = V_{s^\delta}$, seeing that $H^p(\mathsf{U}_{s^\delta},\mathscr{F}) = 0$ for all $p > 0$, and then Proposition 3 shows that

$$\iota'' : H^n(\mathsf{V},\mathscr{F}) \to H^n(\mathsf{U},\mathsf{V};\mathscr{F})$$

is bijective for all $n \geq 0$.

Let $\tau : J \to I$ be a mapping such that $V_j \subset U_{\tau j}$; for the proof of the second part of the Proposition, we need to observe that if f is an n-cocycle of $C(\mathsf{U}, \mathscr{F})$, the cocycles $\iota'(f)$ and $\iota''(\tau f)$ are cohomologous in $C(\mathsf{U}, \mathsf{V}; \mathscr{F})$.

For any integer p, $0 \leq p \leq n-1$, define $g^p \in C^{p,n-p-1}(\mathsf{U}, \mathsf{V}; \mathscr{F})$ by the following formula

$$g^p_{i_0 \ldots i_p, j_0 \ldots j_{n-p-1}} = \rho_p(f_{i_0 \ldots i_p \tau j_0 \ldots \tau j_{n-p}}),$$

ρ_p denoting the restriction defined by the inclusion of

$$U_{i_0 \ldots i_p} \cap V_{j_0 \ldots j_{n-p-1}} \quad \text{in} \quad U_{i_0 \ldots i_p, \tau j_0 \ldots \tau j_{n-p-1}}.$$

We verify by a direct calculation (keeping in mind that f is a cocycle) that we have

$$d''(g^0) = \iota''(\tau f), \ldots, d''(g^p) = d'(g^{p-1}), \ldots d'(g^{n-1}) = (-1)^n \iota'(f)$$

hence $d(g^0 - g^1 + \ldots + (-1)^{n-1} g^{n-1}) = \iota''(\tau f) - \iota'(f)$, which shows that $\iota''(\tau f)$ and $\iota'(f)$ are cohomologous.

Proposition 5. *Suppose that V is finer than U and that $H^q(\mathsf{V}_s, \mathscr{F}) = 0$ for all s and all $q > 0$. Then the homomorphism $\sigma(\mathsf{V}, \mathsf{U}) : H^n(\mathsf{U}, \mathscr{F}) \to H^n(\mathsf{V}, \mathscr{F})$ is bijective for all $n \geq 0$.*

If we apply Proposition 3, switching the roles of U and V, we see that $\iota' : H^n(\mathsf{V}, \mathscr{F}) \to H^n(\mathsf{U}, \mathsf{V}; \mathscr{F})$ is bijective. The Proposition then follows directly from Proposition 4.

Theorem 1. *Let X be a topological space, $\mathsf{U} = \{U_i\}_{i \in I}$ a covering of X, $\mathscr{F}$ a sheaf on X. Assume that there exists a family V^α, $\alpha \in A$ of coverings of X satisfying the following properties:*
(a) *For any covering W of X, there exists an $\alpha \in A$ with $\mathsf{V}^\alpha \prec \mathsf{W}$,*
(b) *$H^q(\mathsf{V}^\alpha_s, \mathscr{F}) = 0$ for all $\alpha \in A$, all simplexes $s \in S(I)$ and every $q > 0$,*
Then $\sigma(\mathsf{U}) : H^n(\mathsf{U}, \mathscr{F}) \to H^n(X, \mathscr{F})$ is bijective for all $n \geq 0$.

Since V^α are arbitrarily fine, we can assume that they are finer than U. In this case, the homomorphism

$$\sigma(\mathsf{V}^\alpha, \mathsf{U}) : H^n(\mathsf{U}, \mathscr{F}) \to H^n(\mathsf{V}^\alpha, \mathscr{F})$$

is bijective for all $n \geq 0$, by Proposition 5. Because V^α are arbitrarily fine, $H^n(X, \mathscr{F})$ is the inductive limit of $H^n(\mathsf{V}^\alpha, \mathscr{F})$, and the theorem follows.

Remarks. (1) It is probable that Theorem 1 remains valid when we replace the condition (b) with the following weaker condition:
(b') $\lim_\alpha H^q(\mathsf{V}^\alpha_s, \mathscr{F}) = 0$ for any simplex s of $S(I)$ and any $q > 0$.

(2) Theorem 1 is analogous to a theorem of Leray on acyclic coverings. Cf. [10] and also [4], statement XVII-7.

Chapter II

Algebraic Varieties – Coherent Algebraic Sheaves on Affine Varieties

From now on, K denotes a commutative algebraically closed field of arbitrary characteristic.

§1 Algebraic varieties

30 Spaces satisfying condition (A)

Let X be a topological space. The condition (A) is the following:

(A) — *Any decreasing sequence of closed subsets of X is stationary.*

In other words, if we have $F_1 \supset F_2 \supset F_3 \supset \dots$, F_i being closed in X, there exists an integer n such that $F_m = F_n$ for $m \geq n$. Or:

(A') — *The set of closed subsets of X, ordered by inclusion, satisfies the minimality condition*

Examples. Equip a set X with the topology where the closed subsets are the finite subsets of X and the whole X; the condition (A) is then satisfied. More generally, any algebraic variety, equipped with Zariski topology, satisfies (A) (cf. n° 34).

Proposition 1. *(a) If X satisfies the condition (A), then X is quasi-compact,*

(b) If X satisfies (A), any subspace of X satisfies it also.

(c) If X is a finite union of Y_i, the Y_i satisfying (A), then X also satisfies (A).

If F_i is a filtering decreasing set of closed subsets of X, and if X satisfies (A'), then there exists an F_i contained in all others; if $F_i = \emptyset$, there is therefore an i such that $F_i = \emptyset$, which shows (a).

Let $G_1 \supset G_2 \supset G_3 \supset \dots$ be a decreasing sequence of closed subsets of a subspace Y of X; if X satisfies (A), there exists an n for which $\bar{G}_m = \bar{G}_n$ for $m \geq n$, hence $G_m = Y \cap \bar{G}_m = Y \cap \bar{G}_n = G_n$, which shows (b).

Let $F_1 \supset F_2 \supset F_3 \supset \dots$ be a decreasing sequence of closed subsets of a space X satisfying (c); since all Y_i satisfy (A), there exists for all i an n_i such that $F_m \cap Y_i = F_{n_i} \cap Y_i$ for $m \geq n_i$; if $n = Sup(n_i)$, we then have $F_m = F_n$ for $m \geq n$, which shows (c).

A space X is said to be *irreducible* if it is not a union of two closed subspaces, distinct from X itself; or equivalently, if any two non-empty open subsets have a non-empty intersection. Any finite family of non-empty open subsets of X then has a non-empty intersection, and any open subset of X is also irreducible.

Proposition 2. *Any space X satisfying the condition (A) is a union of a finite number of irreducible closed subsets Y_i. If we suppose that that Y_i is not contained in Y_j for any pair (i,j), $i \neq j$, the set of Y_i is uniquely determined by X; the Y_i are then called the irreducible components of X.*

The existence of a decomposition $X = \bigcup Y_i$ follows immediately from (A). If Z_k is another such decomposition of X, we have $Y_i = \bigcup Y_i \cap Z_k$, and, since Y_i is irreducible, this implies of an index k such that $Z_k \supset Y_i$; interchanging the roles of Y_i and Z_k, we conclude analogously that there exists an index i' for which $Y_{i'} \supset Z_k$; thus $Y_i \subset Z_k \subset Y_{i'}$, which by the assumption made on Y_i leads to $i = i'$ and $Y_i = Z_k$, hence the uniqueness of the decomposition.

Proposition 3. *Let X be a topological space that is a finite union of non-empty open subsets V_i. Then X is irreducible if and only if all V_i are irreducible and $V_i \cap V_j \neq 0$ for all pairs (i, j).*

The necessity of these conditions was noted above; we show that they are sufficient. If $X = Y \cup Z$, where Y and Z are closed, we have $V_i = (V_i \cap Y) \cup (V_i \cap Z)$, which shows that each V_i is contained either in Y or in Z. Suppose that Y and Z are distinct from X; we can then find two indices i, j such that V_i is not contained in Y and V_j is not contained in Z; according to our assumptions on Y_i, we then have $V_i \subset Z$ and $V_j \subset Y$. Set $T = V_j - V_i \cap V_j$; T is closed in V_j and we have $V_j = T \cup (Z \cap V_j)$; as V_j is irreducible, it follows that either $T = V_j$, which means that $V_i \cap V_j = \emptyset$, or $Z \cap V_j = V_j$, which means that $V_j \subset Z$, and in both cases this leads to a contradiction, q.e.d.

31 Locally closed subsets of an affine space

Let r be an integer ≥ 0 and let $X = K^r$ be the *affine space* of dimension r over the field K. We equip X with the *Zariski topology*; recall that a subset of X is closed in this topology if it is the zero set of a family of polynomials $P^\alpha \in K[X_1, \ldots, X_r]$. Since the ring of polynomials is Noetherian, X satisfies the condition (A) from the preceding n° . Moreover, one easily shows that X is an irreducible space.

If $x = (x_1, \ldots, x_r)$ is a point of X, we denote by $\mathscr{O}_x$ the *local ring* of x; recall that this is the subring of the field $K(X_1, \ldots, X_r)$ consisting of those fractions which can be put in the form:

$R = P/Q$, where P and Q are polynomials and $Q(x) \neq 0$.

Such a fraction is said to be *regular* in x; for all points $x \in X$ for which $Q(x) \neq 0$, the function $x \mapsto P(x)/Q(x)$ is a continuous function with values in K (K being given the Zariski topology) which can be identified with R, the field K being infinite. The $\mathscr{O}_x$, $x \in X$ thus form a subsheaf $\mathscr{O}$ of the sheaf $\mathscr{F}(X)$ of germs of functions on X with values in K (cf. n° 3); the sheaf $\mathscr{O}$ is a sheaf of rings.

We will extend the above to locally closed subspaces of X (we call a subset of a space X *locally closed* in X if it is an intersection of a open subset with a closed subset of X). Let Y be such a subspace and let $\mathscr{F}(Y)$ be the sheaf of germs of functions on Y with values in K; if x is a point of Y, the operation of restriction defines a canonical homomorphism

$$\varepsilon_x : \mathscr{F}(X)_x \to \mathscr{F}(Y)_x.$$

The image of $\mathscr{O}_x$ under ε_x is a subring of $\mathscr{F}(Y)_x$ which we denote by $\mathscr{O}_{x,Y}$; the $\mathscr{O}_{x,Y}$ form a subsheaf $\mathscr{O}_Y$ of $\mathscr{F}(Y)$, which we call the *sheaf of local rings* of Y. A section of $\mathscr{O}_Y$ over an open subset V of Y is thus, by definition, a function $f : V \to K$ which is equal, in the neighborhood of any point $x \in V$, to a restriction to V of a rational function regular at x; such a function is said to be *regular* on V; it is a continuous function if we equip V with the induced topology and K with the Zariski topology. The set of regular functions at all points of V is a ring, the ring $\Gamma(V, \mathscr{O}_Y)$; observe also that, if $f \in \Gamma(V, \mathscr{O}_x)$ and if $f(x) \neq 0$ for all $x \in V$, then $1/f$ also belongs to $\Gamma(V, \mathscr{O}_Y)$.

We can characterize the sheaf $\mathscr{O}_Y$ in another way:

Proposition 4. *Let U (resp. F) be a open (resp. closed) subspace of X and let $Y = U \cap F$. Let $I(F)$ be the ideal $K[X_1, \dots, X_r]$ consisting of polynomials vanishing on F. If x is a point of Y, the kernel of the surjection $\varepsilon_x : \mathscr{O}_x \to \mathscr{O}_{x,Y}$ coincides with the ideal $I(F) \cdot \mathscr{O}_x$ of $\mathscr{O}_x$.*

It is clear that each element of $I(F) \cdot \mathscr{O}_x$ belongs to the kernel of ε_x. Conversely, let $R = P/Q$ be an element of the kernel, P and Q being two polynomials with $Q(x) \neq 0$. By assumption, there exists an open neighborhood W of x such that $P(y) = 0$ for all $y \in W \cap F$; let F' be the complement of W, which is closed in X; since $x \in F'$, there exists, by the definition of the Zariski topology, a polynomial P_1 vanishing on F' and nonzero at x; the polynomial $P \cdot P_1$ belongs to $I(F)$ and we can write $R = P \cdot P_1 / Q \cdot P_1$, which shows that $R \in I(F) \cdot \mathscr{O}_x$.

Corollary. *The ring $\mathscr{O}_{x,Y}$ is isomorphic to the localization of $K[X_1, \dots, X_r]/I(F)$ in the maximal ideal defined by the point x.*

This follows immediately from the construction of localization a quotient ring (cf. for example [8], Chap. XV, §5, th. XI).

32 Regular functions

Let U (resp. V) be a locally closed subspace of K^r (resp. K^s). A function $\phi : U \to V$ is said to be *regular* on U (or simply regular) if:

ϕ is continuous,

If $x \in U$ and $f \in \mathscr{O}_{\phi(x),V}$ then $f \circ \phi \in \mathscr{O}_{x,U}$.

Denote the coordinates of the point $\phi(x)$ by $\phi_i(x)$, $1 \leq i \leq s$. We then have:

Proposition 5. *A map $\phi : U \to V$ is regular on U if and only if $\phi_i : U \to K$ are regular on U for all i, $1 \leq i \leq s$.*

As the coordinate functions are regular on V, the condition is necessary. Conversely, suppose that we have $\phi_i \in \Gamma(U, \mathscr{O}_U)$ for each i; if $P(X_1, \dots, X_s)$ is a polynomial, the function $P(\phi_1, \dots, \phi_s)$ belongs to $\Gamma(U, \mathscr{O}_U)$ since $\Gamma(U, \mathscr{O}_U)$ as a ring; it follows that it is a continuous function on U, thus its zero set is closed, which shows the continuity of ϕ. If we have $x \in U$ and $f \in \mathscr{O}_{\phi(x),V}$, we can write f locally in the form $f = P/Q$, where P and Q are polynomials and

$Q(\phi(x)) \neq 0$. The function $f \circ \phi$ is then equal to $P \circ \phi / Q \circ \phi$ in a neighborhood of x; from what we gave seen, $P \circ \phi$ and $Q \circ \phi$ are regular in a neighborhood of x. As $Q \circ \phi(x) \neq 0$, it follows that $f \circ \phi$ is regular in a neighborhood of x, q.e.d.

A *composition* of two regular maps is regular. A bijection $\phi : U \to V$ is called a *biregular isomorphism* (or simply an isomorphism) if ϕ and ϕ^{-1} are regular; or equivalently, if ϕ is a homeomorphism of U to V which transforms the sheaf $\mathscr{O}_U$ into the sheaf $\mathscr{O}_V$.

33 Products

If r and r' are two nonnegative integers, we identify the affine space K^{r+r^δ} with the product $K^r \times K^{r^\delta}$. The Zariski topology on K^{r+r^δ} is *finer* than the product of the Zariski topologies on K^r and K^{r^δ}; it is even strictly finer if r and r' are positive. In result, if U and U' are locally closed subspaces of K^r and K^{r^δ}, $U \times U'$ is a locally closed subspace of K^{r+r^δ} and the sheaf $\mathscr{O}_{U\times U^\delta}$ is well defined.

On the other hand, let W be a locally closed subspace of K^t, $t \geq 0$ and let $\phi : W \to U$ and $\phi' : W \to U'$ be two maps. As an immediate result of Proposition 5 we have:

Proposition 6. *A map $x \to (\phi(x), \phi'(x))$ is regular from W to $U \times U'$ if and only if ϕ and ϕ' are regular.*

As any *constant* function is regular, the preceding Proposition shows that any *section* $x \mapsto (x, x'_0)$, $x'_0 \in U'$ is a regular function from U to $U \times U'$; on the other hand, the *projections* $U \times U' \to U$ and $U \times U' \to U'$ are obviously regular.

Let V and V' be locally closed subspaces of K^s and K^{s^δ} and let $\psi : U \to V$ and $\psi' : U' \to V'$ be two mappings. The preceding remarks, together with Proposition 6, show that we then have (cf. [1], Chap. IV):

Proposition 7. *A map $\psi \times \psi' : U \times U' \to V \times V'$ is regular if and only if ψ and ψ' are regular.*

Hence:

Corollary. *A map $\psi \times \psi'$ is a biregular isomorphism if and only if ψ and ψ' are biregular isomorphisms.*

34 Definition of the structure of an algebraic variety

Definition. *We call an algebraic variety over K (or simply an algebraic variety) a set X equipped with:*

1° a topology,

2° a subsheaf $\mathscr{O}_x$ of the sheaf $\mathscr{F}(X)$ of germs of functions on X with values in K,

this data being subject to axioms (VA_I) and (VA_{II}) stated below.

First note that if X and Y are equipped with two structures of the above type, we have a notion of *isomorphism* of X and Y: it is a homeomorphism of X to Y which transforms $\mathscr{O}_X$ to $\mathscr{O}_X$. On the other hand, if X' is an open subset of X, we can equip X' with the induced topology and the induced sheaf: we have a notion of an *induced structure* on an open subset. That being said, we can state the axiom (VA_I):

(VA_I) — *There exists a finite open covering* $\mathsf{V} = \{V_i\}_{i\in I}$ *of the space* X *such that each* V_i*, equipped with the structure induced from* X*, is isomorphic to a locally closed subspace* U_i *of an affine space, equipped with the sheaf* $\mathscr{O}_{U_i}$ *defined in n° 31.*

To simplify the language, we call an *prealgebraic variety* a topological space X together with a sheaf $\mathscr{O}_X$ satisfying the axiom (VA_I). An isomorphism $\phi_i : V_i \to U_i$ is called a *chart* of the open subset V_i; the condition (VA_I) means that it is possible to cover X with finitely many open subsets possessing charts. Proposition 1 from n° 30 shows that X satisfies condition (A), thus it is quasi-compact and so are its subspaces.

The topology on X is called the „Zariski topology" and the sheaf $\mathscr{O}_X$ is called the *sheaf of local rings* of X.

Proposition 8. *Let* X *be a set covered by a finite family of subsets* X_j*,* $j \in J$*. Suppose that each* X_j *is equipped with a structure of a prealgebraic variety and that the following conditions are satisfied:*
(a) $X_i \cap X_j$ *is open in* X_i *for all* $i, j \in J$*,*
(b) the structures induced by X_i *and* X_j *on* $X_i \cap X_j$ *coincide for all* $i, j \in J$*.*
Then there exists a unique structure of a prealgebraic variety on X *such that* X_j *are open in* X *and such that the structure induced on each* X_i *is the given structure.*

The existence and uniqueness of the topology on X and the sheaf $\mathscr{O}_X$ are immediate; it remains to check that this topology and this sheaf satisfy (VA_I), which follows from the fact that X_j form a finite family and satisfy (VA_I).

Corollary. *Let* X *and* X' *be two prealgebraic varieties. There exists a structure of a prealgebraic variety on* $X \times X'$ *satisfying the following condition: If* $\phi : V \to U$ *and* $\phi' : V' \to U'$ *are charts (*V *being open in* X *and* V' *being open in* X'*), then* $V \times V'$ *is open in* $X \times X'$ *and* $\phi \times \phi' : V \times V' \to U \times U'$ *is a chart.*

Cover X by a finite number of open V_i having charts $\phi_i : V_i \to U_i$ and let (V'_j, U'_j, ϕ'_j) be an analogous system for X'. The set $X \times X'$ is covered by $V_i \times V'_j$; equip each $V_i \times V'_j$ with the structure of a prealgebraic variety induced from $U_i \times U'_j$ by $\phi_i^{-1} \times \phi_j'^{-1}$; the assumptions (a) and (b) of Proposition 8 are satisfied for this covering of $X \times X'$, by the corollary of Proposition 7. We obtain a structure of a prealgebraic variety on $X \times X'$ which satisfies appropriate conditions.

We can apply the preceding corollary to the particular case $X' = X$; so $X \times X$ has a structure of a prealgebraic variety, and in particular a topology. We can now state the axiom (VA_{II}):

(VA_{II}) — *The diagonal Δ of $X \times X$ is closed in $X \times X$.*

Suppose that X is a prealgebraic variety obtained by the „gluing" procedure of Proposition 8; then the condition (VA_{II}) is satisfied if and only if $X_{ij} = \Delta \cap X_i \times X_j$ is closed in $X_i \times X_j$. Or X_{ij} is the set of (x, x) for $x \in X_i \cap X_j$. Suppose that there exist charts $\phi : X_i \to U_i$ and let $T_{ij} = \phi \times \phi_j(X_{ij})$; T_{ij} is the set of $(\phi_i(x), \phi_j(x))$ for x running over $X_i \cap X_j$. The axiom (VA_{II}) takes therefore the following form:

(VA'_{II}) — *For each pair (i, j), T_{ij} is closed in $U_i \times U_j$.*

In this form we recognize Weil's axiom (A) (cf. [16], p. 167), except that Weil considered only irreducible varieties.

Examples of algebraic varieties: Any locally closed subspace U of an affine space, equipped with the induced topology and the sheaf $\mathscr{O}_U$ defined in n° 31 is an algebraic variety. Any projective variety is an algebraic variety (cf. n° 51). Any algebraic fiber space (cf. [17]) whose base and fiber are algebraic varieties is an algebraic variety.

Remarks. (1) We observe an analogy between condition (VA_{II}) and the condition of *separatedness* imposed on topological, differential and analytic varieties.

(2) Simple examples show that condition (VA_{II}) is not a consequence of condition (VA_I).

35 Regular mappings, induced structures, products

Let X and Y be two algebraic varieties and let ϕ be a function from X to Y. We say that ϕ is *regular* if:

(a) ϕ is continuous.

(b) If $x \in X$ and $f \in \mathscr{O}_{\phi(x),Y}$ then $f \circ \phi \in \mathscr{O}_{x,X}$.

As in n° 32, the composition of two regular functions is regular and a bijection $\phi : X \to Y$ is an isomorphism if and only if ϕ and ϕ^{-1} are regular functions. Regular functions form a family of *morphisms* for the structure of an algebraic variety in the sense of [1], Chap. IV.

Let X be an algebraic variety and let X' be a locally closed subspace of X. We equip X' with the topology induced from X and the sheaf $\mathscr{O}_{X^\flat}$ induced by $\mathscr{O}_X$ (to be precise, for all $x \in X'$ we define $\mathscr{O}_{x,X^\flat}$ as the image of $\mathscr{O}_{x,X}$ under the canonical homomorphism $\mathscr{F}(X)_x \to \mathscr{F}(X')_x$). The axiom (VA_I) is satisfied: if $\phi_i : V_i \to U_i$ is a system of charts such that $X = \ V_i$, we set $V_i' = X' \cap V_i$, $U_i' = \phi_i(V_i')$ and $\phi_i : V_i' \to U_i'$ is a system of charts such that $X' = \ V_i'$. The axiom (VA_{II}) is satisfied as well since the topology of $X' \times X'$ is induced from

$X \times X$ (we could also use (VA'_{II})). We define the structure of an algebraic variety on X' which is *induced* by that of X; we also say that X' is a *subvariety* of X (in Weil [16], the term ,,subvariety" is reserved for what we call here an irreducible closed subvariety). If ι denotes the inclusion of X' in X, ι is a regular mapping; moreover, if ϕ is a function from an algebraic variety Y to X' then $\phi : Y \to X'$ is regular if and only if $\iota \circ \phi : Y \to X$ is regular (which justifies the term ,,induced structure", cf. [1], loc. cit.).

If X and X' are two algebraic varieties, $X \times X'$ is an algebraic variety, called the *product variety*; it suffices to check that the axiom (VA'_{II}) is satisfied, in other words, that if $\phi_i : V_i \to U_i$ and $\phi'_i : V'_i \to U'_i$ are systems of charts such that $X = \bigcup V_i$ and $X' = \bigcup V'_i$, then the set $T_{ij} \times T'_{i'j'}$ is closed in $U_i \times U_j \times U'_{i'} \times V'_{j'}$ (with the notations of n° 34); this follows immediately from the fact that T_{ij} and $T'_{i'j'}$ are closed in $U_i \times U_j$ and $U'_{i'} \times U'_{j'}$ respectively.

Propositions 6 and 7 are valid without change for arbitrary algebraic varieties.

If $\phi : X \to Y$ is a regular mapping, the graph Φ of ϕ is *closed* in $X \times Y$, because it is the inverse image of the diagonal $Y \times Y$ by $\phi \times 1 : X \times Y \to Y \times Y$; moreover, the mapping $\psi : X \to \Phi$ defined by $\psi(x) = (x, \phi(x))$ is an isomorphism: indeed, ψ is a regular mapping, and so is ψ^{-1} (since it is a restriction of the projection $X \times Y \to X$).

36 The field of rational functions on an irreducible variety

We first show two lemmas of purely topological nature:

Lemma 1. *Let X be a connected space, G an abelian group and $\mathscr{G}$ a constant sheaf on X isomorphic to G. The canonical mapping $G \to \Gamma(X, \mathscr{G})$ is bijective.*

An element of $\Gamma(X, \mathscr{G})$ is just a continuous mapping from X to G equipped with the discrete topology. Since X is connected, any such a mapping is constant, hence the Lemma.

We call a sheaf $\mathscr{F}$ on a space X *locally constant* if any point x has an open neighborhood U such that $\mathscr{F}(U)$ is constant on U.

Lemma 2. *Any locally constant sheaf on an irreducible space is constant.*

Let $\mathscr{F}$ be a sheaf, X a space and set $F = \Gamma(X, \mathscr{F})$; it suffices to demonstrate that the canonical homomorphism $\rho_x : F \to \mathscr{F}_x$ is bijective for all $x \in X$, because we would thus obtain an isomorphism of the constant sheaf isomorphic to F with the given sheaf $\mathscr{F}$.

If $f \in F$, the set of points $x \in X$ such that $f(x) = 0$ is open (by the general properties of sheaves) and closed (because $\mathscr{F}$ is locally constant); since an irreducible space is connected, this set is either $\emptyset$ or X, which shows that ρ_x is injective.

Now take $m \in \mathscr{F}_x$ and let s be a section of $\mathscr{F}$ over a neighborhood U of x such that $s(x) = m$; cover X by nonempty open subsets U_i such that $\mathscr{F}(U_i)$ is constant on U_i; since X is irreducible, we have $U \cap U_i \neq \emptyset$; choose a point $x_i U \cap U_i$; obviously there exists a section s_i of $\mathscr{F}$ over U_i such that $s_i(x_i) = s(x_i)$, and since the sections s and s_i coincide in x_i, they coincide on whole $U \cap U_i$, since $U \cap U_i$ is irreducible, hence connected; analogously s_i and s_j coincide on $U_i \cap U_j$, since they coincide on $U \cap U_i \cap U_j \neq \emptyset$; thus the sections s_i define a unique section s of $\mathscr{F}$ over X and we have $\rho_x(s) = m$, which ends the proof.

Now let X be an irreducible algebraic variety. If U is a nonempty open subset of X, set $\mathscr{A}_U = \Gamma(U, \mathscr{O}_X)$; $\mathscr{A}_U$ is an *integral domain*: indeed, suppose that we have $f \cdot g = 0$, f and g being regular functions from U to K; if F (resp. G) denotes the set of $x \in U$ such that $f(x) = 0$ (resp. $g(x) = 0$), we have $U = F \cup G$ and F and G are closed in U, because f and g are continuous; since U is irreducible, it follows that $F = U$ or $G = U$, which means exactly that f or g is zero on U. We can therefore form the field of fractions of $\mathscr{A}_U$, which we denote by $\mathscr{K}_U$; if $U \subset V$, the homomorphism $\rho_U^V : \mathscr{A}_V \to \mathscr{A}_U$ is injective, because U is dense in V, and we have a well defined isomorphism ϕ_U^V of $\mathscr{K}_V$ to $\mathscr{K}_U$; the system of $\{\mathscr{K}_U, \phi_U^V\}$ defines a *sheaf of fields* $\mathscr{K}$; then $\mathscr{K}_x$ is canonically isomorphic with the field of fractions of $\mathscr{O}_{x,X}$.

Proposition 9. *For any irreducible algebraic variety X, the sheaf $\mathscr{K}$ defined above is a constant sheaf.*

By Lemma 2, it suffices to show the Proposition when X is a locally closed subvariety of the affine space K^r; let F be the closure of X in K^r and let $I(F)$ be the ideal in $K[X_1, \ldots, X_r]$ of polynomials vanishing on F (or equivalently on X). If we set $A = K[X_1, \ldots, X_r]/I(F)$, the ring A is an integral domain because X is irreducible; let $K(A)$ be the ring of fractions of A. By corollary of Proposition 4, we can identify $\mathscr{O}_{x,X}$ with the localization of A in the maximal ideal defined by x; we thus obtain an isomorphism of the field $K(A)$ with the field of fractions of $\mathscr{O}_{x,X}$ and it is easy to check that it defines an isomorphism of the constant sheaf equal to $K(A)$ with the sheaf $\mathscr{K}$, which shows the Proposition.

By Lemma 1, the sections of the sheaf $\mathscr{K}$ form a field, isomorphic with $\mathscr{K}_x$ for all $x \in X$, which we denote by $K(X)$. We call it the *field of rational functions* on X; it is an extension of finite type[1] of the field K, whose transcendence degree over K is the *dimension* of X (we extend this definition to reducible varieties by imposing $\dim X = \operatorname{Sup} \dim Y_i$ if X is a union of closed irreducible varieties Y_i). In general, we identify the field $K(X)$ with the field $\mathscr{K}_x$; since we have $\mathscr{O}_{x,X} \subset \mathscr{K}_x$, we see that we can view $\mathscr{O}_{x,X}$ as a *subring* of $K(X)$ (it is the ring of specialization of the point x in $K(X)$ in the sense of Weil, [16], p. 77). If U is an open subset of X, $\Gamma(U, \mathscr{O}_X)$ is the intersection in $K(X)$ of the rings $\mathscr{O}_{x,X}$ for x running over U.

If Y is a subvariety of X, we have $\dim Y \leq \dim X$; if furthermore Y is closed and does not contain any irreducible component of X, we have $\dim Y < \dim X$,

[1] i.e. finitely generated

as shown by reducing to the case of subvarieties of K^r (cf. for example [8], Chap. X, §5, th. II).

§2 Coherent algebraic sheaves

37 The sheaf of local rings on an algebraic variety

Return to the notations of n° 31: let $X = K^r$ and let $\mathscr{O}$ be the sheaf of local rings of X. We have:

Lemma 1. *The sheaf $\mathscr{O}$ is a coherent sheaf of rings, in the sense of n° 15.*

Let $x \in X$, let U be an open neighborhood of x and let $f_1, \dots, f_p$ be sections of $\mathscr{O}$ over U, i.e. rational functions regular at each point of U; we must show that the sheaf of relations between $f_1, \dots, f_p$ is a sheaf of finite type over $\mathscr{O}$. Possibly replacing U by a smaller neighborhood, we can assume that f_i can be written in the form $f_i = P_i/Q$ where P_i and Q are polynomials and Q does not vanish on U. Let now $y \in U$ and $g_i \in \mathscr{O}_y$ such that $\sum_{i=1}^{i=p} g_i f_i$ is zero in a neighborhood of y; we can again write g_i in the form $g_i = R_i/S$ where R_i and S are polynomials and S does not vanish in y. The relationship „$\sum_{i=1}^{i=p} g_i f_i = 0$ in a neighborhood of y" is equivalent to the relationship „$\sum_{i=1}^{i=p} R_i P_i = 0$ in a neighborhood of y", i.e. equivalent to $\sum_{i=1}^{i=p} R_i P_i = 0$. As the module of relations between the polynomials P_i is a module of finite type (because the ring of polynomials is Noetherian), it follows that the sheaf of relations between f_i is of finite type.

Let now V be a closed subvariety of $X = K^r$; for any $x \in X$ let $\mathscr{I}_x(V)$ be the ideal of $\mathscr{O}_x$ consisting of elements $f \in \mathscr{O}_x$ whose restriction to V is zero in a neighborhood of x (we thus have $\mathscr{I}_x(V) = \mathscr{O}_x$ if $x \notin V$). The $\mathscr{I}_x(V)$ form a subsheaf $\mathscr{I}(V)$ of the sheaf $\mathscr{O}$.

Lemma 2. *The sheaf $\mathscr{I}(V)$ is a coherent sheaf of $\mathscr{O}$-modules.*

Let $I(V)$ be the ideal of $K[X_1, \dots, X_r]$ consisting of polynomials P vanishing on V. By Proposition 4 from n° 31, $\mathscr{I}_x(V)$ is equal to $I(V) \cdot \mathscr{O}_x$ for all $x \in V$ and this formula remains valid for $x \notin V$ as shown immediately. The ideal $I(V)$ being generated by a finite number of elements, it follows that the sheaf $\mathscr{I}(V)$ is of finite type, thus coherent by Lemma 1 and Proposition 8 from n° 15.

We shall now extend Lemma 1 to arbitrary algebraic varieties:

Proposition 1. *If V is an algebraic variety, the sheaf $\mathscr{O}_V$ is a coherent sheaf of rings on V.*

The question being local, we can suppose that V is a closed subvariety of the affine space K^r. By Lemma 2, the sheaf $\mathscr{I}(V)$ is a coherent sheaf of ideals, thus the sheaf $\mathscr{O}/\mathscr{I}(V)$ is a coherent sheaf of rings on X, by Theorem 3 from n° 16. This sheaf of rings is zero outside V and its restriction to V is just $\mathscr{O}_V$ (n° 31); thus the sheaf $\mathscr{O}_V$ is a coherent sheaf of rings on V (n° 17, corollary of Proposition 11).

Remark. It is clear that Proposition is valid more generally for any prealgebraic variety.

38 Coherent algebraic sheaves

If V is an algebraic variety whose sheaf of local rings is $\mathscr{O}_V$, we call an *algebraic sheaf* on V any sheaf of $\mathscr{O}_V$–modules, in the sense of n° 6; if $\mathscr{F}$ and $\mathscr{G}$ are two algebraic sheaves, we say that $\phi : \mathscr{F} \to \mathscr{G}$ is an *algebraic homomorphism* (or simply a homomorphism) if it is a $\mathscr{O}_V$–homomorphism; recall that this is equivalent to saying that $\phi_x : \mathscr{F}_x \to \mathscr{G}_x$ is $\mathscr{O}_{x,V}$–linear and that ϕ transforms local sections of $\mathscr{F}$ into local sections of $\mathscr{G}$.

If $\mathscr{F}$ is an algebraic sheaf on V, the cohomology groups $H^q(V, \mathscr{F})$ are modules over $\Gamma(V, \mathscr{O}_V)$, cf. n° 23; in particular, they are *vector spaces* over K.

An algebraic sheaf $\mathscr{F}$ over V is said to be *coherent* if it is a coherent sheaf of $\mathscr{O}_V$–modules, in the sense of n° 12; by Proposition 7 of n° 15 and Proposition 1 above, these sheaves are characterized by the property of being locally isomorphic to the cokernel of an algebraic homomorphism $\phi : \mathscr{O}_V^q \to \mathscr{O}_V^p$.

We shall give some examples of coherent algebraic sheaves (we will see more of them later, cf. in particular n^{os} 48, 57).

39 Sheaf of ideals defined by a closed subvariety

Let W be a closed subvariety of an algebraic variety V. For any $x \in V$, let $\mathscr{J}_x(W)$ be the ideal of $\mathscr{O}_{x,V}$ consisting of elements f whose restriction to W is zero in a neighborhood of x; let $\mathscr{J}(W)$ be the subsheaf of $\mathscr{O}_V$ formed by $\mathscr{J}_x(W)$. We have the following Proposition, generalizing Lemma 2:

Proposition 2. *The sheaf $\mathscr{J}(W)$ is a coherent algebraic sheaf.*

The question being local, we can suppose that V (thus also W) is a closed subvariety of the affine space K^r. It follows from Lemma 2, applied to W, that the sheaf of ideals defined by W in K^r is of finite type; this shows that $\mathscr{J}(W)$, which is its image under the canonical homomorphism $\mathscr{O} \to \mathscr{O}_V$, is also of finite type, thus is coherent by Proposition 8 of n° 15 and Proposition 1 of n° 37.

Let $\mathscr{O}_W$ be the sheaf of local rings of W and let $\mathscr{O}_W^V$ be the sheaf on V obtained by extending $\mathscr{O}_W$ by 0 outside W (cf. n° 5); this sheaf is canonically isomorphic to $\mathscr{O}_V / \mathscr{J}(W)$, in other words, we have an exact sequence:

$$0 \to \mathscr{J}(W) \to \mathscr{O}_V \to \mathscr{O}_W^V \to 0.$$

Let then $\mathscr{F}$ be an algebraic sheaf on W and let $\mathscr{F}^V$ be the sheaf obtained by extending $\mathscr{F}$ by 0 outside W; we can consider $\mathscr{F}^V$ as a sheaf of $\mathscr{O}_W^V$–modules, thus also as a sheaf of $\mathscr{O}_V$–modules whose annihilator contains $\mathscr{J}(W)$. We have:

Proposition 3. *If $\mathscr{F}$ is a coherent algebraic sheaf on W, $\mathscr{F}^V$ is a coherent algebraic sheaf on V. Conversely, if $\mathscr{G}$ is an coherent algebraic sheaf on V whose* *annihilator contains* $\mathscr{J}(W)$*, the restriction of* $\mathscr{G}$ *to* W *is a coherent algebraic sheaf on* W*.*

If $\mathscr{F}$ is a coherent algebraic sheaf on W, $\mathscr{F}^V$ is a coherent sheaf of $\mathscr{O}_W^V$–modules (n° 17, Proposition 11), thus a coherent sheaf of $\mathscr{O}_V$–modules (n° 16, Theorem 3). Conversely, if $\mathscr{G}$ is a coherent algebraic sheaf on V whose annihilator contains $\mathscr{J}(W)$, $\mathscr{G}$ can be considered as a sheaf of $\mathscr{O}_V/\mathscr{J}(W)$–modules, and is a coherent sheaf (n° 16, Theorem 3); the restriction of $\mathscr{G}$ to W is then a coherent sheaf of $\mathscr{O}_W$–modules (n° 17, Proposition 11).

So, any coherent algebraic sheaf on W can by identified with an algebraic coherent sheaf on V (and this identification does not change cohomology groups, by Proposition 8 of n° 26). In particular, any coherent algebraic sheaf on an affine (resp. projective) variety can be considered as a coherent algebraic sheaf on an affine (resp. projective) space; we will frequently use this possibility later.

Remark. Let $\mathscr{G}$ be a coherent algebraic sheaf on V which is zero outside W; the annihilator of $\mathscr{G}$ *does not necessarily contain* $\mathscr{J}(W)$ (in other words, $\mathscr{G}$ not always can be considered as an coherent algebraic sheaf on W); all we can say is that it contains a *power* of $\mathscr{J}(W)$.

40 Sheaves of fractional ideals

Let V be an irreducible algebraic variety and let $K(V)$ denote the constant sheaf of rational functions on V (cf. n° 36); $K(V)$ is an algebraic sheaf which is not coherent if $\dim V > 0$. An algebraic subsheaf $\mathscr{F}$ of $K(V)$ can be called a „sheaf of fractional ideals" since each $\mathscr{F}_x$ is a fractional ideal of $\mathscr{O}_{x,V}$.

Proposition 4. *An algebraic subsheaf* $\mathscr{F}$ *of* $K(V)$ *is coherent if and only if it is of finite type.*

The necessity is trivial. To prove the sufficiency, it suffices to prove that $K(V)$ satisfies condition (b) of definition 2 from n° 12, in other that if $f_1, \ldots, f_p$ are rational functions, the sheaf $\mathscr{R}(f_1, \ldots, f_p)$ is of finite type. If x is a point of V, we can find functions g_i and h such that $f_i = g_i/h$, g_i and h being regular in a neighborhood U of x and h being nonzero on U; the sheaf $\mathscr{R}(f_1, \ldots, f_p)$ is then equal to the sheaf $\mathscr{R}(g_1, \ldots, g_p)$, which is of finite type, since $\mathscr{O}_V$ is a coherent sheaf of rings.

41 Sheaf associated to the total space of a vector bundle

Let E be an algebraic fiber space with a vector space of dimension r as a fiber and an algebraic variety V as a base; by definition, the typical fiber of E is a vector space K^r and the structure group is the linear group $GL(r, K)$ acting on K^r in the usual way (for the definition of an algebraic fiber space, cf. [17]; see also [15], n° 4 for analytic vector bundles).

If U is an open subset of V, let $\mathscr{S}(E)_U$ denote the set of regular sections of E on U; if $V \supset U$, we have the restriction homomorphism $\phi_U^V : \mathscr{S}(E)_V \to \mathscr{S}(E)_U$; thus a sheaf $\mathscr{S}(E)$, called the *sheaf of germs of sections* of E. Since E is a vector bundle, each $\mathscr{S}(E)_U$ is a $\Gamma(U, \mathscr{O}_V)$–module and it follows that $\mathscr{S}(E)$ is an algebraic sheaf on V. If we identify locally E with $V \times K^r$, we have:

Proposition 5. *The sheaf $\mathscr{S}(E)$ is locally isomorphic to $\mathscr{O}_V^r$; in particular, it is a coherent algebraic sheaf.*

Conversely, it is easily seen that any algebraic sheaf $\mathscr{F}$ on V, locally isomorphic to $\mathscr{O}_V^r$, is isomorphic to a sheaf $\mathscr{S}(E)$ where E is determined up to isomorphism (cf. [15] for the analytic case).

If V is a variety without singularities, we can take for E the vector bundle of *p-covectors* tangent to V (p being a nonnegative integer); let Ω^p be the sheaf corresponding to $\mathscr{S}(E)$; an element of Ω_x^p, $x \in V$ is just a differential form of degree p on V, regular in x. If we set $h^{p,q} = \dim_K H^q(V, \Omega^p)$, we know that in the classical case (and if V is projective), $h^{p,q}$ is equal to the dimension of harmonic forms of type (p, q) (theorem of Dolbeault[2] and, if B_n denotes the n-th Betti number of V, we have $B_n = {}_{p+q=n} h^{p,q}$. In the general case, we could take the above formula for the *definition* of the Betti numbers of a nonsingular projective variety (we will see in n° 66 that $h^{p,q}$ are finite). It is convenient to study their properties, in particular to see if they coincide with those involved in the Weil conjectures for varieties over finite fields[3]. We only mention that they satisfy the „Poincaré duality" $B_n = B_{2m-n}$ when V is an irreducible of dimension m.

The cohomology groups $H^q(V, \mathscr{S}(E))$ are also involved in other issues, including the Riemann-Roch, as well as in the classification of algebraic fiber spaces with base V and the affine group $x \mapsto ax + b$ as the structural group (cf. [17], §4, where the case when $\dim V = 1$ is studied).

[2] P. Dolbeault. Sur la cohomologie des variétés analytiques complexes. C. R. Paris, 246, 1953, p. 175-177.
[3] Bulletin Amer. Math. Soc., 55, 1949, p.507

§3 Coherent algebraic sheaves on affine varieties

42 Affine varieties

An algebraic variety V is said to be *affine* if it is isomorphic to a closed subvariety of an affine space. The product of two affine varieties is an affine variety; any closed subvariety of an affine variety is an affine variety.

An open subset U of an algebraic variety V is said to be *affine* if, equipped with the structure of an algebraic variety induced from X, it is an affine variety.

Proposition 1. *Let U and V be two open subsets of an algebraic variety X. If U and V are affine, $U \cap V$ is affine.*

Let Δ be the diagonal of $X \times X$; by n° 35, the mapping $x \mapsto (x, x)$ is a biregular isomorphism from X onto Δ; thus the restriction of this map to $U \cap V$ is a biregular isomorphism of $U \cap V$ onto $\Delta \cap U \times V$. Since U and V are affine varieties, $U \times V$ is also an affine variety; on the other hand, Δ is closed in $X \times X$ by the axiom (VA_{II}), thus $\Delta \cap U \times V$ is closed in $U \times V$, hence affine, q.e.d.

(It is easily seen that this Proposition is false for prealgebraic varieties; the axiom (VA_{II}) plays an essential role).

Let us now introduce a notation which will be used thorough the rest of this paragraph: if V is an algebraic variety and f is a regular function on V, we denote by V_f the open subset of V consisting of all points $x \in V$ for which $f(x) \neq 0$.

Proposition 2. *If V is an affine algebraic variety and f is a regular function on V, the open subset V_f is affine.*

Let W be the subset of $V \times K$ consisting of pairs (x, λ) such that $\lambda \cdot f(x) = 1$; it is clear that W is closed in $V \times K$, thus it is an affine variety. For all $(x, \lambda) \in W$ set $\pi(x, \lambda) = x$; the mapping π is a regular mapping from W to V_f. Conversely, for all $x \in V_f$, set $\omega(x) = (x, 1/f(x))$; the mapping $\omega : V_f \to W$ is regular and we have $\pi \circ \omega = 1$, $\omega \circ \pi = 1$, thus V_f and W are isomorphic, q.e.d.

Proposition 3. *Let V be a closed subvariety of K^r, F be a closed subset of V and let $U = V - F$. The open subsets V_P form a base for the topology of U when P runs over the set of polynomials vanishing on F.*

Let $U' = V - F'$ be an open subset of U and let $x\ inU'$; we must show that there exists a P for which $V_P \subset U'$ and $x \in V_P$; in other words, P has to be zero on F' and nonzero in x; the existence of such a polynomial follows simply from the definition of the topology of K^r.

Theorem 1. *The open affine subsets of an algebraic variety X form an open base for the topology of X.*

The question being local, we can assume that X is a locally closed subspace of an affine space K^r; in this case, the theorem follows immediately from Propositions 2 and 3.

Corollary. *The coverings of X consisting of open affine subsets are arbitrarily fine.*

We note that if $\mathfrak{U} = \{U_i\}_{i\in I}$ is such a covering, the $U_{i_0\ldots i_p}$ are also open affine subsets, by Proposition 1.

43 Some preliminary properties of irreducible varieties

Let V be a closed subvariety of K^r and let $I(V)$ be the ideal of $K[X_1,\ldots,X_r]$ consisting of polynomials vanishing on V; let A be the quotient ring $K[X_1,\ldots,X_r]/I(V)$; we have a canonical homomorphism

$$\iota : A \to \Gamma(V, \mathscr{O}_V)$$

that is injective by the definition of $I(V)$.

Proposition 4. *If V is irreducible, $\iota : A \to \Gamma(V, \mathscr{O}_V)$ is bijective.*

(In fact, this holds for any closed subvariety of K^r, as will be shown in the next n°).

Let $K(V)$ be the field of fractions of A; by n° 36, we can identify $\mathscr{O}_{x,V}$ with the localization of A in the maximal ideal $\mathfrak{m}_x$ consisting of polynomials vanishing in x, and we have $\Gamma(V,\mathscr{O}_V) = A = \bigcap_{x\in V} \mathscr{O}_{x,V}$ (all $\mathscr{O}_{x,V}$ being considered as subrings of $K(V)$). But all maximal ideals of A are $\mathfrak{m}_x$, since K is algebraically closed (Hilbert's theorem of zeros); it follows immediately (cf. [8], Chap. XV, §5, th. X) that $A = \bigcap_{x\in V} \mathscr{O}_{x,V} = \Gamma(V,\mathscr{O}_V)$, q.e.d.

Proposition 5. *Let X be an irreducible algebraic variety, Q a regular function on X and P a regular function on X_Q. Then, for n sufficiently large, the rational function Q^nP is regular on the whole of X.*

By quasi-compactness of X, the question is local; by Theorem 1, we can thus suppose that X is a closed subvariety of K^r. The above Proposition shows that then Q is an element of $A = K[X_1,\ldots,X_r]/(I(X))$. The assumption made on P means that for any point $x \in X_Q$ we can write $P = P_x/Q_x$ with P_x and Q_x in A and $Q_x(x) \neq 0$; if $\mathfrak{a}$ denotes the ideal of A generated by all Q_x, the variety of zeros of $\mathfrak{a}$ is contained in the variety of zeros of Q; by Hilbert's theorem of zeros, this leads to $Q^n \in \mathfrak{a}$ for n sufficiently large, hence $Q^n = \sum R_xQ_x$ and $Q^nP = \sum R_xP_x$ with $R_x \in A$, which shows that Q^nP is regular on X.

(We could also use the fact that X_Q is affine if X is and apply Proposition 4 to X_Q).

Proposition 6. *Let X be an irreducible algebraic variety, Q a regular function on X, $\mathscr{F}$ a coherent algebraic sheaf on X and s a section of $\mathscr{F}$ over X*

whose restriction to X_Q is zero. Then for n sufficiently large the section $Q^n s$ is zero on the whole of X.

The question being again local, we can assume:
(a) that X is a closed subvariety of K^r,
(b) that $\mathscr{F}$ is isomorphic to a cokernel of a homomorphism $\phi : \mathscr{O}_X^p \to \mathscr{O}_X^q$,
(c) that s is the image of a section σ of $\mathscr{O}_X^q$.

(Indeed, all the above conditions are satisfied locally).

Set $A = \Gamma(X, \mathscr{O}_X) = K[X_1, \ldots, X_r]/I(X)$. The section σ can be identified with a system of q elements of A. Let on the other hand

$$t_1 = \phi(1, 0, \ldots, 0), \ldots, t_p = \phi(0, \ldots, 0, 1);$$

the t_i, $1 \leq i \leq p$ are sections of $\mathscr{O}_X^q$ over X, thus can be identified with systems of q elements of A. The assumption made on s means that for all $x \in X_Q$ we have $\sigma(x) \in \phi(\mathscr{O}_{x,X}^p)$, that is, σ can be written in the form $\sigma = \sum_{i=1}^{i=p} f_i \cdot t_i$ with $f_i \in \mathscr{O}_{x,X}$; or, by clearing denominators, that there exist $Q_x \in A$, $Q_x(x) \neq 0$ for which $Q_x \cdot \sigma = \sum_{i=1}^{i=p} R_i \cdot t_i$ with $R_i \in A$. The reasoning used above shows then that, for n sufficiently large, Q^n belongs to the ideal generated by Q_x, hence $Q^n \sigma(x) \in \phi(\mathscr{O}_{x,X}^p)$ for all $x \in X$, which means that $Q^n s$ is zero on the whole of X.

44 Vanishing of certain cohomology groups

Proposition 7. *Let X be an irreducible algebraic variety, Q_i a finite family of regular functions on X that do not vanish simultaneously and U the open covering of X consisting of $X_{Q_i} = U_i$. If $\mathscr{F}$ is a coherent algebraic subsheaf of $\mathscr{O}_X^p$, we have $H^q(\mathsf{U}, \mathscr{F}) = 0$ for all $q > 0$.*

Possibly replacing U by an equivalent covering, we can assume that none of the functions Q_i vanishes identically, in other words that we have $U_i \neq \emptyset$ for all i.

Let $f = (f_{i_0 \ldots i_q})$ be a q-cocycle of U with values in $\mathscr{F}$. Each $f_{i_0 \ldots i_q}$ is a section of $\mathscr{F}$ over $U_{i_0 \ldots i_q}$, thus can be identified with a system of p regular functions on $U_{i_0 \ldots i_q}$; applying Proposition 5 to $Q = Q_{i_0} \ldots Q_{i_q}$ we see that, for n sufficiently large, $g_{i_0 \ldots i_q} = (Q_{i_0} \ldots Q_{i_q})^n f_{i_0 \ldots i_q}$ is a system of p regular functions on X. Choose an integer n for which this holds for all systems $i_0, \ldots, i_q$, which is possible because there is a finite number of such systems. Consider the image of $g_{i_0 \ldots i_q}$ in the coherent sheaf $\mathscr{O}_X^p / \mathscr{F}$; this is a section vanishing on $U_{i_0 \ldots i_q}$; then applying Proposition 6 we see that for m sufficiently large, the product of this section with $(Q_{i_0} \ldots Q_{i_q})^m$ is zero on the whole of X. Setting $N = m + n$, we see that we have constructed sections $h_{i_0 \ldots i_q}$ of $\mathscr{F}$ over X which coincide with $(Q_{i_0} \ldots Q_{i_q})^N f_{i_0 \ldots i_q}$ on $U_{i_0 \ldots i_q}$.

As the Q_i^N do not vanish simultaneously, there exist functions

$$R_i \in \Gamma(X, \mathscr{O}_X)$$

such that $\sum R_i Q_i^N = 1$. Then for any system $i_0, \dots, i_{q-1}$ set

$$k_{i_0 \dots i_{q-1}} = \sum_i R_i h_{i i_0 \dots i_{q-1}} / (Q_{i_0} \dots Q_{i_{q-1}})^N,$$

which makes sense because $Q_{i_0} \dots Q_{i_{q-1}}$ is nonzero on $U_{i_0 \dots i_{q-1}}$.

We have thus defined a cochain $k \in C^{q-1}(\mathsf{U}, \mathscr{F})$. I claim that $f = dk$, which will show the Proposition.

We must check that $(dk)_{i_0 \dots i_q} = f_{i_0 \dots i_q}$; it suffices to show that these two sections coincide on $U = \bigcup U_i$, since they will coincide everywhere, because they are systems of p rational functions on X and $U \neq 0$. Now over U, we can write

$$k_{i_0 \dots i_{q-1}} = \sum_i R_i \cdot Q_i^N \cdot f_{i i_0 \dots i_q},$$

hence

$$(dk)_{i_0 \dots i_q} = \sum_{j=0}^{j=q} (-1)^q \sum_i R_i \cdot Q_i^N \cdot f_{i i_0 \dots \hat{i}_j \dots i_q}$$

and taking into account that f is a cocycle,

$$(dk)_{i_0 \dots i_q} = \sum_i R_i \cdot Q_i^N \cdot f_{i_0 \dots i_q} = f_{i_0 \dots i_q}, \qquad \textit{q.e.d.}$$

Corollary 1. *$H^q(X, \mathscr{F}) = 0$ for $q > 0$.*

Indeed, Proposition 3 shows that coverings of the type used in Proposition 7 are arbitrarily fine.

Corollary 2. *The homomorphism $\Gamma(X, \mathscr{O}_X^p) \to \Gamma(X, \mathscr{O}_X^p / \mathscr{F})$ is surjective.*

This follows from Corollary 1 above and from Corollary 2 to Proposition 6 from n° 24.

Corollary 3. *Let V be a closed subvariety of K^r and let*

$$A = K[X_1, \dots, X_r]/I(V).$$

Then the homomorphism $\iota : A \to \Gamma(V, \mathscr{O}_V)$ is bijective.

We apply Corollary 2 above to $X = K^r$, $p = 1$, $\mathscr{F} = \mathscr{J}(V)$, the sheaf of ideals defined by V; we obtain that every element of $\Gamma(V, \mathscr{O}_V)$ is the restriction of a section of $\mathscr{O}$ on X, that is, a polynomial, by Proposition 4 applied to X.

45 Sections of a coherent algebraic sheaf on an affine variety

Theorem 2. *Let $\mathscr{F}$ be a coherent algebraic sheaf on an affine variety X. For every $x \in X$, the $\mathscr{O}_{x,X}$–module $\mathscr{F}_x$ is generated by elements of $\Gamma(X, \mathscr{F})$.*

Since X is affine, it can be embedded as a closed subvariety of an affine space K^r; by extending the sheaf $\mathscr{F}$ by 0 outside X, we obtain a coherent algebraic sheaf on K^r (cf. n° 39) and we are led to prove the theorem for the new sheaf. In other words, we can suppose that $X = K^r$.

By the definition of a coherent sheaf, there exists a covering of X consisting of open subsets on which $\mathscr{F}$ is isomorphic with a quotient of the sheaf $\mathscr{O}^p$. Applying Proposition 3, we see that there exists a finite number of polynomials Q_i that do not vanish simultaneously and such that on every $U_i = X_{Q_i}$ there exists a surjective homomorphism $\phi_i : \mathscr{O}^{p_i} \to \mathscr{F}$; we can furthermore assume that none of the polynomials is identically zero.

The point x belongs to one U_i, say U_0; it is clear that $\mathscr{F}_x$ is generated by sections of $\mathscr{F}$ over U_0; as Q_0 is invertible in $\mathscr{O}_x$, it suffices to prove the following lemma:

Lemma 1. *If s_0 is a section of $\mathscr{F}$ over U_0, there exists an integer N and a section s of $\mathscr{F}$ over X such that $s = Q_0^N \cdot s_0$ over U_0.*

By Proposition 2, $U_i \cap U_0$ is an affine variety, obviously irreducible; by applying Corollary 2 of Proposition 7 to this variety and to $\phi_i : \mathscr{O}^{p_i} \to \mathscr{F}$, we see that there exists a section σ_{0i} of $\mathscr{O}^{p_i}$ on $U_i \cap U_0$ such that $\phi_i(\sigma_{0i}) = s_0$ on $U_i \cap U_0$; as $U_i \cap U_0$ is the set of points of U_i in which Q_0 does not vanish, we can apply Proposition 5 to $X = U_i$, $Q = Q_0$ and we see that there exists, for n sufficiently large, a section σ_i of $\mathscr{O}^{p_i}$ over U_i which coincides with $Q_0^n \cdot \sigma_{0i}$ over $U_i \cap U_0$; by setting $s'_i = \phi_i(\sigma_i)$, we obtain a section of $\mathscr{F}$ over U_i that coincides with $Q_0^n \cdot s_0$ over $U_i \cap U_0$. The sections s'_i and s'_j coincide on $U_i \cap U_j \cap U_0$; applying Proposition 6 to $s'_i - s'_j$, we see that for m sufficiently large we have $Q_0^m(s'_i - s'_j) = 0$ on the whole of $U_i \cap U_j$. The $Q_0^m \cdot s'_i$ then define a unique section s of $\mathscr{F}$ over X, and we have $s = Q_0^{n+m} s_0$ on U_0, which shows the lemma and completes the proof of Theorem 2.

Corollary 1. *The sheaf $\mathscr{F}$ is isomorphic to a quotient sheaf of the sheaf $\mathscr{O}_X^p$.*

Because $\mathscr{F}_x$ is an $\mathscr{O}_{x,X}$–module of finite type, it follows from the above theorem that there exists a finite number of sections of $\mathscr{F}$ generating $\mathscr{F}_x$; by Proposition 1 of n° 12, these sections generate $\mathscr{F}_y$ for y sufficiently close to x. The space X being quasi-compact, we conclude that there exists a finite number of sections $s_1, \ldots, s_p$ of $\mathscr{F}$ generating $\mathscr{F}_x$ for all $x \in X$, which means that $\mathscr{F}$ is isomorphic to a quotient sheaf of the sheaf $\mathscr{O}_X^p$.

Corollary 2. *Let $\mathscr{A} \xrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C}$ be an exact sequence of coherent algebraic sheaf on an affine variety X. The sequence $\Gamma(X, \mathscr{A}) \xrightarrow{\alpha} \Gamma(X, \mathscr{B}) \xrightarrow{\beta} \Gamma(X, \mathscr{C})$ is also exact.*

We can suppose, as in the proof of Theorem 2, that X is an affine space K^r, thus is irreducible. Set $\mathscr{J} = \Im(\alpha) = \mathrm{Ker}(\beta)$; everything reduces to seeing that $\alpha : \Gamma(X, \mathscr{A}) \to \Gamma(X, \mathscr{J})$ is surjective. Now, by Corollary 1, we can find a surjective homomorphism $\phi : \mathscr{O}_X^p \to \mathscr{A}$ and, by Corollary 2 to Proposition 7, $\alpha \circ \phi : \Gamma(X, \mathscr{O}_X^p) \to \Gamma(X, \mathscr{J})$ is surjective; this is a fortiori the same for $\alpha : \Gamma(X, \mathscr{A}) \to \Gamma(X, \mathscr{J})$, q.e.d.

46 Cohomology groups of an affine variety with values in a coherent algebraic sheaf

Theorem 3. *Let X be an affine variety, Q_i a finite family of regular functions on X that do not vanish simultaneously and let U be the open covering of X consisting of $X_{Q_i} = U_i$. If $\mathscr{F}$ is a coherent algebraic sheaf on X, we have $H^q(\mathsf{U}, \mathscr{F}) = 0$ for all $q > 0$.*

Assume first that X is irreducible. By Corollary 1 to Theorem 2, we can find an exact sequence

$$0 \to \mathscr{R} \to \mathscr{O}_X^p \to \mathscr{F} \to 0.$$

The sequence of complexes: $0 \to C(\mathsf{U}, \mathscr{R}) \to C(\mathsf{U}, \mathscr{O}_X^p) \to C(\mathsf{U}, \mathscr{F}) \to 0$ is *exact*; indeed, this reduces to saying that every section of $\mathscr{F}$ over $U_{i_0 \ldots i_q}$ is the image of a section of $\mathscr{O}_X^p$ over $U_{i_0 \ldots i_q}$, which follows from Corollary 2 to Proposition 7 applied to the irreducible variety $U_{i_0 \ldots i_q}$. This exact sequence gives birth to an exact sequence of cohomology:

$$\ldots \to H^q(\mathsf{U}, \mathscr{O}_X^p) \to H^q(\mathsf{U}, \mathscr{F}) \to H^{q+1}(\mathsf{U}, \mathscr{R}) \to \ldots,$$

and as $H^q(\mathsf{U}, \mathscr{O}_X^p) = H^{q+1}(\mathsf{U}, \mathscr{R}) = 0$ for $q > 0$ by Proposition 7, we conclude that $H^q(\mathsf{U}, \mathscr{F}) = 0$.

We proceed now to the general case. We can embed X as a closed subvariety of an affine space K^r; by Corollary 3 to Proposition 7, the functions Q_i are induced by polynomials P_i; let on the other hand R_j be a finite system of generators of the ideal $I(X)$. The functions P_i, R_j do not vanish simultaneously on K^r, thus define an open covering U' of K^r; let $\mathscr{F}'$ be the sheaf obtained by extending $\mathscr{F}$ by 0 outside X; applying what we have proven to the space K^r, the functions P_i, R_j and the sheaf $\mathscr{F}'$, we see that $H^q(\mathsf{U}', \mathscr{F}') = 0$ for $q > 0$. As we can immediately verify that the complex $C(\mathsf{U}', \mathscr{F}')$ is isomorphic to the complex $C(\mathsf{U}, \mathscr{F})$, it follows that $H^q(\mathsf{U}, \mathscr{F}) = 0$, q.e.d.

Corollary 1. *If X is an affine variety and $\mathscr{F}$ a coherent algebraic sheaf on X, we have $H^q(X, \mathscr{F}) = 0$ for all $q > 0$.*

Indeed, the coverings used in the above theorem are arbitrarily fine.

Corollary 2. *Let $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ be an exact sequence of sheaves on an affine variety X. If the sheaf $\mathscr{A}$ is coherent algebraic, the homomorphism $\Gamma(X, \mathsf{V}) \to \Gamma(X, \mathscr{C})$ is surjective.*

This follows from Corollary 1, by setting $q = 1$.

47 Coverings of algebraic varieties by open affine subsets

Proposition 8. *Let X be an affine variety and let $\mathsf{U} = \{U_i\}_{i \in I}$ be a finite covering of X by open affine subsets. If $\mathscr{F}$ is a coherent algebraic sheaf on X, we have $H^q(\mathsf{U}, \mathscr{F}) = 0$ for all $q > 0$.*

By Proposition 3, there exist regular functions P_j on X such that the covering $\mathsf{V} = \{X_{P_j}\}$ is finer than U. For every $(i_0, \ldots, i_p)$, the covering $\mathsf{V}_{i_0,\ldots,i_p}$ induced by V on $U_{i_0 \ldots i_p}$ is defined by restrictions of P_j to $U_{i_0 \ldots i_p}$; as $U_{i_0 \ldots i_p}$ is an affine variety by Proposition 1, we can apply Theorem 3 to it and conclude that $H^q(\mathsf{V}_{i_0 \ldots i_p}, \mathscr{F}) = 0$ for all $q > 0$. Applying then Proposition 5 of n° 29, we see that

$$H^q(\mathsf{U}, \mathscr{F}) = H^q(\mathsf{V}, \mathscr{F}),$$

and, as $H^q(\mathsf{V}, \mathscr{F}) = 0$ for $q > 0$ by Theorem 3, the Proposition is proven.

Theorem 4. *Let X be an algebraic variety, $\mathscr{F}$ a coherent algebraic sheaf on X and $\mathsf{U} = \{U_i\}_{i \in I}$ a finite covering of X by open affine subsets. The homomorphism $\sigma(\mathsf{U}) : H^n(\mathsf{U}, \mathscr{F}) \to H^n(X, \mathscr{F})$ is bijective for all $n \geq 0$.*

Consider the family V^α of all finite coverings of X by open affine subsets. By the corollary of Theorem 1, these coverings are arbitrarily fine. On the other hand, for every system $(i_0, \ldots, i_p)$ the covering $\mathsf{V}^\alpha_{i_0 \ldots i_p}$ induced by V^α on $U_{i_0 \ldots i_p}$ is a covering by open affine subsets, by Proposition 1; by Proposition 8, we thus have $H^q(\mathsf{V}^\alpha_{i_0 \ldots i_p}, \mathscr{F}) = 0$ for $q > 0$. The conditions (a) and (b) of Theorem 1, n° 29 are satisfied and the theorem follows.

Theorem 5. *Let X be an algebraic variety and $\mathsf{U} = \{U_i\}_{i \in I}$ a finite covering of X by open affine subsets. Let $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ be an exact sequence of sheaves on X, the sheaf $\mathscr{A}$ being coherent algebraic. The canonical homomorphism $H^q_0(\mathsf{U}, \mathscr{C}) \to H^q(\mathsf{U}, \mathscr{C})$ (cf. n° 24) is bijective for all $q \geq 0$.*

It obviously suffices to prove that $C_0(\mathsf{U}, \mathscr{C}) = C(\mathsf{U}, \mathscr{C})$, that is, that every section of $\mathscr{C}$ over $U_{i_0 \ldots i_q}$ is the image of a section of $\mathscr{B}$ over $U_{i_0 \ldots i_q}$, which follows from Corollary 2 of Theorem 3.

Corollary 1. *Let X be an algebraic variety and let $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ be an exact sequence of sheaves on X, the sheaf $\mathscr{A}$ being coherent algebraic. The canonical homomorphism $H^q_0(X, \mathscr{C}) \to H^q(X, \mathscr{C})$ is bijective for all $q \geq 0$.*

This is an immediate consequence of Theorems 1 and 5.

Corollary 2. *We have an exact sequence:*

$$\ldots \to H^q(X, \mathscr{B}) \to H^q(X, \mathscr{C}) \to H^{q+1}(X, \mathscr{A}) \to H^{q+1}(X, \mathscr{B}) \to \ldots$$

§4 Correspondence between modules of finite type and coherent algebraic sheaves

48 Sheaf associated to a module

Let V be an affine variety, $\mathscr{O}$ the sheaf of local rings of V; the ring $A = \Gamma(V, \mathscr{O})$, which will be called *the ring of coordinates* of V, is an algebra over K which has no nilpotent elements but 0. If V is embedded as a closed subvariety of an affine space K^r, we know (cf. n° 44) that A is identified with the quotient algebra of $K[X_1, \ldots, X_r]$ by the ideal of polynomials vanishing on V; it follows that the algebra A is generated by a finite number of elements.

Conversely, we verify easily that if A is a commutative K–algebra without nilpotent elements (other that 0) and is generated by a finite number of elements, there exists an affine variety V such that A is isomorphic to $\Gamma(V, \mathscr{O})$; moreover, V is determined up to isomorphism by this property (we can identify V with the set of characters of A equipped with the usual topology).

Let M be an A–module; M defines a constant sheaf on V which we denote again by M; the same way A defines a constant sheaf, and the sheaf M can be considered as a sheaf of A–modules. Define $\mathscr{A}(M) = \mathscr{O} \otimes_A M$, the sheaf $\mathscr{O}$ being also considered as a sheaf of A–modules; it is clear that $\mathscr{A}(M)$ is an algebraic sheaf on V. Moreover, if $\phi : M \to M'$ is an A–homomorphism, we have a homomorphism $\mathscr{A}(\phi) = 1 \otimes \phi : \mathscr{A}(M) \to \mathscr{A}(M')$; in other words, $\mathscr{A}(M)$ is a covariant functor of the module M.

Proposition 1. *The functor $\mathscr{A}(M)$ is exact.*

Let $M \to M' \to M''$ be an exact sequence of A–modules. We must observe that the sequence $\mathscr{A}(M) \to \mathscr{A}(M') \to \mathscr{A}(M'')$ is exact, in other words, that for all $x \in V$ the sequence:

$$\mathscr{O}_x \otimes_A M \to \mathscr{O}_x \otimes_A M' \to \mathscr{O}_x \otimes_A M''$$

is exact.

Now $\mathscr{O}_x$ is nothing else that the localization A_S of A, S being the set of those $f \in A$ for which $f(x) \neq 0$ (for the definition of localization, cf. [8], [12] or [13]). Proposition 1 is thus a particular case of the following result:

Lemma 1. *Let A be a ring, S a multiplicative system in A not containing 0, A_S the localization of A in S. If $M \to M' \to M''$ is an exact sequence of A–modules, the sequence $A_S \otimes_A M \to A_S \otimes_A M' \to A_S \otimes_A M''$ is exact.*

Denote by M_S the set of fractions m/s with $m \in M$, $s \in S$, two fractions m/s and m'/s' being identified if there exists an $s'' \in S$ such that $s''(s' \cdot m - s \cdot m') = 0$; it is easily seen that M_S is an A_S–module and that the mapping

$$a/s \otimes m \mapsto a \cdot m/s$$

is an isomorphism from $A_S \otimes_A A$ onto M_S; we are thus led to prove that the sequence

$$M_S \to M'_S \to M''_S$$

is exact, which is obvious.

Proposition 2. *$\mathscr{A}(M) = 0$ implies $M = 0$.*

Let m be an element of M; if $\mathscr{A}(M) = 0$, we have $1 \otimes m = 0$ in $\mathscr{O}_x \otimes_A M$ for all $x \in V$. By the discussion above, $1 \otimes m = 0$ is equivalent to existence of an element $s \in A$, $s(x) \neq 0$ such that $s \cdot m = 0$; the annihilator of m in M is not contained in any maximal ideal of A, which implies that it is equal to A, so $m = 0$.

Proposition 3. *If M is an A–module of finite type, $\mathscr{A}(M)$ is a coherent algebraic sheaf on V.*

Because M is of finite type and since A is Noetherian, M is isomorphic to the cokernel of a homomorphism $\phi : A^q \to A^p$ and $\mathscr{A}(M)$ is isomorphic to the cokernel of $\mathscr{A}(\phi) : \mathscr{A}(A^q) \to \mathscr{A}(A^p)$. As $\mathscr{A}(A^p) = \mathscr{O}^p$ and $\mathscr{A}(A^q) = \mathscr{O}^q$, it follows that $\mathscr{A}(M)$ is coherent.

49 Module associated to an algebraic sheaf

Let $\mathscr{F}$ be an algebraic sheaf on V and let $\Gamma(\mathscr{F}) = \Gamma(V, \mathscr{F})$; since $\mathscr{F}$ is a sheaf of $\mathscr{O}$–modules, $\Gamma(\mathscr{F})$ is equipped with a natural structure of an A–module. Any algebraic homomorphism $\phi : \mathscr{F} \to \mathscr{G}$ defines an A–homomorphism $\Gamma(\phi) : \Gamma(\mathscr{F}) \to \Gamma(\mathscr{G})$. If we have an exact sequence of algebraic sheaves $\mathscr{F} \to \mathscr{G} \to \mathscr{K}$, the sequence

$$\Gamma(\mathscr{F}) \to \Gamma(\mathscr{G}) \to \Gamma(\mathscr{K})$$

is exact (n° 45); applying this to an exact sequence $\mathscr{O}^p \to \mathscr{F} \to 0$ we see that $\Gamma(\mathscr{F})$ is an A–module of finite type if $\mathscr{F}$ is coherent.

The functors $\mathscr{A}(M)$ and $\Gamma(\mathscr{F})$ are „inverse" to each other:

Theorem 1. *(a) If M is an A-module of finite type, $\Gamma(\mathscr{A}(M))$ is canonically isomorphic to M.*

(b) If $\mathscr{F}$ is a coherent algebraic sheaf on V, $\mathscr{A}(\Gamma(\mathscr{F}))$ is canonically isomorphic to $\mathscr{F}$.

First let us show (a). Every element $m \in M$ defines a section $\alpha(m)$ of $\mathscr{A}(M)$ by the formula: $\alpha(m)(x) = 1 \otimes m \in \mathscr{O}_x \otimes_A M$; hence a homomorphism $\alpha : M \to \Gamma(\mathscr{A}(M))$. When M is a free module of finite type, α is bijective (it suffices to see this when $M = A$, in which case it is obvious); if M is an arbitrary module of finite type, there exists an exact sequence $L^1 \to L^0 \to M \to 0$ where L^0 and L^1 are free of finite type; the sequence $\mathscr{A}(L^1) \to \mathscr{A}(L^0) \to \mathscr{A}(M) \to 0$

is exact, thus also the sequence $\Gamma(\mathscr{A}(L^1)) \to \Gamma(\mathscr{A}(L^0)) \to \Gamma(\mathscr{A}(M)) \to 0$. The commutative diagram:

$$\begin{array}{ccccccc} L^1 & \longrightarrow & L^0 & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ \Gamma(\mathscr{A}(L^1)) & \longrightarrow & \Gamma(\mathscr{A}(L^0)) & \longrightarrow & \Gamma(\mathscr{A}(M)) & \longrightarrow & 0 \end{array}$$

shows then that $\alpha : M \to \Gamma(\mathscr{A}(M))$ is bijective, which shows (a).

Let now $\mathscr{F}$ be an algebraic coherent sheaf on V. If we associate to every $s \in \Gamma(\mathscr{F})$ an element $s(x) \in \mathscr{F}(X)$, we obtain an A–homomorphism: $\Gamma(\mathscr{F}) \to \mathscr{F}_x$ which extends to an $\mathscr{O}_x$–homomorphism $\beta_x : \mathscr{O}_x \otimes_A \Gamma(\mathscr{F}) \to \mathscr{F}_x$; we easily verify that the β_x form a homomorphism of sheaves $\beta : \mathscr{A}(\Gamma(\mathscr{F})) \to \mathscr{F}$. When $\mathscr{F} = \mathscr{O}^p$, the homomorphism β is bijective; it follows by the same reasoning as above that β is bijective for every coherent algebraic sheaf $\mathscr{F}$, which shows (b).

Remarks. (1) We could also deduce (b) from (a); cf. n° 65, proof of Proposition 6.

(2) We will see in Chapter III how the above correspondence should be modified when one studies coherent sheaves on the projective space.

50 Projective modules and vector bundles

Recall ([6], Chap. I, th. 2.2) that an A–module is called *projective* if it is a direct summand of a free A–module.

Proposition 4. *Let M be an A–module of finite type. Then M is projective if and only if the $\mathscr{O}_x$–module $\mathscr{O}_x \otimes_A M$ is free for every $x \in V$.*

If M is projective, $\mathscr{O}_x \otimes_A M$ is $\mathscr{O}_x$–projective, thus $\mathscr{O}_x$–free since $\mathscr{O}_x$ is a local ring (cf. [6], Chap. VIII, th. 6.1').

Conversely, if all $\mathscr{O}_x \otimes_A M$ are free, we have

$$\dim(M) = \operatorname{Sup}\dim_{x \in V}(\mathscr{O}_x \otimes_A M) = 0 \quad (cf.[6], Chap.VII, Exer.11),$$

from which it follows that M is projective ([6], Chap. VI, §2).

Note that if $\mathscr{F}$ is a coherent algebraic sheaf on V and if $\mathscr{F}_x$ is isomorphic to $\mathscr{O}_x^p$, $\mathscr{F}$ is isomorphic to $\mathscr{O}^p$ in a neighborhood of x; if this property is satisfied in every $x \in V$, the sheaf $\mathscr{F}$ is thus locally isomorphic to the sheaf $\mathscr{O}^p$, the integer p being constant on every connected component of V. Applying this to the sheaf $\mathscr{A}(M)$, we obtain:

Corollary. *Let $\mathscr{F}$ be a coherent algebraic sheaf on a connected affine variety V. The three following properties are equivalent:*

(i) $\Gamma(\mathscr{F})$ is a projective A–module,

(ii) $\mathscr{F}$ *is locally isomorphic to* $\mathscr{O}^p$*,*

(iii) $\mathscr{F}$ *is isomorphic to the sheaf of germs of sections of a vector bundle with base* V*.*

In other words, the mapping $E \mapsto \Gamma(\mathscr{S}(E)$ (E denoting a vector bundle) gives a bijective correspondence between classes of vector bundles and classes of projective A–modules of finite type; in this correspondence, a *trivial* bundle corresponds to a *free* module and conversely.

Note that when $V = K^r$ (in which case $A = K[X_1, \ldots, X_r]$), we do not know if there exist projective A–modules that are not free, or equivalently, if there exist algebraic vector bundles with base K^r that are not trivial.

Chapter III

Coherent Algebraic Sheaves on Projective Varieties

§1 Projective varieties

51 Notations

(The notations introduced below will be used without reference during the whole chapter).

Let r be an integer ≥ 0 and let $Y = K^{r+1} - \{0\}$; the multiplicative group K^* of nonzero elements of K acts on Y by the formula

$$\lambda(\mu_0, \ldots, \mu_r) = (\lambda\mu_0, \ldots, \lambda\mu_r).$$

Two points y and y' will be called equivalent if there exists $\lambda \in K^*$ such that $y' = \lambda y$; the quotient space of Y by this equivalence relation will be denoted by $\mathbb{P}_r(K)$ or simply X; it is the *projective space of dimension r over K*; the canonical projection of Y onto X will be denoted π.

Let $I = \{0, 1, \ldots, r\}$; for every $i \in I$, we denote by t_i the i-th coordinate function on K^{r+1}, defined by the formula:

$$t_i(\mu_0, \ldots, \mu_r) = \mu_i.$$

We denote by V_i the open subset of K^{r+1} consisting of points whose t_i is $\neq 0$ and by U_i the image of V_i by π; the $\{U_i\}$ form a covering U of X. If $i \in I$ and $j \in I$, the function t_j/t_i is regular on V_i and invariant for K^*, thus defines a function on U_i which we denote also by t_j/t_i; for fixed i, the functions t_j/t_i, $j \neq i$ define a bijection $\phi_i : U_i \to K^r$.

We equip K^{r+1} with the structure of an algebraic variety and Y the induced structure. Likewise, we equip X with the quotient topology from Y: a closed subset of X is thus the image by π of a closed cone in K^{r+1}. If U is open in X, we define $A_U = \Gamma(\pi^{-1}(U), \mathscr{O}_Y)$; this is the sheaf of regular functions on $\pi^{-1}(U)$. Let A_U^0 be the subring of A_U consisting of elements invariant for K^* (that is, homogeneous functions of degree 0). When $V \supset U$, we have a restriction homomorphism $\phi_U^V : A_V^0 \to A_U^0$ and the system (A_U^0, ϕ_U^V) defines a sheaf $\mathscr{O}_X$ which can be considered as a subsheaf of the sheaf $\mathscr{F}(X)$ of germs of functions on X. Such a function f, defined in a neighborhood of x belongs to $\mathscr{O}_{x,X}$ if and only if it coincides locally with a function of the form P/Q where P and Q are homogeneous polynomials of the same degree in $t_0, \ldots, t_r$ with $Q(y) \neq 0$ for $y \in \pi^{-1}(x)$ (which we write for brevity as $Q(x) \neq 0$).

Proposition 1. *The projective space $X = \mathbb{P}_r(K)$, equipped with the topology and sheaf above, is an algebraic variety.*

The U_i, $i \in I$ are open in X and we verify immediately that the bijections $\phi_i : U_i \to K^r$ defined above are biregular isomorphisms, which shows that the axiom (VA_I) is satisfied. To show that (VA_{II}) is also satisfied, we must observe that the subset of $K^r \times K^r$ consisting of all pairs $(\psi_i(x), \psi_j(x))$ where $x \in U_i \cap U_j$ is closed, which does not pose difficulties.

In what follows, X will be always equipped with the structure of an algebraic variety just defined; the sheaf $\mathscr{O}_X$ will be simply denoted $\mathscr{O}$. An algebraic variety V is called *projective* if it is isomorphic to a closed subvariety of a projective space. The study of coherent algebraic sheaves on projective varieties can be reduced to the study of coherent algebraic sheaves on $\mathbb{P}_r(K)$, cf. n° 39.

52 Cohomology of subvarieties of the projective space

Let us apply Theorem 4 from n° 47 to the covering $\mathsf{U} = \{U_i\}_{i \in I}$ defined in the preceding n° : it is possible since each U_i is isomorphic to K^r. We thus obtain:

Proposition 2. *If $\mathscr{F}$ is a coherent algebraic sheaf on $X = \mathbb{P}_r(K)$, the homomorphism $\sigma(\mathsf{U}) : H^n(\mathsf{U}, \mathscr{F}) \to H^n(X, \mathscr{F})$ is bijective for all $n \neq 0$.*

Since U consists of $r + 1$ open subsets, we have (cf. n° 20, corollary to Proposition 2):

Corollary. $H^n(X, \mathscr{F}) = 0$ *for* $n > r$.

This result can be generalized in the following way:

Proposition 3. *Let V be an algebraic variety, isomorphic to a locally closed subvariety of the projective space X. Let $\mathscr{F}$ be an algebraic coherent sheaf on V and let W be the subvariety of V such that $\mathscr{F}$ is zero outside W. We then have $H^n(V, \mathscr{F}) = 0$ for $n > \dim W$.*

In particular, taking $W = V$, we see that we have:

Corollary. $H^n(V, \mathscr{F}) = 0$ *for* $n > \dim V$.

Identify V with a locally closed subvariety of $X = \mathbb{P}_r(K)$; there exists an open subset U of X such that V is closed in U. We can clearly assume that W is closed in V, so that W is closed in U. Let $F = X - U$. Before proving Proposition 3, we establish two lemmas:

Lemma 1. *Let $k = \dim W$; there exists $k + 1$ homogeneous polynomials $P_i(t_0, \dots, t_r)$ of degrees > 0, vanishing on F and not vanishing simultaneously on W.*

(By abuse of language, we say that a homogeneous polynomial P vanishes in a point x of $\mathbb{P}_r(K)$ if it vanishes on $\pi^{-1}(x)$).

We proceed by induction on k, the case when $k = -1$ being trivial. Choose a point on each irreducible component of W and let P_1 be a homogeneous polynomial vanishing on F, of degree > 0 and nonvanishing in each of these points (the existence of P_1 follows from the fact that F is closed, given the definition of the topology of $\mathbb{P}_r(K)$). Let W' be a subvariety of W consisting of points $x \in W$ such that $P_1(x) = 0$; by the construction of P_1, no irreducible component of W is contained in W' and it follows (cf. n° 36) that $\dim W' < k$. Applying the induction assumption to W', we see that there exist k homogeneous polynomials $P_2, \ldots, P_{k+1}$ vanishing on F and nonvanishing simultaneously on W'; it is clear that the polynomials $P_1, \ldots, P_{k+1}$ satisfy appropriate conditions.

Lemma 2. *Let $P(t_0, \ldots, t_r)$ be a homogeneous polynomial of degree $n > 0$. The set X_P of all points $x \in X$ such that $P(x) \neq 0$ is an open affine subset of X.*

If we assign to every point $y = (\mu_0, \ldots, \mu_r) \in Y$ the point of the space K^N having for coordinates all monomials $\mu_0^{m_0} \ldots \mu_r^{m_r}$, $m_0 + \ldots + m_r = n$, we obtain, by passing to quotient, a mapping $\phi_n : X \to \mathbb{P}_{N-1}(K)$. It is classical, and also easy to verify, that ϕ_n is a biregular isomorphism of X onto a closed subvariety of $\mathbb{P}^{N-1}(K)$ („Veronese variety"); now ϕ_n transforms the open subset X_P onto the locus of points of $\phi_n(X)$ not lying on a certain hyperplane of $\mathbb{P}_{N-1}(X)$; as the complement of any hyperplane is isomorphic to an affine space, we conclude that X_P is isomorphic to a closed subvariety of an affine space.

We shall now prove Proposition 3. Extend the sheaf $\mathscr{F}$ by 0 on $U - V$; we obtain a coherent algebraic sheaf on U which we also denote by $\mathscr{F}$, and we know (cf. n° 26) that $H^n(U, \mathscr{F}) = H^n(V, \mathscr{F})$. Let on the other hand $P_1, \ldots, P_{k+1}$ be homogeneous polynomials satisfying the conditions of Lemma 1; let $P_{k+2}, \ldots, P_h$ be homogeneous polynomials of degrees > 0, vanishing on $W \cup F$ and not vanishing simultaneously in any point of $U - W$ (to obtain such polynomials, it suffices to take a system of homogeneous coordinates of the ideal defined by $W \cup F$ in $K[t_0, \ldots, t_r]$). For every i, $1 \leq i \leq h$, let V_i be the set of points $x \in X$ such that $P_i(x) \neq 0$; we have $V_i \subset U$ and the assumptions made above show that $\mathsf{V} = \{V_i\}$ is an open covering of U; moreover, Lemma 2 shows that V_i are open affine subsets, so $H^n(\mathsf{V}, \mathscr{F}) = H^n(U, \mathscr{F}) = H^n(V, \mathscr{F})$ for all $n \geq 0$. On the other hand, if $n > k$ and if the indices $i_0, \ldots, i_n$ are distinct, one of the indices is $> k + 1$ and $V_{i_0 \ldots i_n}$ does not meet W; we conclude that the group of alternating cochains $C'^n(\mathsf{V}, \mathscr{F})$ is zero if $n > k$, which shows that $H^n(\mathsf{V}, \mathscr{F}) = 0$, by Proposition 2 of n° 20.

53 Cohomology of irreducible algebraic curves

If V is an irreducible algebraic variety of dimension 1, the closed subsets of V distinct from V are *finite* subsets. If F is a finite subset of V and x a point of F, we set $V_x^p = (V - F) \cup \{x\}$; the V_x^F, $x \in F$ form a finite open covering V^F of V.

Lemma 3. *The coverings V^F of the above type are arbitrarily fine.*

Let $\mathsf{U} = \{U_i\}_{i \in I}$ be an open covering of V, which we may assume to be finite since V is quasi-compact. We can likewise assume that $\rightarrow U_i \neq \emptyset$ for all $i \in I$. If we set $F_i = V - U_i$, F_i is also finite, and so is $F = {}_{i \in I} F_i$. We will show that $\mathsf{V}^F \prec \mathsf{U}$, which proves the lemma. Let $x \in F$; there exists an $i \in I$ such that $x \notin F_i$, since the U_i cover V; we have then $F - \{x\} \supset F_i$, because $F \supset F_i$, which means that $V_x^F \subset U_i$ and shows that $\mathsf{V}^F \prec \mathsf{U}$.

Lemma 4. *Let $\mathscr{F}$ be a sheaf on V and F a finite subset of V. We have*

$$H^n(\mathsf{V}^F, \mathscr{F}) = 0$$

for $n \geq 2$.

Set $W = V - F$; it is clear that $V^F_{x_0} \cap \ldots \cap V^F_{x_n} = W$ if $x_0, \ldots, x_n$ are distinct and if $n \geq 1$. If we put $G = \Gamma(W, \mathscr{F})$, it follows that the alternating complex $C'(\mathsf{V}^F, \mathscr{F})$ is isomorphic, in dimensions ≥ 1, to $C'(S(F), G)$, $S(F)$ denoting the simplex with F for the set of vertices. It follows that

$$H^n(\mathsf{V}^F, \mathscr{F}) = H^n(S(F), G) = 0 \text{ for } n \geq 2,$$

the cohomology of a simplex being trivial.

Lemmas 3 and 4 obviously imply:

Proposition 4. *If V is an irreducible algebraic curve and $\mathscr{F}$ is an arbitrary sheaf in V, we have $H^n(V, \mathscr{F}) = 0$ for $n \geq 2$.*

Remark. I do not know whether an analogous result is true for varieties of arbitrary dimension.

§2 Graded modules and coherent algebraic sheaves on the projective space

54 The operation $\mathscr{F}(\mathsf{n})$

Let $\mathscr{F}$ be an algebraic sheaf on $X = \mathbb{P}_r(K)$. Let $\mathscr{F}_i = \mathscr{F}(U_i)$ be the restriction of $\mathscr{F}$ to U_i (cf. n° 51); if n is an arbitrary integer, let $\theta_{ij}(n)$ be the isomorphism of $\mathscr{F}_j(U_i \cap U_j)$ with $\mathscr{F}_i(U_i \cap U_j)$ defined by multiplication by the function t_j^n/t_i^n; this makes sense, since t_j/t_i is a regular function on $U_i \cap U_j$ with values in K^*. We have $\theta_{ij}(n) \circ \theta_{jk}(n) = \theta_{ik}(n)$ at every point of $U_i \cap U_j \cap U_k$; we can thus apply Proposition 4 of n° 4 and obtain an algebraic sheaf denoted by $\mathscr{F}(n)$, defined by gluing the sheaves $\mathscr{F}_i = \mathscr{F}(U_i)$ using the isomorphisms $\theta_{ij}(n)$.

We have the canonical isomorphisms: $\mathscr{F}(0) \approx \mathscr{F}$, $\mathscr{F}(n)(m) \approx \mathscr{F}(n+m)$. Moreover, $\mathscr{F}(n)$ is locally isomorphic to $\mathscr{F}$, thus coherent if $\mathscr{F}$ is; it also follows that every exact sequence $\mathscr{F} \to \mathscr{F}' \to \mathscr{F}''$ of algebraic sheaves gives birth to exact sequences $\mathscr{F}(n) \to \mathscr{F}'(n) \to \mathscr{F}''(n)$ for all $n \in \mathbb{Z}$.

We can apply the above procedure to the sheaf $\mathscr{F} = \mathscr{O}$ and so obtain the sheaves $\mathscr{O}(n)$, $n \in \mathbb{Z}$. We will give another description of these sheaves: if U is open in X, let A_U^n be the subset of $A_U = \Gamma(\pi^{-1}(U), \mathscr{O}_Y)$ consisting of regular functions of degree n (that is, satisfying the identity $f(\lambda y) = \lambda^n f(y)$ for $\lambda \in K^*$ and $y \in \pi^{-1}(U)$); the A_U^n are A_U^0–modules, thus give birth to an algebraic sheaf, which we denote by $\mathscr{O}'(n)$. An element of $\mathscr{O}'(n)_x$, $x \in X$ can be this identified with a rational function P/Q, P and Q being homogeneous polynomials such that $Q(x) \neq 0$ and $\deg P - \deg Q = n$.

Proposition 1. *The sheaves $\mathscr{O}(n)$ and $\mathscr{O}'(n)$ are canonically isomorphic.*

By definition, a section of $\mathscr{O}(n)$ over an open $U \subset X$ is a system (f_i) of sections of $\mathscr{O}$ over $U \cap U_i$ with $f_i = (t_j^n/t_i^n) \cdot f_j$ on $U \cap U_i \cap U_j$; the f_j can be identified with regular functions, homogeneous of degree 0 over $\pi^{-1}(U) \cap \pi^{-1}(U_i)$; set $g_i = t_i^n \cdot f_i$; we then have $g_i = g_j$ at every point of $\pi^{-1}(U) \cap \pi^{-1}(U_i) \cap \pi^{-1}(U_j)$, thus the g_i are the restrictions of a unique regular function on $\pi^{-1}(U)$, homogeneous of degree n. Conversely, such a function g defines a system (f_i) by setting $f_i = g/t_i^n$. The mapping $(f_i) \mapsto g$ is thus an isomorphism of $\mathscr{O}(n)$ with $\mathscr{O}'(n)$.

Henceforth, we will often identify $\mathscr{O}(n)$ with $\mathscr{O}'(n)$ by means of the above isomorphism. We observe that a section of $\mathscr{O}'(n)$ over X is just a regular function on Y, homogeneous of degree n. If we assume that $r \geq 1$, such a function is identically zero for $n < 0$ and it is a homogeneous *polynomial* of degree n for $n \geq 0$.

Proposition 2. *For every algebraic sheaf $\mathscr{F}$, the sheaves $\mathscr{F}(n)$ and $\mathscr{F} \otimes_{\mathsf{O}} \mathscr{O}(n)$ are canonically isomorphic.*

Since $\mathscr{O}(n)$ is obtained from the $\mathscr{O}_i$ by gluing with respect to $\theta_{ij}(n)$, $\mathscr{F}\otimes\mathscr{O}(n)$ is obtained from $\mathscr{F}_i \otimes \mathscr{O}_i$ by gluing with respect to the isomorphisms $1 \otimes \theta_{ij}(n)$; identifying $\mathscr{F}_i \otimes \mathscr{O}_i$ with $\mathscr{F}_i$ we recover the definition of $\mathscr{F}(n)$.

Henceforth, we will also identify $\mathscr{F}(n)$ with $\mathscr{F} \otimes \mathscr{O}(n)$.

55 Sections of $\mathscr{F}(n)$

Let us first show a lemma on algebraic varieties, that is quite analogous to Lemma 1 of n° 45:

Lemma 1. *Let V be an affine variety, Q a regular function on V and V_Q the set of all points $x \in V$ such that $Q(x) \neq 0$. Let $\mathscr{F}$ be a coherent algebraic sheaf on V and let s be a section of $\mathscr{F}$ over V_Q. Then, for n sufficiently large, there exists a section s' of $\mathscr{F}$ over the whole V such that $s' = Q^n s$ over V_Q.*

Embedding V in an affine space and extending $\mathscr{F}$ by 0 outside V, we are brought to the case where V is an affine space, thus is irreducible. By Corollary 1 to Theorem 2 from n° 45, there exists a surjective homomorphism $\phi : \mathscr{O}_V^p \to \mathscr{F}$; by Proposition 2 of n° 42, V_Q is an open affine subset and thus there exists (n° 44, Corollary 2 to Proposition 7) a section σ of $\mathscr{O}_V^p$ over V_Q such that $\phi(\sigma) = s$. We can identify σ with a system of p regular functions on V_Q; applying Proposition 5 of n° 43 to each of these functions, we see that there exists a section σ' of $\mathscr{O}_V^p$ over V such that $\sigma' = Q^n\sigma$ on V_Q, provided that n is sufficiently large. Setting $s' = \phi(\sigma')$, we obtain a section of $\mathscr{F}$ over V such that $s' = Q^n s$ on V_Q.

Theorem 1. *Let $\mathscr{F}$ be a coherent algebraic sheaf on $X = \mathbb{P}_r(K)$. There exists an integer $n(\mathscr{F})$ such that for all $n \geq n(\mathscr{F})$ and all $x \in X$, the $\mathscr{O}_x$–module $\mathscr{F}(n)_x$ is generated by elements of $\Gamma(X, \mathscr{F}(n))$.*

By the definition of $\mathscr{F}(n)$, a section s of $\mathscr{F}(n)$ over X is a system (s_i) of sections of $\mathscr{F}$ over U_i satisfying the compatibility conditions:

$$s_i = (t_j^n/t_i^n) \cdot s_j \text{ on } U_i \cap U_j;$$

we say that s_i is the *i-th component of s.*

On the other hand, since U_i is isomorphic to K^r, there exists a finite number of sections s_i^α of $\mathscr{F}$ over U_i which generate $\mathscr{F}_x$ for all $x \in U_i$ (n° 45, Corollary 1 to Theorem 2); if for a certain integer n we can find sections s^α of $\mathscr{F}(n)$ whose i-th component is s_i^α, it is clear that $\Gamma(X, \mathscr{F}(n))$ generates $\mathscr{F}(n)_x$ for all $x \in U_i$. Theorem 1 is thus proven if we prove the following Lemma:

Lemma 2. *Let s_i be a section of $\mathscr{F}$ over U_i. For all n sufficiently large, there exists a section s of $\mathscr{F}(n)$ whose i-th component is equal to s_i.*

Apply Lemma 1 to the affine variety $V = U_j$, the function $Q = t_i/t_j$ and the section s_i restricted to $U_i \cap U_j$; this is legal, because t_i/t_j is a regular function on U_j whose zero set is equal to $U_j - U_i \cap U_j$. We conclude that there exists an integer p and a section s'_j of $\mathscr{F}$ over U_j such that $s'_j = (t_i^p/t_j^p) \cdot s_i$ on $U_i \cap U_j$;

for $j = i$, we have $s'_i = s_i$, which allows us to write the preceding formula in the form $s'_j = (t_i^p/t_j^p) \cdot s'_i$.

The s'_j being defined for every index j (with the same exponent p), consider $s'_j - (t_k^p/t_j^p) \cdot s'_k$; it is a section of $\mathscr{F}$ over $U_j \cap U_k$ whose restriction to $U_i \cap U_j \cap U_k$ is zero; by applying Proposition 6 of n° 43 we see that for every sufficiently large integer q we have $(t_i^q/t_j^q)(s'_j - (t_k^p/t_j^p) \cdot s'_k) = 0$ on $U_j \cap U_k$; if we then put $s_j = (t_i^q/t_j^q) \cdot s'_j$ and $n = p + q$, the above formula is written $s_j = (t_k^n/t_j^n) \cdot s_k$ and the system $s = (s_j)$ is a section of $\mathscr{F}(n)$ whose i-th component is equal to s_i, q.e.d.

Corollary. *Every coherent algebraic sheaf $\mathscr{F}$ on $X = \mathbb{P}_r(K)$ is isomorphic to a quotient sheaf of a sheaf $\mathscr{O}(n)^p$, n and p being suitable integers.*

By the above theorem, there exists an integer n such that $\mathscr{F}(-n)_x$ is generated by $\Gamma(X, \mathscr{F}(-n))$ for every $x \in X$; by the quasi-compactness of X, this is equivalent to saying that $\mathscr{F}(-n)$ is isomorphic to a quotient sheaf of a sheaf $\mathscr{O}^p$, p being an appropriate integer ≥ 0. It follows then that $\mathscr{F} \approx \mathscr{F}(-n)(n)$ is isomorphic to a quotient sheaf of $\mathscr{O}(n)^p \approx \mathscr{O}^p(n)$.

56 Graded modules

Let $S = K[t_0, \ldots, t_r]$ be the algebra of polynomials in $t_0, \ldots, t_r$; for every integer $n \geq 0$, let S_n be the linear subspace of S consisting by homogeneous polynomials of degree n; for $n < 0$, we set $S_n = 0$. The algebra S is a direct sum of S_n, $n \in \mathbb{Z}$ and we have $S_p \cdot S_q \subset S_{p+q}$; in other words, S is a *graded algebra*.

Recall that an S–module M is said to be *graded* if there is given a decomposition of M into a direct sum: $M = \bigoplus_{n\in\mathbb{Z}} M_n$, M_n being subgroups of M such that $S_p \cdot M_q \subset M_{p+q}$ for every couple (p, q) of integers. An element of M_n is said to be *homogeneous* of degree n; a submodule N of M is said to be *homogeneous* if it is a direct sum of $N \cap M_n$, in which case it is a graded S–module. If M and M' are two graded S–modules, an S–homomorphism

$$\phi : M \to M'$$

is said to be *homogeneous of degree s* if $\phi(M_n) \subset M'_{n+s}$ for every $n \in \mathbb{Z}$. A homogeneous S–homomorphism of degree 0 is simply called a *homomorphism*.

If M is a graded S–module and n an integer, we denote by $M(n)$ the graded S–module:

$$M(n) = \bigoplus_{p\in\mathbb{Z}} M(n)_p \text{ with } M(n)_p = M_{n+p}.$$

We thus have $M(n) = M$ as S–modules, but a homogeneous element of degree p of $M(n)$ is homogeneous of degree $n + p$ in M; in other words, $M(n)$ is made from M by lowering degrees by n units.

We denote by $\mathscr{C}$ the class of graded S–modules M such that $M_n = 0$ for n sufficiently large. If $A \to B \to C$ is an exact sequence of homomorphisms of graded S–modules, the relations $A \in \mathscr{C}$, $C \in \mathscr{C}$ clearly imply $B \in \mathscr{C}$; in other words, $\mathscr{C}$ is a class in the sense of [14], Chap. I. Generally, we use the terminology introduced in the aforementioned article; in particular, a homomorphism $\phi : A \to B$ is called *$\mathscr{C}$-injective* (resp. *$\mathscr{C}$-surjective*) if $\mathrm{Ker}(\phi) \in \mathscr{C}$ (resp. if $Coker(\phi) \in \mathscr{C}$) and $\mathscr{C}$-bijective if it is both $\mathscr{C}$-injective and $\mathscr{C}$-surjective.

A graded S–module M is said to be *of finite type* if it is generated by a finite number of elements; we say that M *satisfies the condition (TF)* if there exists an integer p such that the submodule $\bigoplus_{n \geq p} M_n$ of M is of finite type; it is the same to say that M is $\mathscr{C}$-isomorphic to a module of finite type. The modules satisfying (TF) form a class containing $\mathscr{C}$.

A graded S–module L is called *free* (resp. *free of finite type*) if it admits a base (resp. a finite base) consisting of homogeneous elements, in other words if it is isomorphic to a direct sum (resp. to a finite direct sum) of the modules $S(n_i)$.

57 The algebraic sheaf associated to a graded S-module

If U is a nonempty subset of X, we denote by $S(U)$ the subset of $S = K[t_0, \dots, t_r]$ consisting of homogeneous polynomials Q such that $Q(x) \neq 0$ for all $x \in U$; $S(U)$ is a multiplicatively closed subset of S, not containing 0. For $U = X$, we write $S(x)$ instead of $S(\{x\})$.

Let M be a graded S–module. We denote by M_U the set of fractions m/Q with $m \in M$, $Q \in S(U)$, m and Q being homogeneous of *the same* degree; we identify two fractions m/Q and m'/Q' if there exists $Q'' \in S(U)$ such that

$$Q''(Q' \cdot m - Q \cdot m') = 0;$$

it is clear that we have defined an equivalence relation between the pairs (m, Q). For $U = x$, we write M_x instead of $M_{\{x\}}$.

Applying this to $M = S$, we see that S_U is the ring of rational functions P/Q, P and Q being homogeneous polynomials of the same degree and $Q \in S(U)$; if M is an arbitrary graded S–module, we can equip M_U with a structure of an S_U–module by imposing:

$$m/Q + m'/Q' = (Q'm + Qm')/QQ'$$
$$(P/Q) \cdot (m/Q') = Pm/QQ'.$$

If $U \subset V$, we have $S(V) \subset S(U)$, hence the canonical homomorphisms

$$\phi_U^V : M_V \to M_U;$$

the system (M_U, ϕ_U^V), where U and V run over nonempty open subsets of X, define thus a sheaf which we denote by $\mathscr{A}(M)$; we verify immediately that

$$\lim_{x \in U} M_U = M_x,$$

that is, that $\mathscr{A}(M)_x = M_x$. In particular, we have $\mathscr{A}(S) = \mathscr{O}$ and as the M_U are S_U–modules, it follows that $\mathscr{A}(M)$ is a sheaf of $\mathscr{A}(S)$–modules, that is, an *algebraic sheaf* on X. Any homomorphism $\phi : M \to M'$ defines in a natural way the S_U-linear homomorphisms $\phi_U : M_U \to M'_U$, thus a homomorphism of sheaves $\mathscr{A}(\phi) : \mathscr{A}(M) \to \mathscr{A}(M')$, which we frequently denote ϕ. We clearly have

$$\mathscr{A}(\phi + \psi) = \mathscr{A}(\phi) + \mathscr{A}(\psi), \quad \mathscr{A}(1) = 1, \quad \mathscr{A}(\phi \circ \psi) = \mathscr{A}(\phi) \circ \mathscr{A}(\psi).$$

The operation $\mathscr{A}(M)$ is thus a *covariant additive functor* defined on the category of graded S–modules and with values in the category of algebraic sheaves on X.

(The above definitions are quite analogous to these of §4, from Chap. II; it should be noted however that S_U *is not* the localization of S in $S(U)$, but only its homogeneous component of degree 0.)

58 First properties of the functor $\mathscr{A}(\mathsf{M})$

Proposition 3. *The functor $\mathscr{A}(M)$ is an exact functor.*

Let $M \xrightarrow{\alpha} M' \xrightarrow{\beta} M''$" be an exact sequence of graded S–modules and show that the sequence $M_x \xrightarrow{\alpha} M'_x \xrightarrow{\beta} M''_x$ is also exact. Let $m'/Q \in M'_x$ be an element of the kernel of β; by the definition of M''_x, there exist $R \in S(x)$ such that $R\beta(m') = 0$; but then there exists $m \in M$ such that $\alpha(m) = Rm'$ and we have $\alpha(m/RQ) = m'/Q$, q.e.d.
(Compare with n° 48, Lemma 1.)

Proposition 4. *If M is a graded S–module and if n is an integer, $\mathscr{A}(M(n))$ is canonically isomorphic to $\mathscr{A}(M)(n)$.*

Let $i \in I$, $x \in U_i$ and $m/Q \in M(n)_x$, with $m \in M(n)_p$, $Q \in S(x)$, $\deg Q = p$. Put:

$$\eta_{i,x}(m/Q) = m/t_i^n Q \in M_x,$$

which is valid because $m \in M_{n+p}$ and $t_i^n Q \in S(x)$. We immediately see that $\eta_{i,x} : M(n)_x \to M_x$ is bijective for all $x \in U_i$ and defines an isomorphism η_i of $\mathscr{A}(M(n))$ to $\mathscr{A}(M)$ over U_i. Moreover, we have $\eta_i \circ \eta_j^{-1} = \theta_{ij}(n)$ over $U_i \cap U_j$. By the definition of the operation $\mathscr{F}(n)$ and Proposition 4 of n° 4, this shows that $\mathscr{A}(M(n))$ is isomorphic to $\mathscr{A}(M)(n)$.

Corollary. *$\mathscr{A}(S(n))$ is canonically isomorphic to $\mathscr{O}(n)$.*

Indeed, it has been said that $\mathscr{A}(S)$ was isomorphic to $\mathscr{O}$.

(It is also clear that $\mathscr{A}(S(n))$ is isomorphic to $\mathscr{O}'(n)$, because $\mathscr{O}'(n)_x$ consists precisely of the rational functions P/Q such that $\deg P - \deg Q = n$ and $Q \in S(x)$.)

Proposition 5. *Let M be a graded S–module satisfying the condition (TF). The algebraic sheaf $\mathscr{A}(M)$ is also a coherent sheaf. Moreover $\mathscr{A}(M) = 0$ if and only if $M \in \mathscr{C}$.*

If $M \in \mathscr{C}$, for all $m \in M$ and $x \in X$ there exists $Q \in S(x)$ such that $Qm = 0$; it suffices to take Q of a sufficiently large degree; we thus have $M_x = 0$, hence $\mathscr{A}(M) = 0$. Let now M be a graded S–module satisfying the condition (TF); there exists a homogeneous submodule N of M, of finite type and such that $M/N \in \mathscr{C}$; applying the above together with Proposition 3, we see that $\mathscr{A}(N) \to \mathscr{A}(M)$ is bijective and it thus suffices to prove that $\mathscr{A}(N)$ is coherent. Since N is of finite type, there exists an exact sequence $L^1 \to L^0 \to N \to 0$ where L^0 and L^1 are *free* modules of finite type. By Proposition 3, the sequence $\mathscr{A}(L^1) \to \mathscr{A}(L^0) \to \mathscr{A}(N) \to 0$ is exact. But, by the corollary to Proposition 4, $\mathscr{A}(L^0)$ and $\mathscr{A}(L^1)$ are isomorphic to finite direct sums of the sheaves $\mathscr{O}(n_i)$ and are thus coherent. It follows that $\mathscr{A}(N)$ is coherent.

Let finally M be a graded S–module satisfying (TF) and such that $\mathscr{A}(M) = 0$; by the above considerations, we can suppose that M is of finite type. If m is a homogeneous element of M, let a_m be the annihilator of m, that is, the set of all polynomials $Q \in S$ such that $Q \cdot m = 0$; it is clear that a_m is a homogeneous ideal. Moreover, the assumption $M_x = 0$ for all $x \in X$ implies that the variety of zeros of a_m in K^{r+1} is reduced to $\{0\}$; Hilbert's theorem of zeros shows that every homogeneous polynomial of sufficiently large degree belongs to a_m. Applying this to the finite system of generators of M, we conclude immediately $M_p = 0$ for p sufficiently large, which completes the proof.

By combining Propositions 3 and 5 we obtain:

Proposition 6. *Let M and M' be two graded S–modules satisfying the condition (TF) and let $\phi : M \to M'$ be a homomorphism of M to M'. Then*

$$\mathscr{A}(\phi) : \mathscr{A}(M) \to \mathscr{A}(M')$$

is injective (resp. surjective, bijective) if and only if ϕ is $\mathscr{C}$-injective (resp. $\mathscr{C}$-surjective, $\mathscr{C}$-bijective).

59 The graded S–module associated to an algebraic sheaf

Let $\mathscr{F}$ be an algebraic sheaf on X and set:

$$\Gamma(\mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathscr{F})_n, \quad \text{with} \quad \Gamma(\mathscr{F})_n = \Gamma(X, \mathscr{F}(n)).$$

The group $\Gamma(\mathscr{F})$ is a graded group; we shall equip it with a structure of an S–module. Let $s \in \Gamma(X, \mathscr{F}(q))$ and let $P \in S_p$; we can identify P with a section of $\mathscr{O}(p)$ (cf. n° 54), thus $P \otimes s$ is a section of $\mathscr{O}(p) \otimes \mathscr{F}(q) = \mathscr{F}(q)(p) = \mathscr{F}(p+q)$, using the homomorphisms from n° 54; we have then defined a section of $\mathscr{F}(p+q)$ which we denote by $P \cdot s$ instead of $P \otimes s$. The mapping $(P, s) \to P \cdot s$ equips $\Gamma(\mathscr{F})$ with a structure of an S–module that is compatible with the grading.

In order to compare the functors $\mathscr{A}(M)$ and $\Gamma(\mathscr{F})$ we define two canonical homomorphisms:

$$\alpha : M \to \Gamma(\mathscr{A}(M)) \quad \text{and} \quad \beta : \mathscr{A}(\Gamma(\mathscr{F})) \to \mathscr{F}.$$

Definition of α. Let M be a graded S–module and let $m \in M_0$ be a homogeneous element of M of degree 0. The element $m/1$ is a well-defined element of M_x that varies continuously with $x \in X$; thus m defines a section $\alpha(m)$ of $\mathscr{A}(M)$. If now m is homogeneous of degree n, m is homogeneous of degree 0 in $M(n)$, thus defines a section $\alpha(m)$ of $\mathscr{A}(M(n)) = \mathscr{A}(M)(n)$ (cf. Proposition 4). This is the definition of $\alpha : M \to \Gamma(\mathscr{A}(M))$ and it is immediate that it is a homomorphism.

Definition of β. Let $\mathscr{F}$ be an algebraic sheaf on X and let s/Q be an element of $\Gamma(\mathscr{F})_x$ with $s \in \Gamma(X, \mathscr{F}(n))$, $Q \in S_n$ and $Q(x) \neq 0$. The function $1/Q$ is homogeneous of degree $-n$ and regular in x, hence a section of $\mathscr{O}(-n)$ in a neighborhood of x; it follows that $1/Q \otimes s$ is a section of $\mathscr{O}(-n) \otimes \mathscr{F}(n) = \mathscr{F}$ in a neighborhood of x, thus defines an element of $\mathscr{F}_x$ which we denote by $\beta_x(s/Q)$, because it depends only on s/Q. We can also define β_x by using the components s_i of s: if $x \in U_i$, $\beta_x(s/Q) = (t_i^n/Q) \cdot s_i(x)$. The collection of the homomorphisms β_x defines a homomorphism $\beta : \mathscr{A}(\Gamma(\mathscr{F})) \to \mathscr{F}$.

The homomorphisms α and β are related by the following Propositions, which are shown by direct computation:

Proposition 7. *Let M be a graded S–module. The composition of the homomorphisms $\mathscr{A}(M) \to \mathscr{A}(\Gamma(\mathscr{A}(M))) \to \mathscr{A}(M)$ is the identity.*

(The first homomorphism is defined by $\alpha : M \to \Gamma(\mathscr{A}(M))$ and the second is β, applied to $\mathscr{F} = \mathscr{A}(M)$.)

Proposition 8. *Let $\mathscr{F}$ be an algebraic sheaf on X. The composition of the homomorphisms $\Gamma(\mathscr{F}) \to \Gamma(\mathscr{A}(\Gamma(\mathscr{F}))) \to \Gamma(\mathscr{F})$ is the identity.*

(The first homomorphism is α, applied to $M = \Gamma(\mathscr{F})$, while the second one is defined by $\beta : \mathscr{A}(\Gamma(\mathscr{F})) \to \mathscr{F}$.)

We will show in n° 65 that $\beta : \mathscr{A}(\Gamma(\mathscr{F})) \to \mathscr{F}$ is bijective if $\mathscr{F}$ is coherent and that $\alpha : M \to \Gamma(\mathscr{A}(M))$ is $\mathscr{C}$-bijective if M satisfies the condition (TF).

60 The case of coherent algebraic sheaves

Let us show a preliminary result:

Proposition 9. *Let $\mathscr{L}$ be an algebraic sheaf on X, a direct sum of a finite number of the sheaves $\mathscr{O}(n_i)$. Then $\Gamma(\mathscr{F})$ satisfies (TF) and $\beta : \mathscr{A}(\Gamma(\mathscr{L})) \to \mathscr{L}$ is bijective.*

It comes down immediately $\mathscr{L} = \mathscr{O}(n)$, then to $\mathscr{L} = \mathscr{O}$. In this case, we know that $\Gamma(\mathscr{O}(p)) = S_p$ for $p \geq 0$, thus we have $S \subset \Gamma(\mathscr{O})$, the quotient belonging to $\mathscr{C}$. It follows first that $\Gamma(\mathscr{O})$ satisfies (TF), then that $\mathscr{A}(\Gamma(\mathscr{O})) = \mathscr{A}(S) = \mathscr{O}$, q.e.d.

(We observe that we have $\Gamma(\mathscr{O}) = S$ if $r \geq 1$; on the other hand, if $r = 0$, $\Gamma(\mathscr{O})$ is not even an S–module of finite type.)

Theorem 2. *For every coherent algebraic sheaf $\mathscr{F}$ on X there exists a graded S–module M, satisfying (TF), such that $\mathscr{A}(M)$ is isomorphic to $\mathscr{F}$.*

By the corollary to Theorem 1, there exists an exact sequence of algebraic sheaves:

$$\mathscr{L}^1 \xrightarrow{\phi} \mathscr{L}^0 \to \mathscr{F} \to 0,$$

where $\mathscr{L}^1$ and $\mathscr{L}^0$ satisfy the assumptions of the above Proposition. Let M be the cokernel of the homomorphism $\Gamma(\phi) : \Gamma(\mathscr{L}^1) \to \Gamma(\mathscr{L}^0)$; by Proposition 9, M satisfies the condition (TF). Applying the functor $\mathscr{A}$ to the exact sequence:

$$\Gamma(\mathscr{L}^1) \to \Gamma(\mathscr{L}^0) \to M \to 0,$$

we obtain an exact sequence:

$$\mathscr{A}(\Gamma(\mathscr{L}^1)) \to \mathscr{A}(\Gamma(\mathscr{L}^0)) \to \mathscr{A}(M) \to 0.$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathscr{A}(\Gamma(\mathscr{L}^1)) & \longrightarrow & \mathscr{A}(\Gamma(\mathscr{L}^0)) & \longrightarrow & \mathscr{A}(M) & \longrightarrow & 0 \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & & & \\
\mathscr{L}^1 & \longrightarrow & \mathscr{L}^0 & \longrightarrow & \mathscr{F} & \longrightarrow & 0
\end{array}$$

By Proposition 9, the two vertical homomorphisms are bijective. It follows that $\mathscr{A}(M)$ is isomorphic to $\mathscr{F}$, q.e.d.

§3 Cohomology of the projective space with values in a coherent algebraic sheaf

61 The complexes $\mathsf{C}_\mathsf{k}(\mathsf{M})$ and $\mathsf{C}(\mathsf{M})$

We preserve the notations of n^{os} 51 and 56. In particular, I will denote the interval $\{0, 1, \ldots, r\}$ and S will denote the graded algebra $K[t_0, \ldots, t_r]$.

Let M be a graded S–module, k and q two integers ≥ 0; we shall define a group $C_k^q(M)$: an element of $C_k^q(M)$ is a mapping

$$(i_0, \ldots, i_q) \mapsto m\langle i_0 \ldots i_q\rangle$$

which associates to every sequence $(i_0, \ldots, i_q)$ of $q+1$ elements of I a homogeneous element of degree $k(q+1)$ of M, depending in an alternating way on $i_0, \ldots, i_q$. In particular, we have $m\langle i_0 \ldots i_q\rangle = 0$ if two of the indices $i_0, \ldots, i_q$ are equal. We define addition in $C_k^q(M)$ in the obvious way. the same with multiplication by an element $\lambda \in K$, and $C_k^q(M)$ is a *vector space over* K.

If m is an element of $C_k^q(M)$, we define $dm \in C_k^{q+1}(M)$ by the formula:

$$(dm)\langle i_0 \ldots i_{q+1}\rangle = \sum_{j=0}^{j=q+1} (-1)^j t_{i_j}^k \cdot m\langle i_0 \ldots \hat{i}_j \ldots i_{q+1}\rangle.$$

We verify by a direct calculation that $d \circ d = 0$; thus, the direct sum $C_k(M) = \bigoplus_{q=0}^{q=r} C_k^q(M)$, equipped with the coboundary operator d, is a *complex*, whose q-th cohomology group is denoted by $H_k^q(M)$.

(We note, after [11], another interpretation of the elements of $C_k^q(M)$: introduce $r+1$ differential symbols $dx_0, \ldots, dx_r$ and associate to every $m \in C_k^q(M)$ a ,,differential form" of degree $q+1$:

$$\omega_m = \sum_{i_0 < \ldots < i_q} m\langle i_0 \ldots i_q\rangle dx_{i_0} \wedge \ldots \wedge dx_{i_q}.$$

If we put $\alpha_k = \sum_{i=0}^{i=r} t_i^k dx_i$, we see that we have:

$$\omega_{dm} = \alpha_k \wedge \omega_m,$$

in other words, the coboundary operation is transformed into the exterior multiplication by the form α_k).

If h is an integer $\geq k$, let $\rho_k^h : C_k^q(M) \to C_h^q(M)$ be the homomorphism defined by the formula:

$$\rho_k^h(m)\langle i_0 \ldots i_q\rangle = (t_{i_0} \ldots t_{i_q})^{h-k} m\langle i_0 \ldots i_q\rangle.$$

We have $\rho_k^h \circ d = d \circ \rho_k^h$ and $\rho_h^l \circ \rho_k^j = \rho_k^l$ if $k \leq h \leq l$. We can thus define a complex $C(M)$, the inductive limit of the system $(C_k(M), \rho_k^h)$ for $k \to +\infty$. The cohomology groups of this complex are denoted $H^q(M)$. Because cohomology commutes with inductive limits (cf. [6], Chap. V, Prop. 9.3*), we have:

$$H^q(M) = \lim_{k \to \infty} H_k^q(M).$$

Every homomorphism $\phi : M \to M'$ defines a homomorphism

$$\phi : C_k(M) \to C_k(M')$$

by the formula: $\phi(m)\langle i_0 \dots i_q \rangle = \phi(m\langle i_0 \dots i_q \rangle)$, hence, by passing to the limit, $\phi : C(M) \to C(M')$; moreover, these homomorphisms commute with boundary and thus define the homomorphisms

$$\phi : Hq_k(M) \to Hq_k(M') \quad \text{and} \quad \phi : H^q(M) \to H^q(M').$$

If we have an exact sequence $0 \to M \to M' \to M'' \to 0$, we have an exact sequence of complexes $0 \to C_k(M) \to C_k(M') \to C_k(M'') \to 0$, hence an exact sequence of cohomology:

$$\dots H_k^q(M') \to H_k^q(M'') \to H_k^{q+1}(M) \to H_k^{q+1}(M') \to \dots$$

The same results for $C(M)$ and $H^q(M)$.

Remark. We shall see later (cf. n° 69) that we can express $H_k^q(M)$ in terms of Ext_S^q.

62 Calculation of $\mathsf{H}_\mathsf{k}^\mathsf{q}(\mathsf{M})$ for certain modules M

Let M be a graded S–module and $m \in M$ a homogeneous element of degree 0. The system of $(t_i^k \cdot m)$ is a 0-cocycle of $C_k(M)$, which we denote by $\alpha^k(m)$ and identify with its cohomology class. We so obtain a K-linear homomorphism $\alpha^k : M_0 \to H_k^0(M)$; as $\alpha^h = \rho_k^h \circ \alpha^k$ if $h \geq k$, the α^k define by passing to the limit a homomorphism $\alpha : M_0 \to H^0(M)$.

Let us introduce two more notations:

If $(P_0, \dots, P_h)$ are elements of S, we denote by $(P_0, \dots, P_h)M$ the submodule of M consisting of the elements $\,_{i=0}^{i=h} P_i \cdot m_i$ with $m_i \in M$; if the P_i are homogeneous, this submodule is homogeneous.

If P is an element of S and N a submodule of M, we denote by $N : P$ the submodule of M consisting of the elements $m \in M$ such that $P \cdot m \in N$; we clearly have $N : P \supset N$; if N and P are homogeneous, so is $N : P$.

Having specified these notations, we have:

Proposition 1. *Let M be a graded S–module and k an integer ≥ 0. Assume that for all $i \in I$ we have:*

$$(t_0^k, \dots, t_{i-1}^k)M : t_i^k = (t_0^k, \dots, t_{i-1}^k)M.$$

Then:
(a) $\alpha^k : M_0 \to H_k^0(M)$ is bijective (if $r \geq 1$),
(b) $H_k^q(M) = 0$ for $0 < q < r$.

(For $i = 0$, the assumption means that $t_0^k \cdot m = 0$ implies $m = 0$.)

This Proposition is a special case of a result of de Rham [11] (the de Rham's result being also valid even if we do not assume that the $m\langle i_0 \dots i_q \rangle$ are homogeneous). See also [6], Chap. VIII, ¶4 for a particular case, sufficient for our purposes.

We now apply Proposition 1 to the graded S–module $S(n)$:

Proposition 2. *Let k be an integer ≥ 0, n an arbitrary integer. Then:*
(a) $\alpha^k : S_n \to H_k^0(S(n))$ is bijective (if $r \geq 1$),
(b) $H_k^q(S(n)) = 0$ for $0 < q < r$,
(c) $H_k^r(S(n))$ admits a base (over K) consisting of the cohomology classes of the monomials $t_0^{\alpha_0} \dots t_r^{\alpha_r}$ with $0 \leq \alpha_i < k$ and $\sum_{i=0}^{i=r} \alpha_i = k(r+1) + n$.

It is clear that the S–module $S(n)$ satisfies the assumptions of Proposition 1, which shows (a) and (b). On the other hand, for every graded S–module M, we have $H_k^r(M) = M_{k(r+1)}/(t_0^k, \dots, t_r^k)M_{kr}$; now the monomials

$$t_0^{\alpha_0} \dots t_r^{\alpha_r}, \alpha_i \geq 0, \sum_{i=0}^{i=r} \alpha_i = k(r+1) + n,$$

form a basis of $S(n)_{k(r+1)}$ and those for which at least α_i is $\geq k$ form a basis of $(t_0^k, \dots, t_r^k)S(n)_{kr}$; hence (c).

It is convenient to write the exponents α_i in the form $\alpha_i = k - \beta_i$. The conditions of (c) are then written:

$$0 < \beta_i \leq k \quad \text{and} \quad \sum_{i=0}^{i=r} \beta_i = -n.$$

The second condition, together with $\beta_i > 0$, implies $\beta_i \leq -n - r$; if thus $k \geq -n - r$, the condition $\beta_i \leq k$ is a consequence of the preceding two. Hence:

Corollary 1. *For $k \geq -n - r$, $H_k^r(S(n))$ admits a basis formed of the cohomology classes of monomials $(t_0 \dots t_r)^k / t_0^{\beta_0} \dots t_r^{\beta_r}$ with $\beta_i > 0$ and $\sum_{i=0}^{i=r} \beta_i = -n$.*

We also have:

Corollary 2. *If $h \geq k \geq -n-r$, the homomorphism*

$$\rho_k^h : H_k^q(S(n)) \to H_k^q(S(n))$$

is bijective for all $q \geq 0$.

For $q \neq r$, this follows from the assertions (a) and (b) of Proposition 2. For $q = r$, this follows from Corollary 1, given that ρ_k^h transforms

$$(t_0 \dots t_r)^k / t_0^{\beta_0} \dots t_r^{\beta_r} \quad \text{into} \quad (t_0 \dots t_r)^h / t_0^{\beta_0} \dots t_r^{\beta_r}.$$

Corollary 3. *The homomorphism $\alpha : S_n \to H^0(S(n))$ is bijective if $r \geq 1$ or if $n \geq 0$. We have $H^q(S(n)) = 0$ for $0 < q < r$ and $H^r(S(n))$ is a vector space of dimension $\binom{-n-1}{r}$ over K.*

The assertion pertaining to α follows from Proposition 2, (a), in the case when $r \geq 1$; it is clear if $r = 0$ and $n \geq 0$. The rest of the Corollary is an obvious consequence of Corollaries 1 and 2 (seeing that the binomial coefficient $\binom{a}{r}$ is zero for $a < r$).

63 General properties of $H^q(M)$

Proposition 3. *Let M be a graded S–module satisfying the condition (TF). Then:*

(a) There exists an integer $k(M)$ such that $\rho_k^h : H_k^q(M) \to H_h^q(M)$ is bijective for $h \geq k \geq k(M)$ and every q.

(b) $H^q(M)$ is a vector space of finite dimension over K for all $q \geq 0$.

(c) There exists an integer $n(M)$ such that for $n \geq n(M)$, $\alpha : M_n \to H^0(M(n))$ is bijective and that $H^q(M(n))$ is zero for all $q > 0$.

This is immediately reduced to the case when M is of finite type. We say that M is of *dimension* $\leq s$ (s being an integer ≥ 0) if there exists an exact sequence:

$$0 \to L^s \to L^{s-1} \to \dots \to L^0 \to M \to 0,$$

where L^i are free graded S–modules of finite type. By the Hilbert syzygy theorem (cf. [6], Chap. VIII, th. 6.5), this dimension is always $\leq r+1$.

We prove the Proposition by induction on the dimension of M. If it is 0, M is free of finite type, i.e. a direct sum of modules $S(n_i)$ and the Proposition follows from Corollaries 2 and 3 and Proposition 2. Assume that M is of dimension $\leq s$ and let N be the kernel of $L^0 \to M$. The graded S–module N is of dimension $\leq s-1$ and we have an exact sequence:

$$0 \to N \to L^0 \to M \to 0.$$

By the induction assumption, the Proposition is true for N and L^0. Applying the five lemma ([7], Chap. I, Lemme 4.3) to the commutative diagram:

$$\begin{array}{ccccccccc} H^q_k(N) & \longrightarrow & H^q_k(L^0) & \longrightarrow & H^q_k(M) & \longrightarrow & H^{q+1}_k(N) & \longrightarrow & H^{q+1}_k(L^0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^q_h(N) & \longrightarrow & H^q_h(L^0) & \longrightarrow & H^q_h(M) & \longrightarrow & H^{q+1}_h(N) & \longrightarrow & H^{q+1}_h(L^0), \end{array}$$

where $h \geq k \geq \mathrm{Sup}(k(N), k(L^0)$, we show (a), thus also (b), because the $H^q_k(M)$ are of finite dimension over K. On the other hand, the exact sequence

$$H^q(L^0(n)) \to H^q(M(n)) \to H^{q+1}(N(n))$$

shows that $H^q(M(n)) = 0$ for $n \geq \mathrm{Sup}(n(L^0), n(N))$. Finally, consider the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_n & \longrightarrow & L_n & \longrightarrow & M_n & \longrightarrow & 0 \\ \downarrow & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \\ 0 & \longrightarrow & H^0(N(n)) & \longrightarrow & H^0(L^0(n)) & \longrightarrow & H^0(M(n)) & \longrightarrow & H^1(N(n)); \end{array}$$

for $n \geq n(N)$, we have $H^1(N(n)) = 0$; we deduce that $\alpha : M_n \to H^0(M(n))$ is bijective for $n \geq \mathrm{Sup}(n(L^0), n(N))$, which completes the proof of the Proposition.

64 Comparison of the groups $\mathsf{H}^q(\mathsf{M})$ and $\mathsf{H}^q(\mathsf{X}, \mathscr{A}(\mathsf{M}))$

Let M be a graded S–module and let $\mathscr{A}(M)$ be the algebraic sheaf on $X = \mathbb{P}_r(K)$ defined by M by the procedure of n° 57. We will now compare $C(M)$ with $C'(\mathsf{U}, \mathscr{A}(M))$, the complex of alternating cochains of the covering $\mathsf{U} = \{U_i\}_{i \in I}$ with values in the sheaf $\mathscr{A}(M)$.

Let $m \in C^q_k(M)$ and let $(i_0, \dots, i_q)$ be a sequence of $q+1$ elements of I. The polynomial $(t_{i_0} \dots t_{i_q})^k$ belongs obviously to $S(U_{i_0 \dots i_q})$, with the notations of n° 57. It follows that $m\langle i_0 \dots i_q \rangle / (t_{i_0} \dots t_{i_q})^k$ belongs to M_U, where $U = U_{i_0 \dots i_q}$, thus defines a section of $\mathscr{A}(M)$ over $U_{i_0 \dots i_q}$. When $(i_0, \dots, i_q)$ varies, the system consisting of this sections is an alternating cochain of U with values in $\mathscr{A}(M)$, which we denote by $\iota_k(m)$. We immediately see that ι_k commutes with d and that $\iota_k = \iota_h \circ \rho^h_k$ if $h \geq k$. By passing to the inductive limit, the ι_k thus define a homomorphism $\iota : C(M) \to C'(\mathsf{U}, \mathscr{A}(M))$, commuting with d.

Proposition 4. *If M satisfies the condition (TF), $\iota : C(M) \to C'(\mathsf{U}, \mathscr{A}(M))$ is bijective.*

If $M \in \mathscr{C}$, we have $M_n = 0$ for $n \geq n_0$, so $C_k(M) = 0$ for $k \geq n_0$ and $C(M) = 0$. As every S–module satisfying (TF) is $\mathscr{C}$-isomorphic to a module of finite type, this shows that we can restrict ourselves to the case when M is of finite type. We can then find an exact sequence $L^1 \to L^0 \to M \to 0$, where L^1 and L^0 are free of finite type. By Propositions 3 and 5 from n° 58, the sequence

$$\mathscr{A}(L^1) \to \mathscr{A}(L^0) \to \mathscr{A}(M) \to 0$$

is an exact sequence of coherent algebraic sheaves; as the $U_{i_0 \dots i_q}$ are affine open subsets, the sequence

$$C'(\mathsf{U}, \mathscr{A}(L^1)) \to C'(\mathsf{U}, \mathscr{A}(L^0) \to C'(\mathsf{U}, \mathscr{A}(M)) \to 0$$

is exact (cf. n° 45, Corollary 2 to Theorem 2). The commutative diagram

$$\begin{array}{ccccccc}
C(L^1) & \longrightarrow & C(L^0) & \longrightarrow & C(M) & \longrightarrow & 0 \\
\downarrow \iota & & \downarrow \iota & & \downarrow \iota & & \downarrow \\
C'(\mathsf{U}, \mathscr{A}(L^1)) & \longrightarrow & C'(\mathsf{U}, \mathscr{A}(L^0)) & \longrightarrow & C'(\mathsf{U}, \mathscr{A}(M)) & \longrightarrow & 0
\end{array}$$

then shows that if the Proposition is true for the module L^1 and L^0, so it is for M. We are thus reduced to the special case of a free module of finite type, then, by the decomposition into direct summands, to the case when $M = S(n)$.

In this case, we have $\mathscr{A}(S(n)) = \mathscr{O}(n)$; a section $f_{i_0 \dots i_q}$ of $\mathscr{O}(n)$ over $U_{i_0 \dots i_q}$ is, by the sole definition of this sheaf, a regular function on $V_{i_0} \cap \dots \cap V_{i_q}$ and homogeneous of degree n. As $V_{i_0} \cap \dots \cap V_{i_q}$ as the set of points of K^{r+1} where the function $t_{i_0} \dots t_{i_q}$ is $\neq 0$, there exists an integer k such that

$$f_{i_0 \dots i_q} = P\langle i_0 \dots i_q \rangle / (t_{i_0} \dots t_{i_q})^k,$$

$P\langle i_0 \dots i_q \rangle$ being a homogeneous polynomial of degree $n + k(q+1)$, that is, of degree $k(q+1)$ in $S(n)$. Thus, every alternating cochain $f \in C'(\mathsf{U}, \mathscr{O}(n))$ defines a system $P\langle i_0 \dots i_q \rangle$ that is an element of $C_k(S(n))$; hence a homomorphism

$$\nu : C'(\mathsf{U}, \mathscr{O}(n)) \to C(S(n)).$$

As we verify immediately that $\iota \circ \nu = 1$ and $\nu \circ \iota = 1$, it follows that ι is bijective, which completes the proof.

Corollary. *ι defines an isomorphism of $H^q(M)$ with $H^q(X, \mathscr{A}(M))$ for all $q \geq 0$.*

Indeed, we know that $H'^q(\mathsf{U}, \mathscr{A}(M)) = H^q(\mathsf{U}, \mathscr{A}(M))$ (n° 20, Proposition 2) and that $H^q(\mathsf{U}, \mathscr{A}(M)) = H^q(X, \mathscr{A}(M))$ (n° 52, Proposition 2, which applies because $\mathscr{A}(M)$ is coherent).

Remark. It is easy to see that $\iota : C(M) \to C'(\mathsf{U}, \mathscr{A}(M))$ is *injective* even if M does not satisfy the condition (TF).

65 Applications

Proposition 5. *If M is a graded S–module satisfying the condition (TF), the homomorphism $\alpha : M \to \Gamma(\mathscr{A}(M))$, defined in n° 59, is $\mathscr{C}$-bijective.*

We must observe that $\alpha : M_n \to \Gamma(X, \mathscr{A}(M(n)))$ is bijective for n sufficiently large. Then, by Proposition 4, $\Gamma(X, \mathscr{A}(M(n)))$ is identified with $H^0(M(n))$; the Proposition follows thus from Proposition 3, (c), given the fact that the homomorphism α is transformed by the above identification to a homomorphism defined at the beginning of n° 62, also denoted by α.

Proposition 6. *Let $\mathscr{F}$ be a coherent algebraic sheaf on X. The graded S–module $\Gamma(\mathscr{F})$ satisfies the condition (TF) and the homomorphism $\beta : \mathscr{A}(\Gamma(\mathscr{F})) \to \mathscr{F}$ defined in n° 59 is bijective.*

By Theorem 2 of n° 60, we can assume that $\mathscr{F} = \mathscr{A}(M)$, where M is a module satisfying (TF). By the above Proposition, $\alpha : M \to \Gamma(\mathscr{A}(M))$ is $\mathscr{C}$-bijective; as M satisfies (TF), it follows that $\Gamma(\mathscr{A}(M))$ satisfies it also. Applying Proposition 6 from n° 58, we see that $\alpha : \mathscr{A}(M) \to \mathscr{A}(\Gamma(\mathscr{A}(M)))$ is bijective. Since the composition $\mathscr{A}(M) \xrightarrow{\alpha} \mathscr{A}(\Gamma(\mathscr{A}(M))) \xrightarrow{\beta} \mathscr{A}(M)$ is the identity (n° 59, Proposition 7), it follows that β is bijective, q.e.d.

Proposition 7. *Let $\mathscr{F}$ be a coherent algebraic sheaf on X. The groups $H^q(X, \mathscr{F})$ are vector spaces of finite dimension over K for all $q \geq 0$ and we have $H^q(X, \mathscr{F}(n)) = 0$ for $q > 0$ and n sufficiently large.*

We can assume, as above, that $\mathscr{F} = \mathscr{A}(M)$ where M is a module satisfying (TF). The Proposition then follows from Proposition 3 and the corollary to Proposition 4.

Proposition 8. *We have $H^q(X, \mathscr{O}(n)) = 0$ for $0 < q < r$ and $H^r(X, \mathscr{O}(n))$ is a vector space of dimension $\binom{-n-1}{r}$ over K, admitting a base consisting of the cohomology classes of the alternating cocycles of* U

$$f_{01\ldots r} = 1/t_0^{\beta_0} \ldots t_r^{\beta_r} \quad \textit{with} \quad \beta_i > 0 \quad \textit{and} \quad \sum_{i=0}^{i=r} \beta_i = -n.$$

We have $\mathscr{O}(n) = \mathscr{A}(S(n))$, hence $H^q(X, \mathscr{O}(n)) = H^q(S(n))$, by the corollary to Proposition 4; the Proposition follows immediately from this and from the corollaries of Proposition 2.

We note that in particular $H^r(X, \mathscr{O}(-r-1))$ is a vector space of dimension 1 over K, with a base consisting of the cohomology class of the cocycle $f_{01\ldots r} = 1/t_0 \ldots t_r$.

66 Coherent algebraic sheaves on projective varieties

Let V be a closed subvariety of the projective space $X = \mathbb{P}_r(K)$ and let $\mathscr{F}$ be a coherent algebraic sheaf on V. By extending $\mathscr{F}$ by 0 outside V, we obtain a coherent algebraic sheaf on X (cf. n° 39) denoted $\mathscr{F}^X$; we know that $H^q(X, \mathscr{F}^X) = H^q(V, \mathscr{F})$. The results of the preceding n° thus apply to the groups $H^q(V, \mathscr{F})$. We obtain immediately (given n° 52):

Theorem 1. *The groups $H^q(V, \mathscr{F})$ are vector spaces of finite dimension over K, zero for $q > \dim V$.*

In particular, for $q = 0$ we have:

Corollary. *$\Gamma(V, \mathscr{F})$ is a vector space of finite dimension over K.*

(It is natural to conjecture whether the above theorem holds for all *complete* varieties, in the sense of Weil [16].)

Let $U'_i = U_i \cap V$; the U'_i form an open covering U' of V. If $\mathscr{F}$ is an algebraic sheaf on V, let $\mathscr{F}_i = \mathscr{F}(U'_i)$ and let $\theta_{ij}(n)$ be the isomorphism of $\mathscr{F}_j(U'_i \cap U'_j)$ to $\mathscr{F}_i(U'_i \cap U'_j)$ defined by multiplication by $(t_j/t_i)^n$. We denote by $\mathscr{F}(n)$ the sheaf obtained by gluing the $\mathscr{F}_i$ with respect to $\theta_{ij}(n)$. The operation $\mathscr{F}(n)$ has the same properties as the operation defined in n° 54 and generalizes it; in particular, $\mathscr{F}(n)$ is canonically isomorphic to $\mathscr{F} \otimes \mathscr{O}_V(n)$.

We have $\mathscr{F}^X(n) = \mathscr{F}(n)^X$. Applying then Theorem 1 of n° 55, together with Proposition 7 from n° 65, we obtain:

Theorem 2. *Let $\mathscr{F}$ be a coherent algebraic sheaf on V. There exists an integer $m(\mathscr{F})$ such that we have, for all $n \geq m(\mathscr{F})$:*
(a) For all $x \in V$, the $\mathscr{O}_{x,V}$–module $\mathscr{F}(n)_x$ is generated by the elements of $\Gamma(V, \mathscr{F}(n))$,
(b) $H^q(V, \mathscr{F}(n)) = 0$ for all $q > 0$.

Remark. It is essential to observe that the sheaf $\mathscr{F}(n)$ *does not depend solely on $\mathscr{F}$ and n*, but also on the *embedding* of V into the projective space X. More precisely, let P be the principal bundle $\pi^{-1}(V)$ with the structural group K^*; with n an integer, we make K^* act on K by the formula:

$$(\lambda, \mu) \mapsto \lambda^{-n}\mu \quad \text{if} \quad \lambda \in K^* \quad \text{and} \quad \mu \in K.$$

Let $E^n = P \times_{K^*} K$ be the fiber space associated to P and the fiber K, equipped with the above action; let $\mathscr{S}(E^n)$ be the sheaf of germs of sections of E^n (cf. n° 41). Taking into account the fact that t_i/t_j form a system of transition maps of P, we verify immediately that $\mathscr{S}(E^n)$ is canonically isomorphic to $\mathscr{O}_V(n)$). The formula $\mathscr{F}(n) = \mathscr{F} \otimes \mathscr{O}_V(n) = \mathscr{F} \otimes \mathscr{S}(E^n)$ shows then that the operation $\mathscr{F} \to \mathscr{F}(n)$ depends only on the *class of the principal bundle P defined by the embedding $V \to X$*. In particular, if V is normal, $\mathscr{F}(n)$ depends only on the class of linear equivalence of hyperplane sections of V in the considered embedding (cf. [17]).

67 A supplement

If M is a graded S–module satisfying (TF), we denote by $M^\natural$ the graded S–module $\Gamma(\mathscr{A}(M))$. We have seen in n° 65 that $\alpha : M \to M^\natural$ is $\mathscr{C}$-bijective. We shall now give conditions for α to be bijective.

Proposition 9. *$\alpha : M \to M^\natural$ is bijective if and only if the following conditions are satisfied:*

(i) If $m \in M$ is such that $t_i \cdot m = 0$ for all $i \in I$, then $m = 0$,

(ii) If elements $m_i \in M$, homogeneous of the same degree, satisfy $t_j \cdot m_i = t_i \cdot m_j = 0$ for every couple (i, j), there exists an $m \in M$ such that $m_i = t_i \cdot m$.

Let us show that the conditions (i) and (ii) are satisfied by $M^\natural$, which will prove the necessity. For (i), we can assume that m is homogeneous, that is, it is a section of $\mathscr{A}(M(n))$; in this case, the condition $t_i \cdot m = 0$ implies that m is zero on U_i, and since this occurs for all $i \in I$, we have $m = 0$. For (ii), let n be the degree of m_i; we thus have $m_i \in \Gamma(\mathscr{A}(M(n)))$; as $1/t_i$ is a section of $\mathscr{O}(-1)$ over U_i, m_i/t_i is a section of $\mathscr{A}(M(n-1))$ over U_i and the condition $t_j \cdot m_i - t_i \cdot m_j$ shows that these various sections are the restrictions of a unique section m of $\mathscr{A}(M(n-1))$ over X; it remains to compare the sections $t_i \cdot m$ and m_i; to show that they coincide on U_j, it suffices to observe that $t_j(t_i \cdot m - m_i) = 0$ on U_j, which follows from the formula $t_j \cdot m_i = t_i \cdot m_j$ and the definition of m.

We will now show that (i) implies that α is injective. For n sufficiently large, we know that $\alpha : M_n \to M_n^\natural$ is bijective and we can thus proceed by descending induction on n. If $\alpha(m) = 0$ with $m \in M_n$, we have $t_i\alpha(m) = \alpha(t_i \cdot m) = 0$ and the induction assumption, applicable since $t_i \cdot m \in M_{n+1}$, shows that $m = 0$. Finally, let us show that (i) and (ii) imply that α is surjective. We can, as before, proceed by descending induction on n. If $m' \in M_n^\natural$, the induction assumption shows that there exist $m_i \in M_{n+1}$ such that $\alpha(m_i) = t_i \cdot m'$; we have $\alpha(t_j \cdot m_i - t_i \cdot m_j) = 0$, hence $t_j \cdot m_i - t_i \cdot m_j = 0$, because α is injective. The condition (ii) then implies that there exists an $m \in M_n$ such that $t_i \cdot m = m_i$; we have $t_i(m' - \alpha(m)) = 0$, which shows that $m' = \alpha(m)$ and completes the proof.

Remarks. (1) The proof shows that the condition (i) is sufficient and necessary for α to be injective.

(2) We can express (i) and (ii) as: the homomorphism $\alpha^1 : M_n \to H^0_q(M(n))$ is bijective for all $n \in \mathbb{Z}$. Besides, Proposition 4 shows that we can identify $M^\natural$ with the S–module $_{n\in\mathbb{Z}} H^0(M(n))$ and it would be easy to provide a purely algebraic proof of Proposition 9 (without using the sheaf $\mathscr{A}(M)$).

§4 Relations with the functors Ext_S^q

68 The functors Ext_S^q

We keep the notations of n° 56. If M and N are two graded S-modules, we denote by $\mathrm{Hom}_S(M,N)_n$ the group of homogeneous S-homomorphisms of degree n from M to N, and by $\mathrm{Hom}_S(M,N)$ the graded group $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_S(M,N)_n$; it is a graded S-module; when M is of finite type it coincides with the S-module of all S-homomorphisms from M to N.

The derived functors (cf. [6], Chapter V) of the functor $\mathrm{Hom}_S(M,N)$ are the functors $\mathrm{Ext}_S^q(M,N)$, $q = 0,1,\ldots$. Let us briefly recall their definition: [1]

One chooses a „resolution" of M, that is, an exact sequence:

$$\ldots \to L^{q+1} \to L^q \to \ldots \to L^0 \to M \to 0,$$

where the L^q are free graded S-modules and the maps are homomorphisms (that is, as usual, homogeneous S-homomorphisms of degree 0). If we set $C^q = \mathrm{Hom}_S(L^q, N)$, the homomorphism $L^{q+1} \to L^q$ defines by transposition a homomorphism $d : C^q \to C^{q+1}$ satisfying $d \circ d = 0$; therefore $C = \bigoplus_{q\geq 0} C^q$ is endowed with a structure of a complex, and the q-th cohomology group of C is just, by definition, equal to $\mathrm{Ext}_S^q(M,N)$; one shows that it does not depend on the chosen resolution. As the C^q are graded S-modules and since $d : C^q \to C^{q+1}$ is homogeneous of degree 0, the $\mathrm{Ext}_S^q(M,N)$ are S-modules graded by the subspaces $\mathrm{Ext}_S^q(M,N)_n)$; the $\mathrm{Ext}_S^q(M,N)$ are the cohomology groups of the complex formed by the $\mathrm{Hom}_S(L^q,N)_n)$, i.e., are the derived functors of the functor $\mathrm{Hom}_S(M,N)_n)$.

Recall the main properties of Ext_S^q:

$\mathrm{Ext}_S^0(M,N) = \mathrm{Hom}_S(M,N)$; $\mathrm{Ext}_S^q(M,N) = 0$ for $q > r+1$ if M is of finite type (due to the Hilbert syzygy theorem, cf. [6], Chapter VIII, theorem 6.5); $\mathrm{Ext}_S^q(M,N)$ is an S-module of finite type if M and N are both of finite type (because we can choose a resolution with the L^q of finite type); for all $n \in \mathbb{Z}$ we have the canonical isomorphisms:

$$\mathrm{Ext}_S^q(M(n),N) \approx \mathrm{Ext}_S^q(M,N(-n)) \approx \mathrm{Ext}_S^q(M,N)(-n).$$

The exact sequences:

$$0 \to N \to N' \to N'' \to 0 \quad \text{and} \quad 0 \to M \to M' \to M'' \to 0$$

give rise to exact sequences:

$$\ldots \to \mathrm{Ext}_S^q(M,N) \to \mathrm{Ext}_S^q(M,N') \to \mathrm{Ext}_S^q(M,N'') \to \mathrm{Ext}_S^{q+1}(M,N) \to \ldots$$

$$\ldots \to \mathrm{Ext}_S^q(M'',N) \to \mathrm{Ext}_S^q(M',M) \to \mathrm{Ext}_S^q(M,N) \to \mathrm{Ext}_S^{q-1}(M'',N) \to \ldots$$

[1] When M is not of finite type, the $\mathrm{Ext}_S^q(M,N)$ defined above can differ from the $\mathrm{Ext}_S^q(M,N)$ defined in [6]: it is due to the fact that $\mathrm{Hom}_S(M,N)$ does not have the same meaning in both cases. However, all the proofs of [6] are valid without change in the case considered here: this is seen either directly or by applying Appendix of [6].

69 Interpretation of $\mathsf{H}_k^q(\mathsf{M})$ in terms of $\mathrm{Ext}_{\mathsf{S}}^{\mathsf{q}}$

Let M be a graded S-module and let k be an integer ≥ 0. Set:

$$B_k^q(M) = \bigoplus_{n \in \mathbb{Z}} H_k^q(M(n)),$$

with the notations of n° 61.

We thus obtain a graded group, isomorphic with the q-th cohomology group of the complex $\bigoplus_{n \in \mathbb{Z}} C_k(M(n))$; this complex can be given a structure of an S-module, compatible with the grading by setting

$$(P \cdot m)\langle i_0 \cdots i_q \rangle = P \cdot m\langle i_0 \cdots i_q \rangle, \text{ if } P \in S_p \text{ and } m\langle i_0 \cdots i_q \rangle \in C_k^q(M(n));$$

as the coboundary operator is a homogeneous S-homomorphism of degree 0, it follows that the $B_k^q(M)$ are themselves graded S-modules.

We put

$$B^q(M) = \lim_{k \to \infty} B_k^q(M) = \bigoplus_{n \in \mathbb{Z}} H^q(M(n)).$$

The $B^q(M)$ are graded S-modules. For $q = 0$ we have

$$B^0(M) = \bigoplus_{n \in \mathbb{Z}} H^0(M(n)),$$

and we recognize the module denoted by $M^\natural$ in n° 67 (when M satisfies the condition (TF)). For each $n \in \mathbb{Z}$, we have defined in n° 62 a linear map $\alpha : M_n \to H^0(M(n))$; we verify immediately that the sum of these maps defines a homomorphism, which we denote also by α, from M to $B^0(M)$.

Proposition 1. *Let k be an integer ≥ 0 and let J_k be the ideal $(t_0^k, \ldots, t_r^k)$ of S. For every graded S-module M, the graded S-modules $B_k^q(M)$ and $\mathrm{Ext}_S^q(J_k, M)$ are isomorphic.*

Let L_k^q, $q = 0, \ldots, r$ be the free graded S-module with a base consisting of the elements $e\langle i_0 \cdots i_q \rangle$, $0 \leq i_0 < i_1 < \ldots < i_q \leq r$ of degree $k(q+1)$; we define an operator $d : L_k^{q+1} \to L_k^q$ and an operator $\varepsilon : L_k^0 \to J_k$ by the formulas:

$$d(e\langle i_0 \cdots i_{q+1} \rangle) = \sum_{j=0}^{j=q+1} (-1)^j t_{i_j}^k \cdot e\langle i_0 \cdots \hat{i}_j \cdots i_{q+1} \rangle,$$

$$\varepsilon(e\langle i \rangle) = t_i^k.$$

Lemma 1. *The sequence of homomorphisms:*

$$0 \to L_k^r \xrightarrow{d} L_k^{r-1} \to \ldots \to L_k^0 \xrightarrow{\varepsilon} J_k \to 0$$

is an exact sequence.

For $k = 1$, this result is well known (cf. [6], Chapter VIII, §4); the general case is shown in the same way (or reduced to it); we can also use the theorem shown in [11].

Proposition 1 follows immediately from the Lemma, if we observe that the complex formed by the $\mathrm{Hom}_S(L_k^q, M)$ and the transposition of d is just the complex $\bigoplus_{n \in \mathbb{Z}} C_k(M(n))$.

Corollary 1. *$H_k^q(M)$ is isomorphic to $\mathrm{Ext}_S^q(J_k, M)_0$.*

Indeed, these groups are the degree 0 components of the graded groups $B_k^q(M)$ and $\mathrm{Ext}_S^q(J_k, M)$.

Corollary 2. *$H^q(M)$ is isomorphic to $\lim_{k\to\infty} \mathrm{Ext}_S^q(J_k, M)_0$.*

We easily see that the homomorphism $\rho_k^h : H_k^q(M) \to H_h^q(M)$ from n° 61 is transformed by the isomorphism from Corollary 1 to a homomorphism from

$$\mathrm{Ext}_S^q(J_k, M)_0 \text{ to } \mathrm{Ext}_S^q(J_h, M)_0$$

induced by the inclusion $J_h \to J_k$; hence the Corollary 2.

Remark. Let M be a graded S-module of finite type; M defines (cf. n° 48) a coherent algebraic sheaf $\mathscr{F}'$ on K^{r+1}, thus on $Y = K^{r+1} - \{0\}$ and we can verify that $H^q(Y, \mathscr{F}')$ is isomorphic to $B^q(M)$.

70 Definition of the functors $T^q(M)$

Let us first define the notion of a *dual module* to a graded S-module. Let M be a graded S-module; for all $n \in \mathbb{Z}$, M_n is a vector space over K, whose dual vector space we denote by $(M_n)'$. Set

$$M^* = \bigoplus_{n \in \mathbb{Z}} M_n^*, \quad \text{with} \quad M_n^* = (M_{-n})'.$$

We give M^* the structure of an S-module compatible with the grading; for all $P \in S_p$, the mapping $m \mapsto P \cdot m$ is a K-linear map from M_{-n-p} to M_{-n}, so defines by transposition a K-linear map from $(M_{-n})' = M_n^*$ to $(M_{-n-p})' = M_{n+p}^*$; this defines the structure of an S-module on M^*. We could also define M^* as $\mathrm{Hom}_S(M, K)$, denoting by K the S-graded module $S/(t_0, \ldots, t_r)$.

The graded S-module M^* is called the module *dual* to M; we have $M^{**} = M$ if every M_n is of finite dimension over K, which holds if $M = \Gamma(\mathscr{F})$, $\mathscr{F}$ being a coherent algebraic sheaf on X, or if M is of finite type. Every homomorphism $\phi : M \to N$ defines by transposition a homomorphism from N^* to M^*. If the sequence $M \to N \to P$ is exact, so is the sequence $P^* \to N^* \to M^*$; in other words, M^* is a *contravariant* and *exact* functor of the module M. When I is a homogeneous ideal of S, the dual of S/I is exactly the ,,inverse system" of I, in the sense of Macaulay (cf. [9], n° 25).

Let now M be a graded S-module and q an integer ≥ 0. In the preceding n° , we have defined the graded S-module $B^q(M)$; the *module dual to* $B^q(M)$ *will be denoted by* $T^q(M)$. We thus have, by definition:

$$T^q(M) = \bigoplus_{n \in \mathbf{Z}} T^q(M)_n, \quad \text{with} \quad T^q(M)_n = (H^q(M(-n)))'.$$

Every homomorphism $\phi : M \to N$ defines a homomorphism from $B^q(M)$ to $B^q(N)$, thus a homomorphism from $T^q(N)$ to $T^q(M)$; thus the $T^q(M)$ are *contravariant* functors of M (we shall see in n° 72 that they can expressed very simply in terms of Ext_S). Every exact sequence:

$$0 \to M \to N \to P \to 0$$

gives rise to an exact sequence:

$$\dots B^q(M) \to B^q(N) \to B^q(P) \to B^{q+1}(M) \to \dots,$$

thus, by transposition, an exact sequence:

$$\dots T^{q+1}(M) \to T^q(P) \to T^q(N) \to T^q(M) \to \dots.$$

The homomorphism $\alpha : M \to B^0(M)$ defines by transposition a homomorphism $\alpha^* : T^0(M) \to M^*$.

Since $B^q(M) = 0$ for $q > r$, we have $T^q(M) = 0$ for $q > r$.

71 Determination of $\mathsf{T}^r(\mathsf{M})$.

(In this n° , and in the following, we assume that we have $r \geq 1$; the case $r = 0$ leads to somehow different, and trivial, statements).

We denote by Ω the graded S-module $S(-r-1)$; this is a free module, with a base consisting of an element of degree $r+1$. We have seen in n° 62 that $H^r(\Omega) = H^r_k(\Omega)$ for k sufficiently large, and that $H^r_k(\Omega)$ admits a base over K consisting of a single element $(t_0 \dots t_r)^k / t_0 \dots t_r$; the image in $H^r(\Omega)$ of this element will be denoted by ξ; ξ is thus a basis of $H^r(\Omega)$.

We will now define a scalar product $\langle h, \phi \rangle$ between elements $h \in B^r(M)_{-n}$ and $\phi \in \mathrm{Hom}_S(M, \Omega)_n$, M being an arbitrary graded S-module. The element ϕ can be identified with an element of $\mathrm{Hom}_S(M(-n), \Omega)_0$, that is, with a homomorphism from $M(-n)$ to Ω; it thus defines, by passing to cohomology groups, a homomorphism from $H^r(M(-n)) = B^r(M)_{-n}$ to $H^r(\Omega)$, which we also denote by ϕ. The image of h under this homomorphism is thus a scalar multiple of ξ, and we define $\langle h, \phi \rangle$ by the formula:

$$\phi(h) = \langle h, \phi \rangle \xi.$$

For every $\phi \in \mathrm{Hom}_S(M, \Omega)_n$, the function $h \mapsto \langle h, \phi \rangle$ is a linear form on $B^r(M)_{-n}$, thus can be identified with an element $\nu(\phi)$ of the dual of $B^r(M)_{-n}$, which is $T^r(M)_n$. We have thus defined a homogeneous mapping of degree 0

$$\nu : \mathrm{Hom}_S(M, \Omega) \to T^r(M),$$

and the formula $\langle P \cdot h, \phi \rangle = \langle h, P \cdot \phi \rangle$ shows that ν is an S-homomorphism.

Proposition 2. *The homomorphism $\nu : \mathrm{Hom}_S(M, \Omega) \to T^r(M)$ is bijective.*

We shall first prove the Proposition when M is a *free* module. If M is a direct sum of homogeneous submodules M^α, we have:

$$\mathrm{Hom}_S(M, \Omega)_n = \bigoplus_\alpha \mathrm{Hom}_S(M^\alpha, \Omega)_n \quad \text{and} \quad T^r(M)_n = \bigoplus_\alpha T^r(M^\alpha)_n.$$

So, if the proposition holds for the M^α, it holds for M, and this reduces the case of free modules to the particular case of a free module with a single generator, that is, to the case when $M = S(m)$. We can identify $\mathrm{Hom}_S(M, \Omega)_n$ with $\mathrm{Hom}_S(S, S(n - m - r - 1))_0$, that is, with the vector space of homogeneous polynomials of degree $n - m - r - 1$. Thus $\mathrm{Hom}_S(M, \Omega)_n$ has for a base the family of monomials $t_0^{\gamma_0} \dots t_r^{\gamma_r}$ with $\gamma_i \geq 0$ and $\sum_{i=0}^{i=r} \gamma_i = n - m - r - 1$. On the other hand, we have seen in n° 62 that $H^r_k(S(m - n))$ has for a base (if k is large enough) the family of monomials $(t_0 \dots t_r)^k / t_0^{\beta_0} \dots t_r^{\beta_r}$ with $\beta_i > 0$ and $\sum_{i=0}^{i=r} \beta_i = n - m$. By setting $\beta_i = \gamma'_i + 1$, we can write these monomials in the form $(t_0 \dots t_r)^{k-1} / t_0^{\gamma'_0} \dots t_r^{\gamma'_r}$, with $\gamma'_i \geq 0$ and $\sum_{i=0}^{i=r} \gamma'_i = n - m - r - 1$. Comparing the definition of $\langle h, \phi \rangle$, we observe that the scalar product

$$\langle (t_0 \dots t_r)^{k-1} / t_0^{\gamma'_0} \dots t_r^{\gamma'_r}, t_0^{\gamma_0} \dots t_r^{\gamma_r} \rangle$$

is always zero, unless $\gamma_i = \gamma'_i$ for all i, in which case it is equal to 1. This means that ν transforms the basis of $t_0^{\gamma_0} \dots t_r^{\gamma_r}$ to the dual basis of $(t_0 \dots t_r)^{k-1} / t_0^{\gamma'_0} \dots t_r^{\gamma'_r}$, thus is bijective, which shows the Proposition in the case when M is free.

Let us now pass to the general case. We choose an exact sequence

$$L^1 \to L^0 \to M \to 0$$

where L^0 and L^1 are free. Consider the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_S(M, \Omega) & \longrightarrow & \mathrm{Hom}_S(L^0, \Omega) & \longrightarrow & \mathrm{Hom}_S(L^1, \Omega) \\
\downarrow \nu & & \downarrow \nu & & \downarrow \nu & & \downarrow \nu \\
0 & \longrightarrow & T^r(M) & \longrightarrow & T^r(L^0) & \longrightarrow & T^r(L^1).
\end{array}$$

The first row of this diagram is an exact sequence, by the general properties of the functor Hom_S; the second is also exact, because it is dual to the sequence

$$B^r(L^1) \to B^r(L^0) \to B^r(M) \to 0,$$

which is exact by the cohomology exact sequence of B^q and the fact that $B^{r+1}(M) = 0$ for any M. On the other hand, the two vertical homomorphisms

$$\nu : \mathrm{Hom}_S(L^0, \Omega) \to T^r(L^0) \quad \text{and} \quad \nu : \mathrm{Hom}_S(L^1, \Omega) \to T^r(L^1)$$

are bijective, as we have just seen. It follows that

$$\nu : \mathrm{Hom}_S(M, \Omega) \to T^r(M)$$

is also bijective, which completes the proof.

72 Determination of $\mathsf{T}^q(\mathsf{M})$.

We shall now prove the following theorem, which generalizes Proposition 2:

Theorem 1. *Let M be a graded S-module. For $q \neq r$, the graded S-modules $T^{r-q}(M)$ and $\mathrm{Ext}_S^q(M, \Omega)$ are isomorphic. Moreover, we have an exact sequence:*

$$0 \to \mathrm{Ext}_S^r(M, \Omega) \to T^0(M) \xrightarrow{\alpha^*} M^* \to \mathrm{Ext}_S^{r+1}(M, \Omega) \to 0.$$

We will use the axiomatic characterization of derived functors given in [6], Chap. III, §5. For this, we first define new functors $E^q(M)$ in the following manner:

$$\begin{aligned} &\text{For } q \neq r, r+1, && E^q(M) = T^{r-q}(M), \\ &\text{For } q = r, && E^r(M) = \mathrm{Ker}(\alpha^*), \\ &\text{For } q = r+1, && E^{r+1}(M) = \mathrm{Coker}(\alpha^*). \end{aligned}$$

The $E^q(M)$ are additive functors of M, enjoying the following properties:

(i) *$E^0(M)$ is isomorphic to $\mathrm{Hom}_S(M, \Omega)$.*

This follows from Proposition 2.

(ii) *If L is free, $E^q(L) = 0$ for $q > 0$.*

It suffices to verify this for $L = S(n)$, in which case it follows from n° 62.

(iii) *To every exact sequence $0 \to M \to N \to P \to 0$ there is associated a sequence of coboundary operators $d^q : E^q(M) \to E^{q+1}(P)$ and the sequence:*

$$\dots E^q(P) \to E^q(N) \to E^q(M) \xrightarrow{d^q} E^{q+1}(P) \to \dots$$

is exact.

The definition of d^q is obvious if $q \neq r-1, r$: this is the homomorphism from $T^{r-q}(M)$ to $T^{r-q-1}(P)$ defined in n° 70. For $q = r-1$ or r, we use the following commutative diagram:

$$\begin{array}{ccccccccc} T^1(M) & \longrightarrow & T^0(P) & \longrightarrow & T^0(N) & \longrightarrow & T^0(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^*} \\ 0 & \longrightarrow & P^* & \longrightarrow & N^* & \longrightarrow & M^* & \longrightarrow & 0. \end{array}$$

This diagram shows immediately that the image of $T^1(M)$ is contained in the kernel of $\alpha^* : T^0(P) \to P^*$, which is just $E^r(P)$. This defines $d^{r-1} : E^{r-1}(M) \to E^r(P)$.

To define $d^r : \mathrm{Ker}(T^0(M) \to M^*) \to \mathrm{Coker}(T^0(P) \to P^*)$, we use the process from [6], Chap. III, Lemma 3.3: if $x \in \mathrm{Ker}(T^0(M) \to M^*)$, there exists $y \in P^*$ and $z \in T^0(N)$ such that x is the image of z and that y and z have the same image in N^*; we then set $d^r(x) = y$.

The exactness of the sequence

$$\ldots \to E^q(P) \to E^q(N) \to E^q(M) \xrightarrow{d^q} E^{q+1}(P) \to \ldots$$

follows from the exactness of the sequence

$$\ldots T^{r-q}(P) \to T^{r-q}(N) \to T^{r-q}(M) \to T^{r-q-1}(P) \to \ldots$$

and from [6], loc. cit.

(iv) *The isomorphism from (i) and the operators d^q from (iii) are ,,natural"*

This follows immediately from the definitions.

As the properties (i) to (iv) characterize the derived functors of the functor $\mathrm{Hom}_S(M, \Omega)$, we have $E^q(M) \approx \mathrm{Ext}_S^q(M, \Omega)$, which proves the Theorem.

Corollary 1. *If M satisfies (TF), $H^q(M)$ is isomorphic to the vector space dual to $\mathrm{Ext}_S^{r-q}(M, \Omega)_0$ for all $q \geq 1$.*

In fact, we know that $H^q(M)$ is a vector space of finite dimension, whose dual is isomorphic to $\mathrm{Ext}_S^{r-q}(M, \Omega)_0$.

Corollary 2. *If M satisfies (TF), the $T^q(M)$ are graded S-modules of finite type for $q \geq 1$, and $T^0(M)$ satisfies (TF).*

We can replace M by a module of finite type without changing the $B^q(M)$, thus $T^q(M)$. The $\mathrm{Ext}_S^{r-q}(M, \Omega)$ are then S-modules of finite type, and we have $M^* \in \mathscr{C}$, hence the Corollary.

§5 Applications to coherent algebraic sheaves

73 Relations between functors Ext^q_S and $\mathrm{Ext}^q_{\mathscr{O}_x}$

Let M and N be two graded S-modules. If x is a point of $X = \mathbb{P}_r(K)$, we have defined in n° 57 the $\mathscr{O}_x$-modules M_x and N_x; we will find relation between $\mathrm{Ext}^q_{\mathcal{O}_x}(M_x, N_x)$ and graded S-module $\mathrm{Ext}^q_S(M, N)$.

Proposition 1. *Suppose that M is of finite type. Then:*

(a) *The sheaf $\mathscr{A}(Hom_S(M, N))$ is isomorphic to the sheaf $Hom_{\mathcal{O}}(\mathscr{A}(M), \mathscr{A}(N))$.*

(b) *For all $x \in X$, the $\mathscr{O}_x$-module $\mathrm{Ext}^q_S(M, N)_x$ is isomorphic to the $\mathscr{O}_x$-module $\mathrm{Ext}^q_{\mathcal{O}_x}(M_x, N_x)$.*

First define a homomorphism $\iota_x : Hom_S(M, N)_x \to Hom_{\mathcal{O}_x}(M_x, N_x)$. An element of first module is a fraction φ/P, with $\varphi \in Hom_S(M, N)_n$, $P \in S(x)$, P is homogeneous of degree n; if m/P' is an element of M_x, $\varphi(m)/PP'$ is an element of N_x which does not depend on φ/P and m/P', and the function $m/P' \to \varphi(m)/PP'$ is a homomorphism $\iota_x(\varphi/P) : M_x \to N_x$; this defines ι_x. After Proposition 5 of n° 14, $Hom_{\mathcal{O}_x}(M_x, N_x)$ can be identified with:

$$Hom_{\mathcal{O}}(\mathscr{A}(M), \mathscr{A}(N))_x;$$

this identification transforms ι_x into:

$$\iota_x : \mathscr{A}(Hom_S(M, N))_x \to Hom_{\mathcal{O}}(\mathscr{A}(M), \mathscr{A}(N))_x,$$

and we easily verify that the family of ι_x is a homomorphism

$$\iota : \mathscr{A}(Hom_S(M, N)) \to Hom_{\mathcal{O}}(\mathscr{A}(M), \mathscr{A}(N)).$$

When M is a free module of finite type, ι_x is a bijection. Indeed, it suffices to regard $M = S(n)$, for which it is obvious.

If now M is any graded S-module of finite type, choose a resolution of M:

$$\ldots \to L^{q+1} \to L^q \to \ldots \to L^0 \to M \to 0$$

where L^q are free of finite type, and consider a complex C formed by $Hom_S(L^q, N)$. The cohomology groups of C are $\mathrm{Ext}^q_S(M, N)$; or else if we denote by B^q and Z^q the submodules of C^q formed respectively by the coboundaries and cocycles, we have the exact sequences:

$$0 \to Z^q \to C^q \to B^{q+1} \to 0$$

and

$$0 \to B^q \to Z^q \to \mathrm{Ext}^q_S(M,N) \to 0.$$

As the functor $\mathscr{A}(M)$ is exact, the sequences

$$0 \to Z^q_x \to C^q_x \to B^{q+1}_x \to 0$$

and

$$0 \to B^q_x \to Z^q_x \to \mathrm{Ext}^q_S(M,N)_S \to 0$$

are also exact.

But after preceding consideration C^q_x is isomorphic to $Hom_{\mathsf{O}_\mathsf{x}}(L^q_x, N_x)$; the $\mathrm{Ext}^q_S(M,N)_x$ are isomorphic to cohomology groups of a complex formed by the $Hom_{\mathsf{O}_\mathsf{x}}(L^q_x, N_x)$ and, because the L^q_x are clearly $\mathscr{O}_x$-free, we get back the definition of $\mathrm{Ext}^q_{\mathsf{O}_\mathsf{x}}(M_x, N_x)$, which shows (b). For $q = 0$ preceding considerations show that ι_x is bijection, so ι is an isomorphism, so (a) holds.

74 Vanishing of cohomology groups $\mathsf{H}^q(\mathsf{X}, \mathscr{F}(-\mathsf{n}))$ for $\mathsf{n} \to +\infty$

Theorem 1. *Let $\mathscr{F}$ be a coherent algebraic sheaf on X and let q be an integer ≥ 0. The following conditions are equivalent:*

(a) $H^q(X, \mathscr{F}(-n)) = 0$ *for n large enough.*

(b) $\mathrm{Ext}^{r-q}_{\mathsf{O}_\mathsf{x}}(\mathscr{F}_x, \mathscr{O}_x) = 0$ *for all $x \in X$.*

After Theorem 2 of n° 60, we can suppose that $\mathscr{F} = \mathscr{A}(M)$, where M is a graded S-module of finite type, and by the n° 64 $H^q(X, \mathscr{F}(-n))$ is isomorphic to $H^q(M(-n)) = B^q(m)_{-n}$, so condition (a) is equivalent to

$$T^q(M)_n = 0$$

for n large enough, that is to say $T^q(M) \in \mathscr{C}$. After Theorem 1 of n° 72 and the fact that $M^\star \in \mathscr{C}$ as M is of finite type, this last condition is equivalent to $\mathrm{Ext}^{r-q}_S(M, \Omega) \in \mathscr{C}$; as $\mathrm{Ext}^{r-q}_S(M, \Omega)$ is a S-module of finite type,

$$\mathrm{Ext}^{r-q}_S(M, \Omega) \in \mathscr{C}$$

is equivalent to $\mathrm{Ext}^{r-q}_S(M, \Omega)_x = 0$ for all $x \in X$, by Proposition 5 of n° 58. Finally the Proposition 1 shows that $\mathrm{Ext}^{r-q}_S(M, \Omega)_x = \mathrm{Ext}^{r-q}_{\mathsf{O}_\mathsf{x}}(M_x, \Omega_x)$ and as M_x is isomorphism to $\mathscr{F}_x$ and Ω_x is isomorphic to $\mathscr{O}(-r-1)_x$, so to $\mathscr{O}_x$, this completes the proof.

For announcing Theorem 2, we will need the notion of *dimension* of an $\mathscr{O}_x$-module. Recall ([6], Chap VI) that $\mathscr{O}_x$-module of finite type P is of dimension $\leq p$ if there is an exact sequence of $\mathscr{O}$-modules:

$$0 \to L_p \to L_{p-1} \to ... \to L_0 \to P \to 0,$$

where each L_p is free (this definition is equivalent to [6], because all projective $\mathscr{O}_x$-modules of finite type are free (cf [6], Chap VIII, Th. 6.1.').

All $\mathscr{O}_x$-modules of finite type are of dimension $\leq r$, by Hilbert's syzygy theorem. (cf. [6], Chap VIII, Th. 6.2').

Lemma 1. *Let P be an $\mathscr{O}_x$-module of finite type and let p be an integer ≥ 0. The following two conditions are equivalent:*

(i) *P is of dimension $\leq p$.*

(ii) $\mathrm{Ext}^m_{\mathsf{O_x}}(P, \mathscr{O}_x) = 0$ *for all $m > p$.*

It is clear that (i) implies (ii). We will show that (ii) implies (i) by induction decreasing on p. For $p \geq r$ the lemma is trivial, because (i) is always true. Now pass from $p+1$ to p; let N be any $\mathscr{O}_x$-module of finite type. We can find an exact sequence $0 \to R \to L \to N \to 0$, where L is free of finite type (because $\mathscr{O}_x$ is Noetherian). The exact sequence:

$$\mathrm{Ext}^{p+1}_{\mathsf{O_x}}(P, L) \to \mathrm{Ext}^{p+1}_{\mathsf{O_x}}(P, N) \to \mathrm{Ext}^{p+2}_{\mathsf{O_x}}(P, R)$$

shows that $\mathrm{Ext}^{p+1}_{\mathsf{O_x}}(P, N) = 0$, so we have $\mathrm{Ext}^{p+2}_{\mathsf{O_x}}(P, L) = 0$ by condition (ii), and $\mathrm{Ext}^{p+2}_{\mathsf{O_x}}(P, R) = 0$ as $\dim P \leq p+1$ by the induction hypothesis. As this property characterizes the modules of finite dimension $\leq p$, the lemma is proved.

By combining Lemma with Theorem 1 we obtain:

Theorem 2. *Let $\mathscr{F}$ be a coherent algebraic sheaf on X, and let p be an integer ≥ 0. The following two conditions are equivalent:*

(i) *$H^q(X, \mathscr{F}(-n)) = 0$ for all n large enough and $0 \leq q < p$.*

(ii) *For all $x \in X$ the $\mathscr{O}_x$-module $\mathscr{F}_x$ is of dimension $\leq r - p$.*

75 Nonsingular varieties

The following results play essential role in extension of the 'duality theorem' [15] to an arbitrary case.

Theorem 3. *Let V be a nonsingular subvariety of projective space $\mathbb{P}_r(K)$. Suppose that all irreducible components of V have the same dimension p. Let $\mathscr{F}$ be a coherent algebraic sheaf on V, such that for all $x \in V$, $\mathscr{F}_x$ is a free module over $\mathscr{O}_{x,V}$. Then we have $H^q(V, \mathscr{F}(-n)) = 0$ for all n large enough and $0 \leq q < p$.*

After Theorem 2, it remains to show that $\mathscr{O}_{x,V}$ considered as $\mathscr{O}_x$-module is of dimension $\leq r - p$. Denote by $g_x(V)$ the kernel of the canonical homomorphism $\epsilon : \mathscr{O}_x \to \mathscr{O}_{x,V}$; since the point x is simple over V, we know (cf. [18], th 1) that this ideal is generated by $r-p$ elements $f_1, ..., f_{r-p}$, and the theorem of Cohen-Macaulay (cf. [13], p. 53, prop 2) shows that we have

$$(f_1, ..., f_{i-1}) : f_i = (f_1, ..., f_{i-1}) \quad \text{for} \quad 1 \leq i \leq r-p.$$

Denote by L_q a free $\mathscr{O}_x$-module which admits a base of elements $e < i_1 ... i_q >$ corresponding to sequence $(i_1, ..., i_q)$ such that

$$1 \leq i_1 < l_2 < ... < i_q \leq r-p;$$

for $q = 0$, take $L_0 = \mathscr{O}_x$ and define:

$$d(e\langle i_1 ... i_q \rangle) = \sum_{j=1}^{q} (-1)^j f_{i,j} e\langle i_1, ... \hat{i}_j ... i_q \rangle$$

$$d(e\langle i \rangle) = f_i$$

After [6], Chap. VIII, prop 4.3, the sequence

$$0 \to L_{r-p} \xrightarrow{d} L_{r-p-1} \xrightarrow{d} ... \xrightarrow{d} L_0 \xrightarrow{\epsilon_x} \mathscr{O}_{x,V} \to 0$$

is exact, which shows that $\dim_{\mathsf{O_x}}(\mathscr{O}_{x,V}) \leq r-p$, QED.

Corollary. *We have $H^q(V, \mathscr{O}_V(-n)) = 0$ for n large enough and $0 \leq q < p$.*

Remark. The above proof applies more generally whenever the ideal $g_x(V)$ admits a system of $r-p$ generators, that is, if the variety V is a *local complete intersection* at all points.

76 Normal Varieties

Lemma 2. *Let M be a $\mathscr{O}_x$ module of finite type and let f be a noninvertible element of $\mathscr{O}_x$, such that the relation $fm = 0$ implies $m = 0$ if $m \in M$. Then the dimension of the $\mathscr{O}_x$-module M/fM is equal to the dimension of M increased by one.*

By assumption, we have an exact sequence $0 \to M \xrightarrow{\alpha} M \to M/fM \to 0$, where α is multiplication by f. If N is a $\mathscr{O}_x$-module of finite type, we have an exact sequence:

$$... \to \mathrm{Ext}^q_{\mathsf{O_x}}(M, N) \xrightarrow{\alpha} \mathrm{Ext}^q_{\mathsf{O_x}}(M, N) \to \mathrm{Ext}^{q+1}_{\mathsf{O_x}}(M/fM, N) \to \mathrm{Ext}^{q+1}_{\mathsf{O_x}}(M, N) \to ...$$

Denote by p the dimension of M. By taking $q = p+1$ in the preceding exact sequence, we see that $\mathrm{Ext}^{p+2}_{\mathsf{O_x}}(M/fM, N) = 0$, which (by [6], Chap. VI, 2) implies that $\dim(M/fM) \leq p+1$. On the other hand, since $\dim M = p$ we can choose N such that $\mathrm{Ext}^p_{\mathsf{O_x}}(M, N) \neq 0$; by taking $q = p$ in the above exact sequence, we see that
$Ext^{p+1}_{\mathsf{O_x}}(M/fM, N)$ can be identified with cokernel of

$$\mathrm{Ext}^p_{\mathsf{O_x}}(M, N) \xrightarrow{\alpha} \mathrm{Ext}^p_{\mathsf{O_x}}(M, N)'$$

as the last homomorphism is nothing else that multiplication by f and that f isn't invertible in the local ring $\mathscr{O}_x$. If follows from [6], Chap. VIII, prop. 5.1' that this cokernel is $\neq 0$, which shows that $\dim M/fM \geq p+1$ and finishes the proof.

We will now show a result, that is related with 'the Enriques-Severi lemma' of Zariski [19]:

Theorem 4. *Let V be an irreducible, normal subvariety of dimension ≥ 2, of projective space $\mathbb{P}_r(K)$. Let $\mathscr{F}$ be a coherent algebraic sheaf on V, such that for all $x \in V$, $\mathscr{F}_x$ is a free module over $\mathscr{O}_{x,V}$. Then we have $H^1(V, \mathscr{F}(-n)) = 0$ for n large enough.*

After Theorem 2, it remains to show that $\mathscr{O}_{x,V}$, considered as $\mathscr{O}_x$-module is of dimension $\leq r-2$. First choose an element $f \in \mathscr{O}_x$ such that $f(x) = 0$ and that the image of f in $\mathscr{O}_{x,V}$ is not zero; this is possible because $\dim V > 0$. As V is irreducible, $\mathscr{O}_{x,V}$ is an integral ring (domain), and we can apply Lemma 2 to the pair $(\mathscr{O}, f)$; we then have:

$$\dim \mathscr{O}_{x,V} = \dim \mathscr{O}_{x,V}/(f) - 1, \quad \text{with} \quad (f) = f\mathscr{O}_{x,V}.$$

As $\mathscr{O}_{x,V}$ is an integrally closed ring, all prime ideals $\mathfrak{p}^\alpha$ of the principal ideal (f) are minimal (cf. [12] p.136, or [9], n° 37), and none of them is equal to the maximal ideal $\mathfrak{m}$ of $\mathscr{O}_{x,V}$ (if not we would have $\dim V \leq 1$). So we can find an element $g \in \mathfrak{m}$, not belonging to any of $\mathfrak{p}^\alpha$; this element g is not divisible by 0 in the quotient ring $\mathscr{O}_{x,V}/(f)$; we denote by $\bar{g}$, a representation of g in $\mathscr{O}_x$. We see that we can apply Lemma to the pair $\mathscr{O}_{x,V}/(f), \bar{g})$; we then have:

$$\dim \mathscr{O}_{x,V}/(f) = \dim \mathscr{O}_{x,V}/(f,g) - 1.$$

But by Hilbert's syzygy theorem, we have $\dim \mathscr{O}_{x,V}/(f,g) \leq r$, so $\dim \mathscr{O}_{x,V} \leq r-1$ and $\dim \mathscr{O}_{x,V} \leq r-2$ QED.

Corollary. *We have $H^1(V, \mathscr{O}_V(-n)) = 0$ for n large enough.*

Remarks.

(1) The reasoning made before is classic in theory of syzygies. Cf. W. Gröbner, *Moderne Algebraische Geometrie*, 152.6 and 153.1.

(2) If the dimension of V is > 2, we can have $\dim \mathscr{O}_{x,V} = r-2$. This is in particular the case when V is a cone which hyperplane section W is a normal and irregular projective variety (i.e., $H^1(W, \mathscr{O}_W) \neq 0$).

77 Homological characterization of varieties k-times of first kind

Let M be a graded S-module of finite type. We show by a reasoning identical to that of Lemma 1:

Lemma 3 dim $\leq k$ *if and only if* $\mathrm{Ext}_S^q(M,S)=0$ *for* $q>k$.

As M is graded. we have $\mathrm{Ext}_S^q(M,\Omega)=\mathrm{Ext}_S^q(M,S)(-r-1)$, so the previous condition is equivalent to $\mathrm{Ext}_S^q(M,\Omega)=0$ for $q>k$. Given Theorem 1 of n° 72, we conclude:

Proposition 2.

(a) *For* $\dim M \leq r$ *it is necessary and sufficient that* $M_n \to H^0(M(n))$ *is injective for all* $n\in\mathbb{Z}$.

(b) *If* k *is an integer* ≥ 1, *for* $\dim M\leq r-k$ *it is necessary and sufficient that* $\alpha: M_n \to H^0(M(n))$ *is bijective for all* $n\in\mathbb{Z}$, *and that* $H^q(M(n))=0$ *for* $0<q<k$ *and all* $n\in\mathbb{Z}$.

Let V be a closed subvariety of $\mathbb{P}_r(K)$, and let $I(V)$ be an ideal if homogeneous polynomials, which are zero on V.

Denote $S(V)=S/I(V)$, this is a graded S-module whose associated sheaf is $\mathscr{O}_V$. We say[2] that V is a variety "k-times of first kind" of $\mathbb{P}_r(K)$ if the dimension of S-module $S(V)$ is $\leq r-k$. It is obvious that $\alpha: S(V)_n \to H^0(V,\mathscr{O}_V(n))$ is injective for all $n\in\mathbb{Z}$, so all varieties are 0-times of first kind. Using preceding proposition to $M=S(V)$, we obtain:

Proposition 3. *Let* k *be an integer* ≥ 1. *For a subvariety* V *to be a* k-*times of first kind, it is necessary and sufficient that the following conditions are satisfied for all* $n\in\mathbb{Z}$:

(i) $\alpha: S(V)_n \to H^0(V,\mathscr{O}_V(n))$ *is bijective.*

(ii) $H^q(V,\mathscr{O}_V(n))=0$ *for* $0<q<k$.

(The condition (i) can also be expressed by saying that linear series cut on V by forms of degree n is complete, which is well known.)

By comparing with Theorem 2 (or by direct reasoning), we obtain:

Corollary. *If* V *is* k-*times of first kind, we have* $H^q(V,\mathscr{O}_V)=0$ *for* $0<q<k$ *and, for all* $x\in V$, *the dimension of* $\mathscr{O}_x$-*module* $\mathscr{O}_{x,V}$ *is* $\leq r-k$.

[2] Cf. P. Dubreil, Sur la dimension des idéaux de polynômes, J . Math. Pures App., 15, 1936, p. 271-283. See also W . Gröbner, hloderne Algebraische Geometrie, §5.

If m is an integer ≥ 1, denote by φ_m the embedding of $\mathbb{P}_r(K)$ into a projective space of convenient dimension, given by the monomials of degree m (cf. [8], Chap. XVI, 6, or n° 52, proof of Lemma 2). So the preceding corollary admits following converse:

Proposition 4. *Let k be an integer ≥ 1, and let V be a connected and closed subvariety of $\mathbb{P}_r(K)$. Suppose that $H^q(V, \mathscr{O}_V) = 0$ for $0 < q < k$, and that for all $x \in V$ the dimension of $\mathscr{O}_x$-module $\mathscr{O}_{x,V}$ is $\leq r - k$.*
Then for all m large enough, $\varphi_m(V)$ is a subvariety k-times of first kind.

Because V is connected, we have $H^0(V, \mathscr{O}_V) = K$. So, if V is irreducible, it's evident (if not, $H^0(V, \mathscr{O}_V)$ contains a polynomial algebra and is not of finite dimension over K); if V is reducible, all elements $f \in H^0(V, \mathscr{O}_V)$ induce a constant on each of irreducible components of V, and this constants are the same, because of connectivity of V.

By the fact that $\dim \mathscr{O}_{x,V} \leq r - 1$, the algebraic dimension of each of irreducible components of V is at least equal to 1. So it follows that

$$H^0(V, \mathscr{O}_V(-n)) = 0$$

for all $n > 0$ (because if $f \in H^0(V, \mathscr{O}_V(-n))$ and $f \neq 0$, the $f^k g$ with $g \in S(V)_{nk}$ form a vector subspace of $H^0(V, \mathscr{O}_V)$ of dimension > 1).

That being said, denote by V_m the subvariety $\varphi_m(V)$; we obviously have:

$$\mathscr{O}_{V_m}(n) = \mathscr{O}_V(nm).$$

For m large enough the following conditions are satisfied:

(a) *$\alpha : S(V)_{nm} \to H^0(V, \mathscr{O}_V(nm))$ is bijective for all $n \geq 1$.*

This follows from Proposition 5 of n° 65.

(b) *$H^q(V, \mathscr{O}_V(mn)) = 0$ for $0 < q < k$ and for all $n \geq 1$.*

This follows from Proposition of n° 65.

(c) *$H^q(V, \mathscr{O}_V(nm)) = 0$ for $0 < q < k$ and for all $n \leq -1$.*

This follows from Theorem 2 of n° 74, and hypothesis made on $\mathscr{O}_{x,V}$.

On the other hand, we have $H^0(V, \mathscr{O}_V) = K$, $H^0(V, \mathscr{O}_V(nm)) = 0$ for all $n \leq -1$, and $H^q(V, \mathscr{O}_V) = 0$ for $0 < q < k$, by the hypothesis. It follows that V_m satisfies all the hypothesis of Proposition 3, QED.

Corollary. *Let k be an integer ≥ 1, and let V be a projective variety without singularities, of dimension $\geq k$. For V being birationally isomorphic to a subvariety k-times of first kind of a convenient projective space, it is necessary and sufficient that V is connected and that $H^q(V, \mathscr{O}_V) = 0$ for $0 < q < k$.*

The necessity is evident, by Proposition 3. To show sufficiency, it suffices to remark that $\mathscr{O}_{x,V}$ is of dimension $\leq r - k$ (cf. n° 75) and to apply the previous proposition.

78 Complete intersections

A subvariety V of dimension p of projective space $\mathbb{P}_r(K)$ is a *complete intersection* if the ideal $I(V)$ of polynomials zero at V admits a system of $r-p$ generators $P_1, ..., P_{r-p}$; in this case, all irreducible components of V have the dimension p, by the theorem of Macaulay (cf. [9], n° 17). It is known, that this variety is p-times of first kind, which implies that $H^q(V, \mathscr{O}_V(n)) = 0$ for $0 < q < p$, as we have just seen. We will determine $H^p(V, \mathscr{O}_V(n))$ as a function of degree $m_1, ..., m_{r-p}$ of homogeneous polynomials $P_1, ..., P_{r-p}$.

Let $S(V) = S/I(V)$ be a ring of projective coordinates of V. By theorem 1 of n° 72 all it is left, is to determine the S-module $\mathrm{Ext}_S^{r-p}(S(V), \Omega)$. We have a resolution, analogous to that of n° 75: we take L^q the graded free S-module, admitting for a base the elements $e\langle i_1, ..., i_q\rangle$, corresponding to sequences $(i_1, ..., i_q)$ such that $1 \leq i_1 < i_2 < ... < i_q \leq r-p$ and of degree $\sum_{j=1}^q m_j$; for L^0 we take S. We set:

$$d(e\langle i_1, ..., i_q\rangle) = \sum_{j=1}^{q} (-1)^i P_{i_j}\, e\langle i_1 ... \hat{i}_j ... i_q\rangle$$
$$d(e\langle i\rangle) = P_i.$$

The sequence $0 \to L^{r-p} \xrightarrow{d} ... \xrightarrow{d} L^0 \to S(V) \to 0$ is exact ([6], Chap. VIII, Prop. 4.3). It follows that the $\mathrm{Ext}_S^q(S(V), \Omega)$ are the cohomology groups of the complex formed by the $Hom_S(L^q, \Omega)$; but we can identify an element of $Hom_S(L^q, \Omega)_n$ with a system $f\langle i_1, ... i_q\rangle$, where the $f\langle i_1, ..., i_q\rangle$ are homogeneous polynomials of degree $m_{i_1} + ... + m_{i_q} + n - r - 1$; after this identification is made, the operator of coboundary is given by usual formula:

$$(df)\langle i_1 ... i_{q+1}\rangle = \sum_{j=1}^{q} (-1)^j P_{i_j}\, f\langle i_1 ... \hat{i}_j ... i_{q+1}\rangle.$$

The theorem of Macaulay implies that we are in conditions of [11], and we obtain that $\mathrm{Ext}_S^q(S(V), \Omega) = 0$ for $q \neq r-p$. On the other hand, $\mathrm{Ext}_S^{r-p}(S(V), \Omega)_n$ is isomorphic to a vector subspace of $S(V)$ formed by homogeneous elements of degree $N+n$, where $N = \sum_{i=1}^{r-p} m_i - r - 1$. Using Theorem 1 of n° 72 we obtain:

Proposition 5. *Let V be a complete intersection, defined by the homogeneous polynomials $P_1, ..., P_{r-p}$ of degrees $m_1, ..., m_{r-p}$.*

(a) The function $\alpha : S(V)_n \to H^0(V, \mathscr{O}_V(n))$ is bijective for all $n \in \mathbb{Z}$.

(b) $H^q(V, \mathscr{O}_V(n)) = 0$ for $0 < q < p$ and all $n \in \mathbb{Z}$.

(c) $H^q(V, \mathscr{O}_V(n))$ is isomorphic to a dual vector space to $H^0(V, \mathscr{O}_V(N-n))$, with $N = \sum_{i=1}^{r-p} m_i - r - 1$.

We see that in particular $H^p(V, \mathscr{O}_V)$ is zero if $N < 0$.

§6 The characteristic function and arithmetic genus

79 Euler-Poincare characteristic

Let V be a projective variety and $\mathscr{F}$ a coherent algebraic sheaf on V. Let

$$h^q(V, \mathscr{F}) = dim_K H^q(V, \mathscr{F}).$$

We have seen (n° 66, Theorem 1) that $h^q(V, \mathscr{F})$ are *finite* for all integer q and zero for $q > dim V$. So we can define an integer $\chi(V, \mathscr{F})$ by:

$$\chi(V, \mathscr{F}) = \sum_{q=0}^{\infty} (-1)^q \; h^q(V, \mathscr{F}).$$

This is the Euler-Poincare characteristic of V with coefficient in $\mathscr{F}$.

Lemma 1. *Let $0 \to L_1 \to ... \to L_p \to 0$ be an exact sequence, with L_i being finite dimensional vector spaces over K, and homomorphisms $L_i \to L_{i+1}$ being K-linear. Then we have:*

$$\sum_{q=1}^{p} (-1)^q \; dim_K L_q \; = \; 0.$$

We proceed by induction on p. The lemma is evident if $p \leq 3$. If L'_{p-1} is the kernel of $L_{p-1} \to L_p$, we have two exact sequences:

$$0 \to L_1 \to ... \to L'_{p-1} \to 0$$

$$0 \to L'_{p-1} \to L_{p-1} \to L_p \to 0.$$

Applying induction hypothesis to each sequence, we see that $\sum_{q=1}^{p-2} (-1)^q dim L_q + (-1)^{p-1} dim L'_{p-1} = 0$, and

$$dim L'_{p-1} - dim L_{p-1} + dim L_p = 0,$$

which proves the lemma.

Proposition 1. *Let $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ be an exact sequence of coherent algebraic sheaves on a projective variety V, with homomorphisms $\mathscr{A} \to \mathscr{B}$ and $\mathscr{B} \to \mathscr{C}$ being K-linear. Then we have:*

$$\chi(V, \mathscr{B}) = \chi(V, \mathscr{A}) + \chi(V, \mathscr{C}).$$

By Corollary 2 of Theorem 5 of n° 47, we have an exact sequence of cohomology:

$$... \to H^q(V, \mathscr{B}) \to H^q(V, \mathscr{C}) \to H^{q+1}(V, \mathscr{A}) \to H^{q+1}(V, \mathscr{B}) \to ...$$

Applying Lemma to this exact sequence of vector spaces we obtain the Proposition.

Proposition 2. *Let $0 \to \mathscr{F}_1 \to ... \to \mathscr{F}_p \to 0$ be an exact sequence of coherent algebraic sheaves on a projective variety V, with homomorphisms $\mathscr{F}_i \to \mathscr{F}_{i+1}$ being algebraic. Then we have:*

$$\sum_{q=1}^{p} (-1)^q \ \chi(V, \mathscr{F}_q) = 0.$$

We proceed by induction on p. The proposition is a particular case of Proposition 1 if $p \leq 3$. If we define $\mathscr{F}'_{p-1}$ to be the kernel of $\mathscr{F}_{p-1} \to \mathscr{F}_p$, the sheaf $\mathscr{F}'_{p-1}$ is coherent algebraic because $\mathscr{F}_{p-1} \to \mathscr{F}_p$ is an algebraic homomorphism. So we can applicate the induction hypothesis to two exact sequences

$$0 \to \mathscr{F}_1 \to ... \to \mathscr{F}'_{p-1} \to 0$$

$$0 \to \mathscr{F}'_{p-1} \to \mathscr{F}_{p-1} \to \mathscr{F}_p,$$

and the Proposition follows.

80 Relation with characteristic function of a graded S-module

Let $\mathscr{F}$ be a coherent algebraic sheaf on the space $\mathbb{P}_r(K)$. We write $\chi(\mathscr{F})$ instead of $\chi(\mathbb{P}_r(K), \mathscr{F})$. We have:

Proposition 3. *$\chi(\mathscr{F}(n))$ is a polynomial of n of degree $\leq r$.*

By Theorem 2 of n° 60, there exists a graded S-module M of finite type, such that $\mathscr{A}(M)$ is isomorphic to $\mathscr{F}$. Applying the Hilbert's syzygy theorem to M we obtain an exact sequence of graded S-modules:

$$0 \to L^{r+1} \to ... \to L^0 \to M \to 0,$$

where L^q are free of finite type. Applying the functor $\mathscr{A}$ to this sequence, we obtain an exact sequence of sheaves:

$$0 \to \mathscr{L}^{r+1} \to ... \to \mathscr{L}^0 \to \mathscr{F} \to 0,$$

where each $\mathscr{L}^q$ is isomorphic to a finite direct sum of shaves $\mathscr{O}(n_i)$. The proposition 2 implies that $\chi(\mathscr{F}(n))$ is equal to an alternating sum of $\chi(\mathscr{L}^0(n))$, which brings us to case of the sheaf $\mathscr{O}(n_i)$. Now it follows from n° 62 that we have $\chi(\mathscr{O}(n)) = \binom{n+r}{r}$, which is a polynomial on n of the degree $\leq r$. This implies the Proposition.

Proposition 4. *Let M be a graded S-module satisfying condition (TF), and let $\mathscr{F} = \mathscr{A}(M)$. For all n large enough, we have $\chi(\mathscr{F}(n)) = dim_K M_n$.*

We know (by n° 65) that for n large enough, the homomorphism $\alpha : M_n \to H^0(X, \mathscr{F}(n))$ is bijective, and $H^q(X, \mathscr{F}(n)) = 0$ for $q > 0$. So we have:

$$\chi(\mathscr{F}(n)) = h^0(X, \mathscr{F}(n)) = dim_K M_n.$$

We use a well known fact, that $dim_K M_n$ is a polynomial of n for n large enough. This polynomial, which we denote by P_M is called the *characteristic function* of M. For all $n \in \mathbb{Z}$ we have $P_M(n) = \chi(\mathscr{F}(n))$, and in particular for $n = 0$, we see that the *constant term of* P_M *is equal to* $\chi(\mathscr{F})$.

Apply this to $M = S/I(V)$, $I(V)$ being a homogeneous ideal of S of polynomials which are zero on a closed subvariety V of $\mathbb{P}_r(K)$. The constant term of P_M is called in this case the *arithmetic genus* of V (cf. [19]). Since on the other hand we have $\mathscr{A}(M) = \mathscr{O}_V$, we obtain:

Proposition 5. *The arithmetic genus of a projective variety V is equal to*

$$\chi(V, \mathscr{O}_V) = \sum_{q=0}^{\infty} (-1)^q dim_K H^q(V, \mathscr{O}_V).$$

Remarks.

(1) *The preceding Proposition makes evident the fact, that the arithmetic genus is independent of an embedding of V into a projective space, since it's true for $H^q(V, \mathscr{O})$.*

(2) *The virtual arithmetic genus (defined by Zariski in [19]) can also be reduced to Euler-Poincare characteristic. We return to this question later, by Riemann-Roch theorem.*

(3) *For the reason of convenience, we have adopted the definition of arithmetic genus different from the classical one (cf. [19]). If all irreducible components of V have the same dimension p, two definitions are related by the following formula: $\chi(V, \mathscr{O}_V) = 1 + (-1)^p p_a(V)$.*

81 The degree of the characteristic function

If $\mathscr{F}$ is a coherent algebraic sheaf on an algebraic variety V, we call the support of $\mathscr{F}$, and denote by $Supp(\mathscr{F})$, the set of points $x \in V$ such that $\mathscr{F}_x \neq 0$. By the fact that $\mathscr{F}$ is a sheaf of finite type, this set is closed. If we have $\mathscr{F}_x = 0$, the zero section generates $\mathscr{F}_x$, then also $\mathscr{F}_y$ for y in neighborhood of x (n° 12, Proposition 1), which means that the complement of $Supp(\mathscr{F})$ is open.

Let M be a graded S-module of finite type, and let $\mathscr{F} = \mathscr{A}(M)$ be a sheaf defined by M on $\mathbb{P}_r(K) = X$. We can determine $Supp(\mathscr{F})$ from M in the following manner:

Let $0 = \bigcap_{\alpha} M^{\alpha}$ be a decomposition of 0 as an intersection of homogeneous primary submodules M^{α} of M. M^{α} correspond to homogeneous primary ideals $\mathfrak{p}^{\alpha}$ (cf. [12], Chap. IV). We suppose that this decomposition is 'the shortest possible', i.e. that non of M^{α} is contained in an intersection of others. For all $x \in X$, each $\mathfrak{p}$ defines a primary ideal $\mathfrak{p}_x^{\alpha}$ of a local ring $\mathscr{O}_x$, and we have $\mathfrak{p}_x^{\alpha} = \mathscr{O}$ if and only if x is not an element of a variety V^{α} defined by an ideal $\mathfrak{p}^{\alpha}$. We have also $0 = \bigcap_{\alpha} M_x^{\alpha}$ in M_x, and we verify easily that we thereby obtain a primary decomposition of 0 in M_x. The M_x^{α} correspond to primary ideals $\mathfrak{p}_x^{\alpha}$; if $x \notin V^{\alpha}$, we have $M_x^{\alpha} = M_x$, and if we restrict ourself to consider M_x^{α} such that $x \in V^{\alpha}$, we obtain 'the shortest possible decomposition' (cf. [12], Chap IV, th 4.). We conclude that $M_x \neq 0$ if and only if x is an element of V^{α}, thus $Supp(\mathscr{F}) = \bigcup_{\alpha} V^{\alpha}$.

Proposition 6. *If $\mathscr{F}$ is a coherent algebraic sheaf on $\mathbb{P}_r(K)$, the degree of $\chi(\mathscr{F}(n))$ is equal to the dimension of $Supp(\mathscr{F})$.*

We proceed by induction on r. The case $r = 0$ is trivial. We can suppose that $\mathscr{F} = \mathscr{A}(M)$, where M is a graded S-module of finite type. Using notation introduced below, we have to show that $\chi(\mathscr{F}(n))$ is an polynomial of degree $q = Sup\ dim\ V^{\alpha}$.

Let t be a linear homogeneous form, which do not appear in any of proper prime ideals $\mathfrak{p}^{\alpha}$. Such a form exists because the field K is infinite. Let E be a hyperplane of X with equation $t = 0$. Consider the exact sequence:

$$0 \to \mathscr{O}(-1) \to \mathscr{O} \to \mathscr{O}_K \to 0,$$

where $\mathscr{O} \to \mathscr{O}_E$ is a restriction homomorphism, while $\mathscr{O}(-1) \to \mathscr{O}$ is a homomorphism $f \mapsto tf$. Applying tensor product with $\mathscr{F}$, we obtain an exact sequence:

$$\mathscr{F}(-1) \to \mathscr{F} \to \mathscr{F}_E \to 0, \quad \text{with} \quad \mathscr{F}_E = \mathscr{F} \otimes_{\mathscr{O}} \mathscr{O}_E.$$

On U_i, we can identify $\mathscr{F}(-1)$ with $\mathscr{F}$, and this identification transforms the homomorphism $\mathscr{F}(-1) \to \mathscr{F}$ defined above to the multiplication by t/t_i. Because t was chosen outside $\mathfrak{p}^{\alpha}$, t/t_i don't belong to any prime ideal of $M_x = \mathscr{F}_x$ if $x \in U_i$, and the preceding homomorphism is injective (cf. [12], p. 122, th. 7, b")). So we have an exact sequence:

$$0 \to \mathscr{F}(-1) \to F \to \mathscr{F}_E \to 0,$$

from which, for all $n \in \mathbb{Z}$ the exact sequence:

$$0 \to \mathscr{F}(n-1) \to \mathscr{F}(n) \to \mathscr{F}_K(n) \to 0.$$

Applying Proposition 1, we see that:

$$\chi(\mathscr{F}(n)) - \chi(\mathscr{F}(n-1)) = \chi(\mathscr{F}_E(n)).$$

But the sheaf $\mathscr{F}_E$ is a coherent sheaf of $\mathscr{O}_E$-modules, which means that it is a coherent algebraic sheaf on E, which is a projective space of dimension $r-1$. Moreover $\mathscr{F}_{x,E} = 0$ means that the endomorphism of $\mathscr{F}_x$ defined by multiplication by t/t_i is surjective, which leads to $F_x = 0$ (cf. [6], Chap VIII, prop 5.1'). It follows that $Supp(\mathscr{F}_K) = E \cap Supp(\mathscr{F})$, and because E does not contain any of varieties V^α, if follows by a known fact, that the dimension of $Supp(\mathscr{F}_E)$ is equal to $q-1$. By the induction hypothesis $\chi(\mathscr{F}_E(n))$ is a polynomial of degree $q-1$. As this difference is prime to the function $\chi(\mathscr{F}(n))$, the latter is a polynomial of degree q.

Remarks.

(1) Proposition 6 was well known for $\mathscr{F} = \mathscr{O}/\mathscr{I}$, $\mathscr{I}$ being a coherent sheaf of ideals. Cf. [9] n° 24.

(2) The above proof does not use Proposition 3 and shows it once again.

Bibliography

[1] N. Bourbaki, *Théorie des Ensembles.* Paris (Hermann).

[2] H. Cartan, *Séminaire E.N.S.*, 1950-1951.

[3] H. Cartan, *Séminaire E.N.S.*, 1951-1952.

[4] H. Cartan, *Séminaire E.N.S.*, 1953-1954.

[5] H. Cartan, *Variétes analytiques complexes et cohomologie.* Colloque de Bruxelles, (1953), p. 41-55.

[6] H. Cartan and S. Eilenberg, *Homological Algebra*. Princeton Math. Ser., n° 19.

[7] S. Eilenberg and N. E. Steenrod, *Foundations of Algebraic Topology*, Princeton Math. Ser., n° 15.

[8] W. V. D. Hodge and D. Pedoe, *Methods of Algebraic Geometry*, Cambridge Univ. Press.

[9] W. Krull, *Idealtheorie*, Ergenbisse IV-3. Berlin (Springer).

[10] J. Leray, *L'anneau spectral et l'anneau filtré d'homologie d'un espace localement compact et d'une application continue*, J. Math. Pures App., 29, (1950), p. 1-139.

[11] G. de Rham, *Sur la division de formes et de courants par une forme linéaire*, Comment. Math. Helv., 28, (1954), p. 346-352.

[12] P. Samuel, *Commutative Algebra* (Notes by D. Hertzig), Cornell Univ., 1953.

[13] P. Samuel, *Algèbre locale*, Mém. Sci. Math., CXXIII, Paris, 1953.

[14] J-P. Serre, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math., 58, (1953), p. 258-294.

[15] J-P. Serre, *Un théorème de dualité*, Comment. Math. Helv., 29, (1955), p. 9-26.

[16] A. Weil, *Foundations of Algebraic Geometry*, Colloq. XXIX.

[17] A. Weil, *Fibre-spaces in Algebraic Geometry* (Notes by A. Wallace), Chicago Univ., 1952.

[18] O. Zariski, *The concept of a simple point of an abstract algebraic variety*, Trans. Amer. Math. Soc., 62, (1947), p. 1-52.

[19] O. Zariski, *Complete linear systems on normal varieties and a generalization of a lemma of Enriques-Severi*, Ann. of Math., 55, (1952), p. 552-592.