UNIVERSITY of ROCHESTER

MTH150/A: Discrete Mathematics

Midterm Exam 1 Solutions

October 23, 2009

Name (please print legibly): ________________________

University ID Number: ________________________

Andrew Ledoan MWF 11:00 am –11:50 am

- Do all 10 problems in the space provided and explain your work carefully. The quality of your writeup counts. Each problem counts 10 points.
- Please show all your work. You may use the backs of pages if necessary. A correct answer with no work shown will not receive full credit. Please label and circle your final answers.
- You are responsible for checking that this exam has all 11 pages. Please tell us immediately if your exam is missing a page. Missing pages will not contribute to your total score.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. (10 points) The following

$$[p \wedge (p \to q) \wedge r] \to [(p \vee q) \to r]$$

is a valid argument. Establish its validity by means of a truth table. Determine which rows of the table are crucial for assessing the validity of the argument and which rows can be ignored.

Solution. We have

p	q	r	$p \to q$	$p \vee q$	$(p \vee q) \to r$
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1

The validity of the argument follows from the results in the last row. The first seven rows may be ignored.

2. (10 points) Prove using the contrapositive: If the product of two integers x and y is odd, then both x and y are odd.

Solution. Suppose that x and y are not odd and are therefore even. Then we can find integers m and n such that $x = 2m$ and $y = 2n$. Then $xy = 4mn = 2(2mn)$ which is even. Thus if x and y are not odd, their product is not odd.

3. (10 points) Let A_1, A_2, and A_3 be subsets of $\mathbb{R}$ defined as follows:

$$A_1 = \{x : -10 < x < 10\},$$
$$A_2 = \{x : 0 < x < 20\},$$
$$A_3 = \{x : 5 < x < 25\}.$$

Find each of the the following sets. (Hint: Rewrite the sets in parts (c) and (d).)

(a) $A_1 \cup A_2 \cup A_3 = \{x : -10 < x < 25\}$

(b) $A_1 \cap A_2 \cap A_3 = \{x : 5 < x < 10\}$

(c) $(A_1 \cap A_2) \cup A_3 = (A_1 \cup A_3) \cap (A_2 \cup A_3) = \{x : 0 < x < 25\}$

(d) $A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3) = \{x : 0 < x < 10\}$

4. (10 points) If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find the function $g \circ f$ and its domain.

Solution. We find that

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2 - \sqrt{x}}.$$

For $\sqrt{x}$ to be defined we must have $x \geq 0$. For $\sqrt{2 - \sqrt{x}}$ to be defined we must have $2 - \sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

5. (10 points) Consider the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 3x + 7$ for all $x \in \mathbb{R}$. Show that f is one-to-one.

Solution. For all $x_1, x_2 \in \mathbb{R}$, we find that

$$f(x_1) = f(x_2)$$

implies that

$$3x_1 + 7 = 3x_2 + 7,$$

which, in turn, implies that

$$3x_1 = 3x_2,$$

and so

$$x_1 = x_2.$$

Hence the given function f is one-to-one.

6. (10 points)

(a) Find the inverse of the function $f : \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^{2x+5}$.

Solution. Let

$$y = e^{2x+5}.$$

Taking the natural logarithms of both sides, we find that

$$\ln y = \ln e^{2x+5} = (2x+5)\ln e = 2x+5.$$

Solving for x, we find that

$$x = \frac{1}{2}(\ln y - 5).$$

Then interchanging the roles of x and y, we obtain

$$y = \frac{1}{2}(\ln x - 5).$$

Hence

$$f^{-1}(x) = \frac{1}{2}(\ln x - 5).$$

(b) Show that $f \circ f^{-1} = 1_{\mathbb{R}^+}$. In other words, verify that $(f \circ f^{-1})(x) = x$.

Solution. For $x \in \mathbb{R}^+$, we have

$$(f \circ f^{-1})(x) = f\left(\frac{1}{2}(\ln x - 5)\right) = e^{2[(\ln x - 5)/2]+5} = e^{\ln x} = x.$$

7. (10 points) Evaluate the double sum

$$\sum_{i=-1}^{1}\sum_{j=0}^{2}(2i+3j).$$

Solution. We write that

$$\begin{aligned}
\sum_{i=-1}^{1}\sum_{j=0}^{2}(2i+3j) &= \sum_{i=-1}^{1}\left[\sum_{j=0}^{2}(2i+3j)\right] \\
&= \sum_{i=-1}^{1}[(2i+3\cdot 0)+(2i+3\cdot 1)+(2i+3\cdot 2)] \\
&= \sum_{i=-1}^{1}(6i+9) \\
&= [6\cdot(-1)+9]+(6\cdot 0+9)+(6\cdot 1+9) \\
&= 27.
\end{aligned}$$

8. (10 points) Find the best "big-Oh" estimate for each of the following functions $f : \mathbb{Z}^+ \to \mathbb{R}$.

(a) $f(n) = 7 + \cos\left(\frac{1}{n}\right) = O(1)$.

(b) $f(n) = n^2 + (n-1)^3 = O(n^3)$

(c) $f(n) = 5n^2 + 3n\log n = O(n^2)$

(d) $f(n) = \frac{(n+1)(n+2)}{n+3} = O(n)$

9. (10 points) Give a "big-Oh" estimate for $f(n) = 5n \log n! + (n^2 + 11) \log n$.

Solution. We have

$$n! = O(n^n),$$

so that

$$\log n! = O(n \log n).$$

Thus

$$5n \log n! = O(n^2 \log n).$$

Since

$$(n^2 + 11) \log n = O(n^2 \log n),$$

we have

$$f(n) = 5n \log n! + (n^2 + 11) \log n = O(n^2 \log n).$$

10. (10 points) If $a, b \in \mathbb{Z}^+$, and both are odd, prove that $2 \mid (a^2 + b^2)$ but $4 \nmid (a^2 + b^2)$.

Solution. Since a is odd, we have $a = 2l + 1$ for some $l \in \mathbb{Z}$. Since b is odd, we have $b = 2k + 1$ for some $k \in \mathbb{Z}$. Then

$$\begin{aligned} a^2 + b^2 &= (2l + 1)^2 + (2k + 1)^2 \\ &= 4l^2 + 4l + 1 + 4k^2 + 4k + 1 \\ &= 4l^2 + 4l + 4k^2 + 4k + 2 \\ &= 2(2l^2 + 2l + 2k^2 + 2k + 1), \end{aligned}$$

where $2l^2 + 2l + 2k^2 + 2k + 1 \in \mathbb{Z}$. It is clear that $2 \mid (a^2 + b^2)$ but $4 \nmid (a^2 + b^2)$.