Samelson products

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Abstract

This is an expository paper on Samelson products. We shall see that the theory of Samelson products depends on three crucial properties of homotopy uniqueness.

Contents

1 Samelson products, commutators and Lie identities

1. definition of external samelson products via commutators [9, 11, 12, 7]: cofibration sequence says that it is the unique homotopy class which factors the cimmutator through the smash.

George Whitehead [11] had the fundamental insight that the Lie identities for Samelson products are a consequence of certain analogous Lie identities for groups. In this section we present these Lie identities for groups. The treatment here is heavily influenced by that found in Serre's book. [10]

Let x, y be elements of a group G and define

1) The congugate homomorphisms are $x^y = y^{-1}xy$. Recall that $(xy)^z = x^z y^z$ and $(x^y)^{y^{-1}} = x$.

2) The commutators are $[x,y] = xyx^{-1}y^{-1}$. Thus, $[x,y]^z = [x^z, y^z]$.

The Lie identities in groups are the following formulas:

Proposition 1.5.1: For elements x, y, z in a group G,
1) exponentiation modulo a commutator:

$$x^y = x[x^{-1}, y^{-1}]$$

2) inverse of a commutator:

$$[x,y]^{-1} = [y,x], \qquad [x^{-1}, y] = [y,x]^x$$

3) commutativity modulo commutators:

$$xy = [x,y]yx$$

4) bilinearity modulo commutators:

$$[x, yz] = [x,y]\ [x,z]^{(y^{-1})}, \qquad [xy, z] = [y,z]^{(x^{-1})}\ [x,z]$$

5) Jacobi identity modulo commutators

$$[x^{(y^{-1})}, [z,y]]\ [y^{(z^{-1})}, [x,z]]\ [z^{(x^{-1})}, [y,x]] = 1$$

It may be difficult to discover some of the above formulas but there can be no doubt that they are straightforward to prove. Merely write them in the form $c = 1$ and reduce the word c to the identity via successive applications of
1) $wdw^{-1} = 1$ if and only if $d = 1$.
2) $ww^{-1} = 1$.
This amounts to reducing a word in a free group to the identity.
the lie identities hold in maps to groups, therefore 0) addition is defined by multiplication, negation is defined by inverse 1) addition is commutative if the domain is an abelian co H space. 2) bilinearity of samelson products hold on the smash of abelian co-H spaces. 3) anticommutativity holds on the smash 4) the jacobi identity holds on the triple smash

2 Internal Samelson products and the coproduct

definition of the internal samelson product. existence, uniqueness, commutativity, and associativity

1)spheres: the smash product subgroups [1, 8], coabelian spaces and nilpotence length, the commutator identities [10], the lie identities [4]

2)moore spaces, the splitting of the smash, the 2 primary cases, the odd primary cases, commutativity and associativity, the lie identities, the prime 3, the failure of the jacobi identity

3 Reduction and Bockstein maps

compatibility of these maps with the samelson products
the lie identities and reduction and bockstein maps
samelson products and the bockstein spectral sequence

4 Relative Samelson products

the normal subgroup analog, the existence of the relative samelson product for the loops on a fibration sequence. the universal model for the relative samelson product. [5, 6] existence of a unique universal choice for relative samelson products implies the lie identities are satisfied, jacobi identities are valid if the prime is greater than 3.

5 Samelson products over the loops on an H-space

the normal subgroup with abelian quotient analog. universal models for samelson products over the loops on an h-space. [5, 6] the choice depends on the multiplication of the base h-space. the lie identities are satisfied with a twist which goes away if the multiplication on the base is homotopy commutative. the jacobi identities are satisfied if the prime is greater than 3.

6 Homotopy commutative structures at odd primes

localized at an odd prime, h space structures can be averaged to be homotopy commutative.

7 Application to higher torsion in homotopy groups

higher torsion in moore spaces, [3] in the fibre of the pinch map, and in the fibre of the mod p pinch map [2]

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