Contemporary Mathematics

Odd primary exponents of Moore spaces

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Dedicated to the memory of John Coleman Moore

Contents

1. Introduction

If p is a prime and A is an abelian group, then A is said to have exponent p^n at the prime p if p^n annihilates the p torsion of A and it is the least such power. A topological space X has exponent p^n if if this is the least power which annihilates the p torsion in all the homotopy groups of X.

The proof of the exact result for the homotopy exponent of an odd primary Moore space is one which always made me feel uneasy. You know you are on dangerous ground when you have trouble recalling why some of things you have written are true. It was with some trepidation that I dealt with Brayton Gray's questions about this work. Some serious mistakes were found. But they were not fatal, except for the 3-primary case. Modulo the 3-primary case, the proof has survived this test and it has even improved under Brayton's insightful questioning.

2000 *Mathematics Subject Classification.* Primary 55-02, 55Q40,55Q15; Secondary 55R05.
Key words and phrases. Samelson products, homotopy groups, exponents.

His questioning has led to a simplification and generalization of the main technical result which I call the semi-splitting. I am very grateful to Brayton for bringing this about.

In fact, Brayton's suggestion of an improved semi-splitting has enabled me to take advantage of a last gift to me from the late John Moore. I was sorting through his old files of mathematical reprints when I found an old unpublished reprint of mine which I had completely forgotten about. The combination of the main idea of that reprint with the improved semi-splitting gives an upper bound for the homotopy exponent of an odd primary Moore space which is within one factor of p of the best possible. It applies even to the 3-primary case.

In summary, this paper will show that, when p is a prime greater than 3, p^{r+1} is the best possible homotopy exponent of a mod p^r Moore space of dimension at least 3. But p^{r+2} is an upper bound in the case of all odd primes.

Norman Steenrod once said to me: "If you are having trouble explaining something, it could be that you don't understand it well enough." I hope this paper demonstrates that this is no longer true about this work.

Let us begin by reviewing some definitions and theorems.

An abelian group A is said to have exponent $\leq p^n$ at a prime p if p^n annihilates the $p-$ torsion in A. It is said to have exponent exactly p^n if this is the least such power. A topological space X is said to have homotopy exponent $\leq p^n$, respectively, homotopy exponent exactly p^n, if the direct sum of all of its homotopy groups, equivalently the localiization of these homotopy groups at p, has this exponent.

For homotopy associative $H-$ spaces X we say that X has geometric exponent $\leq p^n$ if the power map

$$p^n : X \to X, \quad x \mapsto x^n$$

is null homotopic. It has geometric exponent exactly p^n if this is the least such null homotopic power map. It is clear that geometric exponents imply that the homotopy groups are $p-$torsion with the same or lesser homotopy exponents.

Note that, if X has a geometric exponent, then its loop space ΩX has the same or lesser geometric exponent.

The reverse implication is not true. In fact, it is possible that ΩX has a geometric exponent when an $H-$space X has none. This means that it might be possible that a space X has a homotopy exponent but no iterated loop space $\Omega^k X$ has any geometric exponent.

Hence, given any space X we can ask the following questions at a prime p:

1. Does X have a homotopy exponent at a prime p and, if so, what is the least such.

2. Does any iterated loop space $\Omega^k X$ or a localization at p have a geometric exponent. If so what is the least such power? And, do the geometric and homotopy exponents coincide.

These questions are of particular interest in the cases of spheres, Moore spaces, and their localizations. There are substantial differences between the prime 2 and the odd primes.

The first such result is due to James [**9**] who showed that the odd dimensional spheres S^{2n+1} have 2-primary homotopy exponents $\leq 4^n$. For $n = 1$, the $3-$dimensional sphere, this result is best possible, that is, there are elements of order 4 in the homotopy of S^3 but no elements of order 8. In general, James showed that the 2-primary homotopy exponent increases at most by a factor of 4 as one passes from S^{2n-1} to S^{2n+1}. But, already for $n = 2$, this result is not best possible. Selick [**22**] has shown that the 2-primary homotopy exponent increases by at most a factor of 2 as one passes from S^{4n-1} to S^{4n+1}. The 5−dimensional sphere has 2-primary homotopy exponent 8.

There is a conjecture of Barratt and Mahowald that the exact 2-primary exponent for S^{2n+1} is given by λ times the exact exponent for S^{2n-1} where λ is either 2, 4, 2,or 1, depending on whether n is congruent to 0, 1, 2, or 3 modulo 4, respectively. For example, the 2-primary exponents for the homotopy groups of $S^1, S^3, S^5, S^7, S^9, S^{11}, \ldots$ would be $1, 4, 8, 8, 16, 64, \ldots$. This result is not known to be true although Mahowald by constructing homotopy classes had shown that, if true, it is best possible. [**11**]

These results were given geometric form in a Princeton course by Moore who showed that, when the $2n+1$ connected cover $S^{2n+1} < 2n+1 >$ is localized at 2, then the 4^n power map is null on the iterated loop space $\Omega^{2n+1}(S^{2n+1} < 2n+1 >)$. [**23, 18**]

All of the above 2-primary results were proved by study of Hopf invariants and the EHP sequence. Toda studied Hopf invariants and introduced a new Hopf invariant which enabled him to prove the odd primary analog of the results of James: The spheres S^{2n+1} have homotopy exponent $\leq p^{2n}$ at an odd prime p. [**26, 27, 28**] Once again, Moore gave these results geometric form by showing that, localized at an odd prime p, the iterated loop space $\Omega^{2n+1}(S^{2n+1} < 2n+1 >)$ has a null homotopic p^{2n} power map.

Moore's Princeton course led almost immediately to improvements in the odd primary exponents of spheres, in fact to the best possible results. First, Selick [**21**] in his thesis proved Barratt's conjecture that the homotopy exponent of the 3-dimensional sphere is p if p is an odd prime. Second, Cohen, Moore, and Neisendorfer [**4, 3**] generalized Selick's result by showing that the homotopy exponent of S^{2n+1} is p^n at all primes greater than 3. Later, Neisendorfer [**14**] was able to show that this was also true for the prime 3.

Gray [**7**] had already shown that the above results were best possible.

The odd primary results have the geometric form that the iterated loop space $\Omega^{2n-1}(S^{2n+1} < 2n+1 >)$ has a null homotopic p^n power map localized at an odd prime p. [**18**] In fact, this result is best possible in terms of the number of loops since $\Omega^{2n-2}(S^{2n+1} < 2n+1 >)$, no matter how large the power of p, has no null homotopic power maps localized at any prime p. This last result is a consequence of the localization due to Bousfield and Dror-Farjoun.

Consider the special case of the above localization in which the map $M(Z[\frac{1}{p}], 1) \vee BZ/pZ \to *$ is made into an equivalence. The effect of the Moore space $M(Z[\frac{1}{p}], 1)$ is to insure that these localizations are complete and the effect of the classifying space BZ/pZ is to insure that completions of finite complexes are local but connected covers of them are usually not. [**17, 18**] This localization preserves products and has the following property: Suppose Y is any connected cover of X where X is either a finite complex or an iterated loop space of a finite complex and suppose that X is simply connected and has at most torsion in $\pi_2(X)$. Then this localization $L(Y)$ has the homotopy type of the p−completion $\hat{X}_p$.

Applying L to

$$Y = \Omega^{2n-2}(S^{2n+1} < 2n+1 >) = (\Omega^{2n-2}S^{2n+1}) < 3 >$$

shows that $L(Y) \simeq \hat{X}_p =$ the p-completion of $\Omega^{2n-2}(S^{2n+1})$. Since L preservers products this implies that, if Y had a null homotopic power map, the same would be true for $\hat{X}_p$. But $\pi_3(\hat{X}_p) = \hat{Z}_p$ which is torsion free. Hence, neither Y nor $\hat{X}_p$ can have a null homotopic power map.

Scholium: Joe Roitberg coined the phrase $1\frac{1}{2}$ connected for a space which is simply connected and has a torsion second homotopy group. The localization functor does not produce the same results when we apply it to a connected cover of a space which does not have a torsion second homotopy group, for example, to a connected cover of the $2n-1$ fold loops on a $2n+1$ dimensional sphere, or even more simply, to a connected cover of S^2. Note that S^2 has the 3-connected cover $S^2 < 3 >= S^3$. Thus, $L(S^2 < 3 >) = \hat{S^3}_p \neq \hat{S^2}_p$. In these cases, $\pi_2 = Z$ is not torsion, that is, neither the above iterated loop space nor S^2 is $1\frac{1}{2}$ connected. This localization property is based on a lemma of Zabodsky and on Miller's theorem, that is, on the Sullivan conjecture. [**12**] It is a remarkable coincidence that Miller's theorem gives exactly the range of applicability to prove that the $2n-2$ fold loop space has no geometric exponent at any prime.

The early work on exponents of homotopy groups of spheres focused on Hopf invariants. The work of Cohen, Moore, and Neisendorfer focused on Samelson products and the relationship between spheres and the p-torsion Moore spaces $P^n(p^r) = S^{n-1} \cup_{p^r} e^n$ with one nontrivial integral cohomology group isomorphic to Z/p^rZ in dimension n.. [**4, 3, 18, 20**] In fact, their work on homotopy exponents was originally focused on the homotopy groups of odd primary Moore spaces. In so doing, it was discovered that these odd primary Moore spaces were a key to understanding the odd primary homotopy theory of spheres. At first it was discovered, if p is an odd prime and $n \geq 3$, then the mod p^r Moore spaces $P^n(p^r)$ had infinitely many elements of order p^{r+1}. [**4**] (The original arguments were in fact valid only for primes greater than 3 but these results were extended to all odd primes in [**14**].) In [**5**] it was shown that the odd primary Moore spaces $P^n(p^r)$ with $n \geq 3$ had a homotopy exponent no greater than p^{2r+1}. In [**16**], the homotopy exponent p^{r+1} was achieved, except for a mistake which was easily repaired unless $p = 3$.. At least for $p > 3$, this was the best possible result in several ways. For $p = 3$ we at least have the upper bound of p^{r+2}.

One, the restriction to dimensions $n \geq 3$ was necessary since the universal cover of $P^2(p^r)$ has the homotopy type of a bouguet of $p^r - 1$ copies of the 2-dimensional sphere, Unless $p^r = 2$, the Hilton-Milnor theorem shows that this space has no exponent at any prime.

Two, when $p > 3$, the fact that the homotopy exponent is exactly p^{r+1} is a consequence of the existence of infinitely many elements of order p^{r+1} in the homotopy groups of $P^n(p^r)$ with p odd and $n \geq 3$.

Three, it was shown that the double loop space $\Omega^2 P^n(p^r),\ n \geq 3,\ p > 3$, has a geometric exponent of p^{r+1}. When $p = 3$ this double loop space has null homotopic p^{r+2} power map. The single loop space $\Omega P^n(p^r)$ has no geometric exponent whatever, that is, it has no null homotopic power maps. This is a consequence of the fact that, no matter how large s is, the power maps $p^s : \Omega P^n(p^r) \to \Omega P^n(p^r)$ do not induce zero in mod p homology. [**5**] Thus, when $p > 3$ this geometric exponent is best possible both in the power required and in the number of loops required.

On the other hand, the situation for the homotopy exponents of 2-primary Moore spaces is not so well understood. Not everything is known about the existence or non-existence of these exponents. But Theriault [**25**] has shown that $2^{r+1}\pi*(P^m(2^r)) = 0$ for $m \geq 4,\ r \geq 6$. And Cohen [**2**] has shown that this result is best possible in this case. For $3 \leq r \leq 5$ Theriault has shown that 2^{r+2} is an upper bound for the homotopy exponent and for $r = 2$ an upper bound is 32. But it is not known whether the mod 2 Moore space $P^m(2)$ has any homotopy exponent.

Michael Barratt has an unproven conjecture which would resolve these issues by establishing a connection between the additive order of a double suspension and the multiplicative order of its double loop space. He conjectured that: Let p be any prime. If $\Sigma^2 X$ is a double suspension with p^r times the identity null homotopic, then the multiplicative power map p^{r+1} is null homotopic on the double loop space $\Omega^2\Sigma^2 X$.

2. Tools for odd primary exponent theory

A starting point for odd primary exponent theory can be found in the papers [**4, 3**], the book [**18**], or in the survey article [**20**]. We also note that all this can be extended to the prime 3 by [**14**]. In fact, since the Jacobi identity remains valid for Samelson products in homotopy groups with mod 3^r coefficients with $r \geq 2$, the methods are valid without change for mod 3^r Moore spaces with $r \geq 2$. The case of the mod 3 Moore spaces are more subtle since the Jacobi identity fails for the mod 3 coefficients. But, in the end, it can be made to work.

On the other hand, the problem with the proof of the homotopy exponent of a 3 primary Moore space has nothing to do with the failure of the Jacobi identity. It has to do with the lack of homotopy associativity in the H-space structure of the fibre of the degree 3^r map and occurs as far as we know for all $r \geq 1$.

Scholium: The reference for the properties of Samelson products with 3-primary coefficients is [**19**]. Unfortunately the published version contains a mistake which was pointed out to me by Brayton Gray and subsequently corrected in the version which appears on my website.

First of all, we note that it is sufficient in odd primary exponent theory for both spheres and Moore spaces to restrict to the odd dimensional case since we have the two splitting theorems:

Theorem 2.1. *Serre: Localized at an odd prime, there is a homotopy equivalence*

$$S^{2n+1} \times \Omega S^{4n+3} \simeq \Omega S^{2n+2}.$$

Theorem 2.2. [**4**] *If p is an odd prime and*

$$S^{2n+1}\{p^r\} \to S^{2n+1} \xrightarrow{p^r} S^{2n+1}$$

is a fibration sequence up to homotopy, then there is a homotopy equivalence

$$S^{2n+1}\{p^r\} \times \Omega \bigvee_{k\geq 0} P^{4n+2kn+3}(p^r) \simeq \Omega P^{2n+2}(p^r).$$

That is, we have split fibration sequences up to homotopy

$$\Omega S^{4n+3} \to \Omega\Sigma S^{2n+1} \to S^{2n+1}$$

and

$$\Omega \bigvee_{k\geq 0} P^{4n+2kn+3}(p^r) \to \Omega\Sigma P^{2n+1}(p^r) \to S^{2n+1}\{p^r\}$$

where the projection maps of these fibrations are the multiplicative extensions of the respective maps $S^{2n+1} \to S^{2n+1}$ and $P^{2n+1}(p^r) \to S^{2n+1}\{p^r\}$. In both of the above theorems, the left hand loop factors are mapped to the right by the multiplicative extensions of the respective Samelson products $[\iota,\iota]$ and the iterated $ad^k(\mu)([\nu,\nu]$ where

$$\iota \in \pi_{2n}(\Omega S^{2n+2}),\ \nu \in \pi_{2n+1}(\Omega P^{2n+2}(p^r); Z/p^rZ),\quad \mu \in \pi_{2n}(\Omega P^{2n+2}(p^r); Z/p^rZ)$$

are generators of these homotopy groups.

In both cases the multiplicative extensions are the same as the looping of the corresponding adjoint Whitehead products. And in both cases the left hand product spaces are mapped to the right by multiplying maps of the individual factors.

Scholium: When p is an odd prime, we know that the base spaces in the above two fibrations sequences are both homotopy commutative H-spaces. When $p > 3$, we know that they are also homotopy associative. [**1, 15**]. It follows that multiplicative extensions into these spaces exist and, since Samelson products vanish in the base spaces, it is clear that any Samelson product into the total spaces of these fibrations factors through the respective fibres, as do their multiplicative extensions. This is an important technical point. When $p > 3$, the total spaces of these fibrations modulo loops on Whitehead products are the same as the base spaces.

The odd dimensional theory for odd primary Moore spaces starts with the analysis of the fibration sequence of the pinch map $F^{2n+1}\{p^r\} \to P^{2n+1}(p^r) \to S^{2n+1}$. The main results of [**4, 3**] are the description of the commutative diagrams (localized at p) below in which the rows and columns are all loop maps and fibration sequences (as always, up to homotopy):

$$\begin{array}{ccccc}
\Omega E^{2n+1}\{p^r\}) & \to & \Omega P^{2n+1}(p^r) & \to & \Omega S^{2n+1}\{p^r\} \\
\downarrow & & \downarrow = & & \downarrow \\
\Omega F^{2n+1}\{p^r\} & \to & \Omega P^{2n+1}(p^r) & \to & \Omega S^{2n+1} \\
\downarrow & & \downarrow & & \downarrow \Omega p^r \\
\Omega^2 S^{2n+1} & \to & P\Omega S^{2n+1} & \to & \Omega S^{2n+1}.
\end{array}$$

The bottom row is just the sequence of the path fibration and the right column is the loop of the defining fibration for the fibre of the degree p^r map on a sphere, The most important result is splitting theorem that identifies the left hand column as a fibration sequence:

THEOREM 2.3. *In the above diagram the left hand column is the fibration sequence*

$$\Omega\Sigma \bigvee_{\alpha} P^{n_\alpha}(p^r) \times \Pi_{r+1} \times C(n) \to \Omega\Sigma \bigvee_{\alpha} P^{n_\alpha}(p^r) \times \Pi_{r+1} \times S^{2n-1} \to \Omega^2 S^{2n+1}$$

which is the result of multiplying the total space and fibre in the fibration sequence of the double suspension $C(n) \to S^{2n-1} \xrightarrow{\Sigma^2} \Omega S^{2n+1}$ *by the space* $\Omega\Sigma \bigvee_\alpha P^{n_\alpha}(p^r) \times \Pi_{r+1}$. *The map* $\Omega\Sigma \bigvee_\alpha P^{n_\alpha}(p^r) \to \Omega P^{2n+1}(p^r)$ *is a multiplicative extension of*

Samelson products. The space $\Pi_{r+1} = \Pi_{k\geq 1} S^{2p^k n-1}\{p^{r+1}\}$ *is an infinite product of fibres of degree* p^{r+1} *maps on spheres.*

This is important enough that it deserves to be expanded below in the diagram in which all rows and columns are fibration sequences:

$$\begin{array}{ccccc}
\Omega\Sigma\bigvee_\alpha P^{n_\alpha}(p^r)\times\Pi_{r+1}\times C(n) & \to & \Omega P^{2n+1}(p^r) & \to & \Omega S^{2n+1}\{p^r\} \\
\downarrow & & \downarrow = & & \downarrow \\
\Omega\Sigma\bigvee_\alpha P^{n_\alpha}(p^r)\times\Pi_{r+1}\times S^{2n-1} & \to & \Omega P^{2n+1}(p^r) & \to & \Omega S^{2n+1} \\
\downarrow & & \downarrow & & \downarrow \Omega p^r \\
\Omega^2 S^{2n+1} & \to & P\Omega S^{2n+1} & \to & \Omega S^{2n+1}.
\end{array}$$

We recommend the survey article [**20**] as the quickest introduction to the above results which we will not prove here.

Scholium: If one extends the above diagrams to the left by fibration sequences in the standard way, then one gets almost immediately the factorization

$$\Omega^2 p^r : \Omega^2 S^{2n+1} \xrightarrow{\pi_r} S^{2n-1} \xrightarrow{\Sigma^2} \Omega^2 S^{2n+1}$$

where $\pi_r = \pi\circ\partial : \Omega^2 S^{2n+1} \to S^{2n-1}$ is the composition of the connecting morphism in the fibration sequence with the projection on the sphere factor in the product. Taking $r = 1$ gives that p times the homotopy is in the image of the double suspension, which is a strong form of the odd primary exponent theorem for the homotopy groups of spheres.

3. Decompositions of suspensions into bouquets

Perhaps the simplest decomposition of a suspension into a bouquet is the decomposition of the suspension of a product of two spaces:

$$\Sigma(X\times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X\wedge Y).$$

Iteration of this gives that $\Sigma(X_1\times X_2\times\cdots\times X_k)$ splits into a bouquet for which the "top" piece is $\Sigma(X_1\wedge X_2\wedge\cdots\wedge X_k)$.. The James decomposition [**8**] of the suspension of the loop suspension is an immediate consequence of this, that is:

Theorem 3.1. *James:*

$$\bigvee_{kgeq1} \Sigma X^{\wedge k} \simeq \Sigma\Omega\Sigma X$$

for all connected spaces X.

Proof. Take the suspension map, the adjoint of the identity, $\Sigma : X \to \Omega\Sigma X$, and multiply it by itself k times to get maps $X\times X\times\cdots\times X \to \Omega\Sigma X$. Now suspend these maps and restrict to the top piece of the bouquet decomposition. Then add them up to get a map

$$\bigvee_{k\geq 1} \Sigma X^k \to \Sigma\Omega\Sigma X.$$

By the Bott-Samelson theorem this is a homology equivalence with all field coefficients, hence, it an integral homology equivalence of simply connected spaces. Hence it is a weak homotopy equivalence. □

Scholium: Since we are in the category of spaces with the homotopy type of a CW complex, we can conclude that homology equivalences of simply connected spaces are not just weak equivalences but actually homotopy equivalences. [**13**] Or we can just work in the category of simplicial sets.

For an odd primary Moore spaces we have [**18**]:

LEMMA 3.2. *If p is an odd prime, there is a homotopy equivalence*

$$P^n(p^r) \wedge P^m(p^r) \simeq P^{n+m-1}(p^r) \vee P^{n+m}(p^r).$$

Combining this with the James decomposition gives:

COROLLARY 3.3. *If p is an odd prime and X is a bouquet of mod p^r Moore spaces with r fixed, then $\Sigma\Omega\Sigma X$ is homotopy equivalent to a bouquet of mod p^r Moore spaces. In particular, the mod p^r Hurewicz map is surjective.*

Now suppose that X and Y are both bouquets of mod p^r Moore spaces with p an odd prime and with r fixed. Then the following is true:

THEOREM 3.4. *Given a map $f : \Omega\Sigma Y \to \Omega\Sigma X$ which is a mod p^r homology monomorphism, there is a space W and a map $g : W \to \Sigma\Omega\Sigma X$ such that*

$$\Sigma f \vee g : \Sigma\Omega\Sigma Y \vee W \to \Sigma\Omega\Sigma X$$

is a homotopy equivalence.

PROOF. Over the ring Z/p^rZ, any free module is projective [**10**] and injective [**6**]. Hence, if $B \subset A$ is a free submodule of a free module, there is a complementary free module C such that $A = B \oplus C$.

We apply this to the map induced in mod p^r homology by the suspension Σf. Note that the mod p^r homologies of the domain and range are both acyclic with respect to the $r-th$ Bockstein differential β^r associated to the short exact coefficient sequence $0 \to Z/p^rZ \to Z/p^{2r}Z \to Z/p^rZ \to 0$. Hence, the complement of the image is also acyclic with respect to β^r. Therefore, the complement has an acyclic basis, that is, a basis consisting of pairs x_α and $\beta^r x_\alpha$. Since the mod p^r Hurewicz map is surjective, we can pick maps $g_\alpha : P^{m_\alpha}(p^r) \to \Sigma\Omega\Sigma X$ such that $g_{\alpha*}(e_\alpha) = x_\alpha$ where e_α is a generator of the mod p^r homology of $P^{m_\alpha}(p^r)$ in the top dimension $m_\alpha =$ the dimension of x_α.

Setting $W = \bigvee_\alpha P^{m_\alpha}(p^r)$ gives us a map g defined on W so that

$$\Sigma f \vee g : \Sigma\Omega\Sigma Y \vee W \to \Sigma\Omega\Sigma X$$

is a mod p^r homology isomorphism. Since all the spaces are bouquets of mod p^r Moore spaces, it follows easily that this map is an isomorphism in integral homology between simply connected spaces. Hence it is a weak equivalence. $\square$

We conclude this section with

THEOREM 3.5. *Let p be any prime and let $S^m\{p^r\}$ be the fibre of the degree p^r map $p^r : S^m \to S^m$. Then there is a homotopy equivalence*

$$\bigvee_{k \geq 0} P^{m+k(m-1)+1}(p^r) \to \Sigma S^m\{p^r\}.$$

Proof. The first right translate of the defining fibration sequence is the fibration sequence $\Omega S^m \to S^m\{p^r\} \to S^m$ which is totally nonhomologous to zero mod p^r and such that the homology of the total space is a module via the action of the fibre ΩS^m. Hence, $H(S^m\{p^r\}; Z/p^r Z)$ is a free $H(\Omega S^m; Z/p^r Z) = T(\iota_{m-1})$ module on $H(S^m : Z/p^r Z) =< 1.e_m >$. That is,

$$H(S^m\{p^r\}; Z/p^r Z) =< 1, \iota, e, \iota^2, e\iota, \ldots, \iota^k, e\iota^{k-1}, \cdots >$$

with dimension $\iota = m-1$, dimension $e = m$, and r-th Bockstein being the derivation given by $\beta^r\iota = 0,\ \beta^r e = \iota$.

Now let $f : P^m(p^r) \to S^m\{p^r\}$ be a map which is mod p^r homology isomorphism in dimensions $m-1$ and m. That is, f hits ι, e in mod p^r homology.

Restricting the action of the fibre to $S^{m-1} \subset \Omega S^m$ and iterating this action k times gives maps $S^{m-1} \times \cdots \times S^{m-1} \times P^m(p^r) \to S^m\{p^r\}$. Suspend these maps and restrict to the top bouquet piece to get maps

$$\Sigma(S^{m-1} \wedge \cdots \wedge S^{m-1} \wedge P^m(p^r)) = P^{m+k(m-1)+1}(p^r) \to \Sigma S^m\{p^r\}.$$

Adding these maps together gives the required homotopy equivalence. $\square$

4. Splittings which lead to exponents for odd primary Moore spaces

For the remainder of this paper, p will always be an odd prime.

The following splitting result requires no computation and is the key to establishing a homotopy exponent for odd primary Moore spaces.

Theorem 4.1. *Suppose that $f : \Omega Y \to X$ is such that the suspension $\Sigma f : \Sigma\Omega Y \to \Sigma X$ has a left inverse $g : \Sigma X \to \Sigma\Omega Y$, that is g is a retraction, $g \circ \Sigma f \simeq 1$. Then f has a retraction h, $\quad h \circ f \simeq 1$.*

Proof. It is an easy exercise in adjoint functors that the composition

$$h : X \xrightarrow{\Sigma} \Omega\Sigma X \xrightarrow{\Omega g} \Omega\Sigma\Omega Y \xrightarrow{\Omega\, eval} \Omega Y$$

is a left inverse. $\square$

Corollary 4.2. *Suppose that $f : \Omega Y \to \Omega X$ is a mod p^r homology monomorphism where X and Y are both bouquets of mod p^r Moore spaces. Then f has a left inverse, that is, there is a retraction $g : \Omega X \to \Omega Y, g \circ f \simeq 1$.*

Proof. Apply the above theorem and the theorem in the previous section. $\square$

Scholium: When I tried to explain the above result to John Moore by a very complicated argument using Hopf invariants, he said to me: "If it is true, then it cannot be that complicated." And he proceeded to immediately invent the argument you have just seen. I was amazed!

In the CMN splittings, the maps

$$\Omega\Sigma \bigvee_\alpha P^{n_\alpha}(p^r) \to \Omega E^{2n+1}\{p^r\} \to \Omega F^{2n+1}\{p^r\} \to \Omega P^{2n+1}(p^r)$$

are the loopings of Whitehead product maps

$$\Sigma \bigvee_\alpha P^{n_\alpha}(p^r) \to E^{2n+1}\{p^r\} \to F^{2n+1}\{p^r\} \to P^{2n+1}(p^r).$$

More precisely, these maps are canonical compressions of the Whitehead product maps, respectively, the H-based Whitehead product into $E^{2n+1}\{p^r\}$, the relative Whitehead product into $F^{2n+1}\{p^r\}$, and the usual mod p^r Whitehead product into $P^{2n+1}(p^r)$. Details for the Samelson products which are adjoint forms of these Whitehead products can be found in [**18, 20**]. The above corollary says that the following fibration row sequences which define the spaces

$$V^{2n+1}\{p^r\},\ W^{2n+1}\{p^r|\},\ T^{2n+1}\{p^r|\}$$

are all split:

$$\begin{array}{ccccccccc}
\Omega\Sigma\bigvee_\alpha P^{n_\alpha}(p^r) & \to & \Omega E^{2n+1}\{p^r\} & \to & V^{2n+1}\{p^r\} & \to & \Sigma\bigvee_\alpha P^{n_\alpha}(p^r) & \to & E^{2n+1}\{p^r\} \\
\downarrow= & & \downarrow & & \downarrow & & \downarrow= & & \downarrow \\
\Omega\Sigma\bigvee_\alpha P^{n_\alpha}(p^r) & \to & \Omega F^{2n+1}\{p^r\} & \to & W^{2n+1}\{p^r\} & \to & \Sigma\bigvee_\alpha P^{n_\alpha}(p^r) & \to & F^{2n+1}\{p^r\} \\
\downarrow= & & \downarrow & & \downarrow & & \downarrow= & & \downarrow \\
\Omega\Sigma\bigvee_\alpha P^{n_\alpha}(p^r) & \to & \Omega P^{2n+1}(p^r) & \to & T^{2n+1}\{p^r\} & \to & \Sigma\bigvee_\alpha P^{n_\alpha}(p^r) & \to & P^{2n+1}(p^r)
\end{array}$$

In more detail, the product splittings below are all compatible:

$$\begin{array}{ccc}
\Omega E^{2n+1}\{p^r\} & \simeq & \Omega\Sigma\bigvee_\alpha P^{n_\alpha}(p^r)\times V^{2n+1}\{p^r\} \\
\downarrow & & \downarrow \\
\Omega F^{2n+1}\{p^r\} & \simeq & \Omega\Sigma\bigvee_\alpha P^{n_\alpha}(p^r)\times W^{2n+1}\{p^r\} \\
\downarrow & & \downarrow \\
\Omega P^{2n+1}(p^r) & \simeq & \Omega\Sigma\bigvee_\alpha P^{n_\alpha}(p^r)\times T^{2n+1}\{p^r\}
\end{array}$$

In the diagram below the rows and the right hand length 3 rows are fibration sequences:

$$\begin{array}{ccccccc}
\Pi_{r+1}\times C(n) & \simeq & V^{2n+1}\{p^r\} & \to & T^{2n+1}\{p^r\} & \to & \Omega S^{2n+1}\{p^r\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Pi_{r+1}\times S^{2n-1} & \simeq & W^{2n+1}\{p^r\} & \to & T^{2n+1}\{p^r\} & \to & \Omega S^{2n+1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \Omega p^r \\
\Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} & \to & P\Omega S^{2n+1} & \to & \Omega S^{2n+1}
\end{array}$$

We refer to [**5, 16, 18**] for details on these fibration sequences and their compatibility. It is based on the expansion of a 2×2 commutative cube into a larger cube with rows and columns fibration sequences.

Hint: Expand a cube with path spaces at coordinates $(0,0,0),\ (0,1,0)$, Moore spaces at $(1,0,1),\ (1,1,1)$, loops on a bouquet of Moore spaces at $(0,0,1),\ (0,,1,1)$, a sphere at $(1,0,0)$ and a fibre of degree p^r map at $(1,1,0)$. Vertical maps go down and horizontal maps go to the left.

The point of the above diagram is that it is a cut down version of the original fibration diagrams of CMN in which the space $\Omega P^{2n+1}(p^r)$ has been replaced by the "core" $T^{2n+1}\{p^r\}$ of the loops on a Moore space, that is, the factor $\Omega\Sigma\bigvee_\alpha P^{n_\alpha}(p^r)$ has been excised.

Remark: Consider the maps of fibration sequences:

$$\begin{array}{ccccc}
\Pi_{r+1}\times C(n) & \to & \Omega E^{2n+1}\{p^r\} & \to & V^{2n+1}\{p^r\} \\
\downarrow & & \downarrow & & \downarrow \\
\Pi_{r+1}\times S^{2n-1} & \to & \Omega F^{2n+1}\{p^r\} & \to & W^{2n+1}\{p^r\} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^2 S^{2n+1} & \xrightarrow{=} & \Omega^2 S^{2n+1} & \xrightarrow{=} & \Omega^2 S^{2n+1}
\end{array}$$

The composite in the middle row is a mod p homology equivalence since the fibration sequence $\Omega\Sigma \bigvee_\alpha P^{n_\alpha}(p^r)) \to \Omega F^{2n+1}\{p^r\} \to W^{2n+1}\{p^r\}$ has a section and therefore

$$H(\Pi_{r+1} \times C(n)) \cong Z/pZ \otimes_{H(\Omega\Sigma \bigvee_\alpha P^{n_\alpha}(p^r))} H\Omega F^{2n+1}\{p^r\} \cong H(W^{2n+1}\{p^r\}).$$

By the comparison theorem, the composite in the top row is also a mod p homology equivalence. Since all the spaces are localized and finite type over the ring $Z_{(p)}$ of integers localized at p, these mod p equivalences are local equivalences.

Corollary 4.3. *If all the spaces $\Omega T^{2\ell+1}\{p^r\}$ have the same geometriic exponent p^{r+s} then all of double loop spaces $\Omega^2 P^n(p^r)$ with $n \geq 3$ have this geometric exponent.*

Proof. It follows from the decomposition of the loops on an even dimensional odd primary Moore space, the bouquet decomposition of the smash of two Moore spaces, and from the Hilton-Milnor theorem [**29, 18**] that all these spaces are infinite products of the spaces $T^{2\ell+1}\{p^r\}$ and the spaces $\Omega S^{2\ell+1}\{p^r\}$. Once looped these splittings are multiplicative, that is, they are splittings as H-spaces. Since the latter fibres of degree p^r maps, even before they are looped, have geometric exponent p^r [**15**], this completes the argument. □

It is useful to record [**15**]

Lemma 4.4. *If p is an odd prime, the fibres of degree p^r maps $S^{2\ell-1}\{p^r\}$ are all H-spaces with a null homotopic power map p^r. They are always homotopy commutative and, if $p > 3$, they are homotopy associative.*

Theorem 4.5. *Semi-splitting: a. Let Y be an $n-1$ connected space, $n \geq 2$, where ΣY has the homotopy type of a bouquet of mod p^r Moore spaces. Suppose that p is an odd prime.*

b. Suppose that

$$f : X \to Y$$

is a mod p^r homology monomorphism where

$$X = S^{\ell_1} \times \cdots \times S^{\ell_k} \times Z$$

and where ΣZ has the homotopy type of a bouquet of mod p^{r+s} Moore spaces with $s \geq 0$, but the s may vary.

c. Let G be a homotopy associative H-space with $p^r\pi_*(G) = 0$, that is all the integral homotopy groups of G are annihilated by p^r.

Let $g : X \to G$ be any map which is null on the 2-skeleton. Then there exists a map

$$h : Y \to G$$

such that $h \circ f = g$.

Loosely speaking, h is an "extension" of g.

Remark: This theorem was previously stated only for the special case $G = S^{2m-1}\{p^r\}$ with $p > 3$. And the statement includes the case where X has no sphere factors or where Z is a point. Observe that the hypothesis implies that ΣX has the homotopy type of a bouquet of spheres and mod p^{r+s} Moore spaces, $s \geq 0$.

5. Proof of the geometric exponent theorems for odd primary Moore spaces

First, we prove the upper bound exponent theorem proved in [**5**].

Consider the fibration sequence

$$\Pi_{r+1} \times C(n) \to T^{2n+1}\{p^r\} \to \Omega S^{2n+1}\{p^r\}.$$

Once looped this sequence is a multiplicative fibration sequence. Hence the total space has geometric exponent bounded by the product of the geometric exponents of the ends. Since $S^{2n+1}\{p^r\}$ has geometric exponent p^r, $C(n)$ has geometric exponent p, and $\Pi_{r+1} = \Pi_{k \geq 1} S^{2p^k n-1}\{p^{r+1}\}$ has geometric exponent p^{r+1} [**4, 3, 15**] we get

Theorem 5.1. *All the spaces $\Omega T^{2n+1}\{p^r\}$ have geometric exponent bounded above by p^{2r+1} and hence so do all the double loops on mod p^r Moore spaces $\Omega^2 P^m(p^r)$ with p an odd prime and $m \geq 3$.*

Corollary 5.2. *If p is an odd prime and $m \geq 3$, then p^{2r+1} annihilates the homotopy groups $\pi_*(P^m(p^r))$.*

In order to prove the sharp geometric exponent theorem for odd primary Moore spaces, we recall some facts about mod p^r homology of the spaces involved. As usual, all the results will be at an odd prime p.

Theorem 5.3. *All the mod p^r homologies $H = H(\quad, Z/p^r Z)$ are free over the ring $Z/p^r Z$:*

a) $H(S^{2n+1}\{p^{r+1}\}) = E(x_{2n+1}) \otimes P(y_{2n})$ with $\beta^{r+1} x = y$.

b) the four fibrations

$$\Omega F^{2n+1}\{p^r\} \to \Omega P^{2n+1}(p^r) \to \Omega S^{2n+1}$$

$$\Omega \bigvee_\alpha P^{\alpha+1}(p^r) \to \Omega F^{2n+1}\{p^r\} \to W^{2n+1}\{p^r\}$$

$$\Omega \bigvee_\alpha P^{\alpha+1}(p^r) \to \Omega P^{2n+1}\{p^r\} \to T^{2n+1}\{p^r\}$$

$$W^{2n+1}\{p^r\} \to T^{2n+1}\{p^r\} \to \Omega S^{2n+1}$$

are totally non homologous to zero mod p^r with the first three being principal fibrations.

Proof. Part a) is a consequence of the fact that the fibration $\Omega S^{2n+1} \to S^{2n+1}\{p^{r+1}\} \to S^{2n+1}$ is totally nonhomologous to zero mod p^r.

The fact that the first fibration in part b) is totally nonhomologous to zero is part of the main computation in [**4**]. The second and third fibrations in part b) are both split, hence they are both totally nonhomologous to zero.

Since the Serre spectral sequence of the first fibration in part b) maps surjectively to the Serre spectal sequence of the fourth fibration, the fact that the first is totally nonhomologous to zero implies that the fourth one is also.

□

Scholium: Since all of the above spaces have mod p^r homologies which are free over $Z/p^r Z$, it follows that the corresponding mod p homologies are gotten from these by simply tensoring with Z/pZ.

Furthermore, the fact that $T^{2n+1}\{p^r\}$ is a retract of $\Omega P^{2n+1}(p^r)$ implies that $\Sigma T^{2n+1}\{p^r\}$ is a bouquet of mod p^r Moore spaces. Since $V^{2n-1}\{p^r\} \simeq \Pi_{r+1} \times S^{2n-1}$ with $\Sigma\Pi_{r+1}$ having the homotopy type of a bouquet of mod p^{r+1} Moore spaces, the semi-splitting theorem applies when $G = \Pi_r$ is homotopy associative, that is, when $p > 3$.

Hence, any map

$$g : V^{2n+1}\{p^r\} \to G = \Pi_r = \Pi_k S^{2p^k n-1}\{p^r\}$$

extends to a map $h : T^{2n+1}\{p^r\} \to G$.

We choose $g : \Pi_{r+1} \times S^{2n-1} \to G = \Pi_r$ to be defined by projection onto Π_{r+1}, then followed by the projection in the fibration sequence [**16**]

$$\Pi_1 \to \Pi_{r+1} \to \Pi_r.$$

Thus, when $p > 3$, a special case of 4.5 is the following:

Corollary 5.4. *Let $f : \Pi_{r+1} \times S^{2n-1} \to T^{2n+1}\{p^r\}$ be the map which occurs in the cut down version of the main diagram and let $g : \Pi_{r+1} \times S^{2n-1} \xrightarrow{proj} \Pi_{r+1} \xrightarrow{\rho} \Pi_r$ be a map where the fibre of ρ is Π_1. Then, if $p > 3$, there is a map $h : T^{2n+1}\{p^r\} \to \Pi_r$ such that the composition $h \cdot f = g$.*

Remark: Since the fibre of the double suspension $C(n)$ factors through the sphere, it is automatic that the sphere S^{2n-1} may be replaced by $C(n)$ in the above corollary. The map $\Omega h : \Omega T^{2n+1}\{p^r\} \to \Omega\Pi_r$ is the "extension" of Ωg. If Ωh is restricted to $\Omega W^{2n+1}\{p^r\}$, it is just the loop of the map

$$\Pi_{r+1} \times S^{2n-1} \xrightarrow{proj} \Pi_{r+1} \xrightarrow{\Pi\pi_k} \Pi_r$$

If we restrict further to the space $\Omega V^{2n+1}\{p^r\}$, it is the loop of the map

$$\Pi_{r+1} \times C(n) \xrightarrow{proj} \Pi_{r+1} \xrightarrow{\Pi\pi_k} \Pi_r.$$

Hence we have two fibration sequences as follows;

Theorem 5.5. *If $p > 3$, there is a fibration sequence*

$$\Omega\Pi_{r+1} \times \Omega C(n) \to \Omega T^{2n+1} \to \Omega^2 S^{2n+1}\{p^r\}$$

and there is a map

$$\Omega h : \Omega T^{2n+1}\{p^r\} \to \Omega\Pi_r,$$

"the loop of the semi-splitting," which restricts to $\Omega V^{2n+1}\{p^r\} = \Omega\Pi_{r+1} \times \Omega C(n)$ to define a fibration sequence

$$\Omega\Pi_1 \times \Omega C(n) \to \Omega\Pi_{r+1} \times \Omega C(n) \to \Omega\Pi_r$$

.

Corollary 5.6. *For $m \geq 3$ and $p > 3$, the double loop space $\Omega^2 P^m(p^r)$ has geometric exponent p^{r+1} and hence p^{r+1} annihilates the homotopy groups $\pi(P^m(p^r))$.*

Proof. The fibrations in the preceding theorem are multiplicative. Let α be the identity map on $\Omega T^{2n+1}\{p^r\}$. Then $p^r\alpha$ goes to zero in $\Omega^2 S^{2n-1}\{p^r\}$. Hence, $p^r\alpha$ comes from a map δ into $\Omega C(n) \times \Omega\Pi_{r+1}$. Now δ goes to $p^r\epsilon$ in $\Omega\Pi_r$ where ϵ is the image of α. Since this is zero, δ comes from $\Omega C(n) \times \Omega\Pi_1$. Hence δ has order p and α has order p^{r+1}. Since the geometric exponent of $\Omega T^{2n+1}\{p^r\}$ is p^{r+1} for all n, the geometric exponent of $\Omega^2 P^n(p^r)$ is also p^{r+1}. ☐

6. An exponent for the odd primary Moore spaces which works for all odd primes

Since the spaces $S^{2n+1}\{p^r\}$ are known to be homotopy associative only for primes $p > 3$, the homotopy exponent of p^{r+1} for a mod p^r Moore space is not known to be valid for mod 3^r Moore spaces. In this section, we replace these spaces by the double loop spaces $\Omega^2 S^{2n+1}\{p^r\}$ which are certainly homotopy associative and achieve the homotopy exponent p^{r+2} for all mod p^r Moore spaces of dimension ≥ 3 and all odd primes.

Consider the commutative diagram below which defines the double suspension $\Sigma^2 : S^{2n-1}\{p^r\} \to \Omega^2 S^{2n+1}\{p^r\}$ for fibres of degree p^r maps. In this diagram all rows and all columns are fibration sequences.

$$\begin{array}{ccccc}
C(n) \times \Omega C(n) & \to & S^{2n-1}\{p^r\} & \xrightarrow{\Sigma^2} & \Omega^2 S^{2n+1}\{p^r\} \\
\downarrow & & \downarrow & & \downarrow \\
C(n) & \to & S^{2n-1} & \xrightarrow{\Sigma^2} & \Omega^2 S^{2n+1} \\
\downarrow p^r & & \downarrow p^r & & \downarrow p^r \\
C(n) & \to & S^{2n-1} & \xrightarrow{\Sigma^2} & \Omega^2 S^{2n+1}
\end{array}$$

The left vertical column is a fibration sequence since p^r is null on $C(n)$ and the top horizontal row is the version of the double suspension for fibres of degree p^r maps.

The product gives a map

$$\Sigma^2 : \Pi_r = \prod_{k \geq 1} S^{2p^k - 1}\{p^r\} \to \prod_{k \geq 1} \Omega^2 S^{2p^k + 1}\{p^r\} = \Omega^2_r.$$

We now replace corollary 5.4 by the following form of the semi-splitting:

Corollary 6.1. *Let $f : \Pi_{r+1} \times S^{2n-1} \to T^{2n+1}\{p^r\}$ be the map which occurs in the cut down version of the main diagram and compose the previous map with the double suspension, that is, let g be the composition*

$$g : \Pi_{r+1} \times S^{2n-1} \xrightarrow{proj} \Pi_{r+1} \xrightarrow{\rho} \Pi_r \xrightarrow{\Sigma^2} \Omega^2_r.$$

Then, if p is any odd prime, there is a map $h : T^{2n+1}\{p^r\} \to \Omega^2_r$ such that the composition $h \cdot f = g$.

Just as in section 5, it is automatic that the sphere S^{2n-1} may be replaced by $C(n)$ in the above corollary. The map $\Omega h : \Omega T^{2n+1}\{p^r\} \to \Omega\Omega^2_r$ is the "extension" of Ωg. If Ωh is restricted to $\Omega W^{2n+1}\{p^r\}$, it is just the loop of the map

$$\Pi_{r+1} \times S^{2n-1} \xrightarrow{proj} \Pi_{r+1} \xrightarrow{\Pi\pi_k} \Pi_r \xrightarrow{\Sigma^2} \Omega^2_r$$

If we restrict further to the space $\Omega V^{2n+1}\{p^r\}$, it is the loop of the map

$$\Pi_{r+1} \times C(n) \xrightarrow{proj} \Pi_{r+1} \xrightarrow{\Pi\pi_k} \Pi_r \xrightarrow{\Sigma^2} \Omega^2_r.$$

Hence we have two fibration sequences as follows:

Theorem 6.2. *If p is an odd prime, there is a fibration sequence*

$$\Omega\Pi_{r+1} \times \Omega C(n) \to \Omega T^{2n+1} \to \Omega^2 S^{2n+1}\{p^r\}$$

and there is a map

$$\Omega h : \Omega T^{2n+1}\{p^r\} \to \Omega\Pi_r \xrightarrow{\Sigma^2} \Omega\Omega^2_r,$$

"the loop of the semi-splitting," which restricts to $\Omega V^{2n+1}\{p^r\} = \Omega\Pi_{r+1} \times \Omega C(n)$ *to define the maps*

$$\Omega\Pi_{r+1} \times \Omega C(n) \to \Omega\Pi_{r+1} \to \Omega\Pi_r \xrightarrow{\Omega\Sigma^2} \Omega\Omega_r^2$$

where

$$\Omega\Pi_1 \times \Omega C(n) \to \Omega\Pi_{r+1} \times \Omega C(n) \to \Omega\Pi_r$$

is a fibration sequence.

Corollary 6.3. *For* $m \geq 3$ *and* p *an odd prime, the double loop space* $\Omega^2 P^m(p^r)$ *has geometric exponent* p^{r+2} *and hence* p^{r+2} *annihilates the homotopy groups* $\pi(P^m(p^r))$.

Proof. The fibrations in the preceding theorem are multiplicative. Let α be the identity map on $\Omega T^{2n+1}\{p^r\}$. Then $p^r\alpha$ goes to zero in $\Omega^2 S^{2n-1}\{p^r\}$. Hence, $p^r\alpha$ comes from a map δ into $\Omega C(n) \times \Omega\Pi_{r+1}$. Now δ goes to $p^r\epsilon$ in $\Omega\Omega_r^2$ where ϵ is the image of α. This is zero since $\Omega\Omega_r^2$ has geometric exponent p^r. Hence, δ goes through the fibre of $\Omega\Pi_r \to \Omega\Omega_r^2$. Since the fibre of this map has geometric exponent p, $p\delta$ goes to zero in $\Omega\Pi_r$.

Since this is zero, $p\delta$ comes from $\Omega C(n) \times \Omega\Pi_1$. Hence $p\delta$ has order p, that is, δ has order p^2 and α has order p^{r+2}. Since $\Omega T^{2n+1}\{p^r\}$ has geometric exponent $\leq p^{r+2}$ for all n, $\Omega^2 P^m(p^r)$ has geometric exponent $\leq p^{r+2}$ for all $m \geq 3$. $\square$

7. The adjoint form of the semi-splitting

Let G be an H-space. Then there is an extension of the suspension map $\Sigma : G \to \Omega\Sigma G$ to a retraction $\pi : \Omega\Sigma G \to G$. If G is homotopy associative, π can be chosen to be an H-map. [**24**]

In any case, let $F(G) \to \Omega\Sigma G \to G$ be a fibration sequence up to homotopy. $F(G)$ is called a "universal Samelson product."

Remark: When G is homotopy associative and homotopy commutative, the name universal Samelson product is justified, for example, when $G = S^{2n+1}\{p^r\}$, $G = \Pi_r$ with $p > 3$ [**15**], or $G = \Omega^2\Pi_r = \Omega_r^2$. In these cases, all Samelson products in $\Omega\Sigma G$ vanish upon projection to G and hence factor through the fibre $F(G)$.

Since the fibration sequence is split, there is a homotopy equivalence

$$\Omega\Sigma G \simeq G \times F(G).$$

If $\overline{g} : \Sigma Z \to \Sigma G$ is a map with adjoint $g : Z \to \Omega\Sigma G$, we may uniquely decompose this as a sum $g = g_1 + w$ where g_1 factors through G and w factors through $F(G)$.

Remark: If $\Sigma Z \simeq \Sigma\overline{Z}$ is a homotopy equivalence which is not a suspension, it is possible that the above decomposition $g = g_1 + w$ depends on the choice of Z or $\overline{Z}$. For example, if $\Sigma Z \simeq \Sigma \bigvee_\alpha P^{n_\alpha}(p^r)$, it might be convenient to use the bouquet in order to prove something about Z. But care must be taken.

Given another such map $\overline{h} : \Sigma Z \to \Sigma G$ with decomposition of the adjoint $h = h_1 + w'$, we say that $\overline{g}$ is congruent modulo Whitehead products to $\overline{h}$ if the summands g_1 and h_1 are equal. We may write $\overline{g} \equiv_w \overline{h}$ for this congruence. With this notion of modulo Whitehead products, suspension is erased, that is,

$$[\Sigma Z, \Sigma G]_* = [Z, \Omega\Sigma G]_* = [Z, G]_* \times [Z, F(G)]_* \equiv_w [Z, G]_*$$

Remark: In general, we need to make a distinction between the above notion of "modulo Whitehead products" and the subsequent notion of "up to addition of a sum of Whitehead products" in sections 9 and 10. In the case when G is a homotopy commutative and homotopy associative H-space, the second notion is a special case of the first one and no distinction need be made.

Scholium: Recall the split fibration sequence which we choose to write as

$$\Omega \bigvee_{k\geq 0} P^{4m+2k(m-1)-1} \to \Omega P^{2m}(p^r) \to S^{2m-1}\{p^r\}.$$

The inclusion of the fibre is the loops on a bouquet of Whitehead products and if $p > 3$ it is universal in the sense that any Samelson product into the total space factors through this fibre. In the special case where $G = S^{2m-1}\{p^r\}$ with $p > 3$, we can use this fibration to replace the Hopf fibration just described.

The semi-splitting theorem as described above is implied by the adjoint form:

THEOREM 7.1. *Semi-splitting: Let Y be an $n-1$ connected space, $n \geq 2$, where ΣY has the homotopy type of a bouquet of mod p^r Moore spaces. Suppose that*

$$f : X \to Y$$

is a mod p^r homology monomorphism where ΣX has the homotopy type of a bouquet of spheres and mod p^{r+s} Moore spaces, $s \geq 0$, but the s may vary. Suppose also that G is a homotopy associative H-space for which all the homotopy groups $\pi_* G$ are annihilated by p^r.

Let

$$\overline{g} : \Sigma X \to \Sigma G$$

be any map which is null on the 3-skeleton. Then, if $\overline{f} = \Sigma f : \Sigma X \to \Sigma Y$, there exists a map

$$\overline{h} : \Sigma Y \to \Sigma G$$

such that $\overline{h} \circ \overline{f} = \overline{g}$ modulo Whitehead products, that is, $\overline{h} \circ \overline{f} \equiv_w \overline{g}$.

In other words, if $\underline{g} : X \to \Omega\Sigma G$ and $\underline{h} : Y \to \Omega\Sigma G$ are adjoints, then the following is commutative

$$\begin{array}{ccccc} Y & \xrightarrow{h} & \Omega\Sigma G & \xrightarrow{\pi} & G \\ \uparrow f & & & \nearrow \pi & \\ X & \xrightarrow{g} & \Omega\Sigma G & & \end{array}$$

8. A cofibration sequence

LEMMA 8.1. *There is a cofibration sequence*

$$P^n(p^{r+s}) \xrightarrow{p^r} P^n(p^{r+s}) \xrightarrow{\epsilon} P^n(p^r) \vee P^{n+1}(p^r)$$

where the induced map ϵ_* in mod p^r homology is:

a) on a generator e in dimension n,

$$\epsilon_*(e) = (0, \beta^{r+s} e),\ s > 0,\ \epsilon_*(e) = (e, 0),\ s = 0,$$

and b) on a generator $\beta^{r+s} e$ in dimension n-1,

$$\epsilon_*(\beta^{r+s} e) = (\beta^r e, 0)),\ s \geq 0$$

Furthermore, the map ϵ is uniquely determined up to Whitehead products by its induced map in mod p^r homology. If $n > 2$, then it is uniquely determined.

PROOF. The cofibre is identified by the diagram below in which all rows and columns are cofibration sequences

$$\begin{array}{ccccc}
S^{m-1} & \xrightarrow{p^r} & S^{m-1} & \to & P^m(p^r) \\
\downarrow p^{r+s} & & \downarrow p^{r+s} & & \downarrow p^{r+s} = * \\
S^{m-1} & \xrightarrow{p^r} & S^{m-1} & \to & P^m(p^r) \\
\downarrow & & \downarrow & & \downarrow \\
P^m(p^{r+s}) & \xrightarrow{p^r} & P^m(p^{r+s}) & \xrightarrow{\epsilon} & P^m(p^r) \vee P^{m+1}(p^r).
\end{array}$$

In mod p^r homology, the induced map $HP^m(p^{r+s}) \xrightarrow{p^r} HP^m(p^{r+s})$ is zero. Hence,

$$0 \to \overline{H}P^m(p^{r+s}) \xrightarrow{\epsilon_*} \overline{H}P^m(p^r) \oplus \overline{H}P^{m+1}(p^r) \to \overline{H}P^{m+1}(p^{r+s}) \to 0$$

is split exact. If $s > 0$, the fact that ϵ_* commutes with the $r-$th Bockstein forces the homology computation (at least up to changing the generators by multiplication by a unit). If s $= 0$, then the cofibration sequence is split and we may choose the bouquet decomposition to get the homology computation in this form.

The fact that ϵ is determined by its induced map in mod p^r homology is a consequence of Proposition 6.4.3 in [**18**]. □

9. Acyclic envelopes

LEMMA 9.1. *Let G be an H-space such that $p^r\pi_*G = 0$. Given any map $\overline{g}$: $P^d(p^{r+s}) \to \Sigma G$, there is a factorization modulo Whitehead products*

$$P^d(p^{r+s}) \xrightarrow{\epsilon} P^d(p^r) \vee P^{d+1}(p^r) \xrightarrow{\overline{h}} \Sigma G \qquad \overline{h} \circ \epsilon \equiv_w \overline{g}.$$

PROOF. Consider the adjoint $g : P^{d-1}(p^{r+s}) \to \Omega\Sigma G$ and the split fibration sequence

$$F(G) \to \Omega\Sigma G \to G$$

The base is an H-space with integral homotopy groups annihilated by p^r. Since the universal coefficient homotopy sequences are split at odd primes [**18**]

$$\pi_*(G; Z/p^{r+s}Z) = \pi_*(G) \otimes Z/p^{r+s}Z \oplus Tor(\pi_{*-1}(G), Z/p^{r+s}Z),$$

all maps $P^{d-1}(p^{r+s}) \to G$ are annihilated by p^r.

Thus, $g = g_1 + w$ where g_1 factors through G and w factors though $F(G)$. In terms of adjoints, $\overline{g} = \overline{g_1} + \overline{w}$. Since $g_1 \circ p^r = 0$, g_1 factors through $P^{d-1}(p^r) \vee P^d(p^r)$. Hence, its adjoint $\overline{g_1}$ factors factors through $P^d(p^r) \vee P^{d+1}(p^r)$. That is, $\overline{g} - \overline{w} = \overline{g_1} = \overline{h} \circ \epsilon$ factors as desired. □

If $s > 0$ and the differential is the $r - th$ Bockstein β^r, then the above cofibre map ϵ is the geometric model for the following definition.

DEFINITION 9.2. Let $A = A_1 \oplus A_2$ be a differential graded submodule of a differential graded module B and suppose that A_1 consists entirely of cycles and A_2 is acyclic. An acyclic envelope of A is a minimal acyclic submodule $E(A)$ of B which contains A. It is clear that acyclic envelopes always exist if the ambient module B is acyclic. However, acyclic envelopes are not unique. But they always are of the form $A_1 \oplus sA_1 \oplus A_2$ with the differential $d : sA_1 \xrightarrow{\cong} A_1$ being an isomorphism.

Remark: For any module A we have $dE(A) \subset A$. Furthermore, $E(A_1 \oplus A_2) = E(A_1) \oplus E(A_2)$ is a choice for an acyclic envelope for the direct sum.

The following lemma and corollary are easy to see

LEMMA 9.3. *If A_1 is a differential submodule and A_2 has zero differential in an ambient acyclic module, then $A_1 \cap A_2 = 0$ implies that $A_1 \cap E(A_2) = 0$ for any choice of an acyclic envelope.*

COROLLARY 9.4. *If $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ is a direct sum of differential submodules, then so is an acyclic envelope*

$$E(A_1 \oplus A_2 \oplus \cdots \oplus A_k) = E(A_1) \oplus E(A_2) \oplus \cdots \oplus E(A_k).$$

Scholium: We are primarily interested in differential graded modules over the ring $R = Z/p^r Z$. Over the ring $R = Z/p^r Z$, projective modules are both free and injective. Hence, if $A \subset B$ is a free submodule of a free module, then $B = A \oplus C$ for some free module C.

COROLLARY 9.5. *Let A be a differential graded module over the ring $Z/p^r Z$ and suppose A is free in all degrees. Suppose the differential is split, that is,*

$$0 \to ZA \to A \to BA \to 0$$

is split exact. Then ZA, BA, HA are all free modules and

$$A = ZA \oplus C = HA \oplus BA \oplus C$$

with $BA \oplus C$ acyclic, that is, $d : C \to BA$ is an isomorphism.

The above corollary shows that, in the situation which occurs in the semi-splitting, the explicit description that we have given for acyclic envelopes is valid.

10. Geometric acyclic envelopes

Now suppose we are in the geometric situation of the semi-splitting, that is, we have a mod p^r homology monomorphism $\overline{f} : \Sigma X \to \Sigma Y$ where ΣY is a bouquet of n connected mod p^r Moore spaces, $n \geq 2$ and ΣX is bouquet of spheres and mod p^{r+s} Moore spaces with possibly varying $s \geq 0$. We are given a map $\overline{g} : \Sigma X \to \Sigma G$ where G is an H-space for which all of the integral homotopy groups are annihillated by p^r. We seek to extend this map modulo Whitehead products to a map $\overline{h} : \Sigma Y \to \Sigma G$, that is, we seek $\overline{h} \circ \overline{f} \equiv_w \overline{g}$.

We choose a basis for the mod p^r homology of ΣX made up of pairs of generators $x_\alpha, \beta^{r+s} x_\alpha$ for the mod p^r homologies of all the mod p^{r+s} Moore spaces in the bouquet decomposition of ΣX. If a sphere ΣS^ℓ is also a part of this bouquet decomposition, we include in this basis a generator e_ℓ of dimension $\ell + 1$.

Via the embedding of mod p^r homologies, we now extend this basis to a basis for the mod p^r homology of ΣY so that it includes a basis for the acyclic envelopes.

That is, if $s > 0$, each pair of generators has in this basis two associated generators y_α, z_α with $\beta^r y_\alpha = x_\alpha, \beta^r z_\alpha = \beta^{r+s} x_\alpha$. If $s = 0$, we just have the two generators x_α and $\beta^r x_\alpha$. If e_ℓ occurs, we also have a generator z with $\beta^r z = e_\ell$. Thus, a basis for the mod p^r homology of ΣY includes the acyclic pairs

a) y_α, x_α, and $z_\alpha, \beta^{r+s} x_\alpha$, when $s \geq 0$.

b) $x_\alpha, \beta^r x_\alpha$ when $s = 0$.

c) z, e_ℓ

In every such pair, the $r-th$ Bockstein of first element is the second element.

Furthermore, the span of all these generators is an acyclic submodule of the mod p^r homology of ΣY. Since this homology is also β^r acyclic, we can choose an acyclic complement to this span and add acycllc pairs $z_\gamma, \beta^r z_\gamma$ so that the totality of the pairs is a basis for the reduced mod p^r homology of ΣY.

Since ΣY is a bouquet of mod p^r Moore spaces, the mod p^r Hurewicz map is surjective. Hence we can choose these mod p^r Moore spaces so that the generators for each of the mod p^r Moore spaces are a part of this basis.

In other words, we have chosen the bouquet decomposition of ΣY so that the resulting basis for mod p^r homology is consistent with the acyclic envelopes in the bouquet decomposition of ΣX.

Scholium: It follows from the Hilton-Milnor theorem [**29, 18**] that any map into a bouquet $\Sigma A \to \Sigma B \vee \Sigma C$ is a sum of three terms, the projections onto the two summands plus a sum of Whitehead products. Thus, modulo Whitehead products it is sum of the two projections. Of course, if the dimension of A is small enough and the connectivities of B and C are great enough, then the Whitehead products can be ignored. And in the conclusion of the semi-splitting theorem we are going to prove, we require an extension only modulo Whitehead products anyway.

We begin with a first approximation to the notion of geometric acyclic envelope.

We shall call $\Sigma X \to E(\Sigma X) = E$ a total geometric acyclic envelope if E is a subbouquet of ΣY and the induced map in mod p^r homology is an acyclic envelope. Unfortunately, ϵ is not usually the same as the "embedding" $\overline{f}$. It merely agrees with it in mod p^r homology. But homology cannot see everything. Two paragraphs below we give a more precise definition of the geometric acyclic envelope when we restrict to the spheres and mod p^{r+s} Moore space in the bouquet decomposition of ΣX.

Here is the precise definition of a geometric acyclic envelope in terms of the bouquet decomposition of ΣX into a bouquet of spheres and mod p^{r+s} Moore spaces $P^d(p^{r+s})$. There are 3 cases:

1) If the sphere S^d occurs in the bouquet decomposition of ΣX, the geometric acyclic envelope is $E(S^d) = P^{d+1}(p^r)$ where the mod p^r homology of $E(S^d)$ has the acyclic basis z, e_d as above and the canonical map $\epsilon : S^d \to E(S^d)$ is the inclusion of the bottom cell..

2) If the mod p^r Moore space $P^d(p^r)$ occurs in the bouquet decomposition of ΣX, the geometric acyclic envelope is $E(P^d(p^r)) = P^d(p^r)$ where the mod p^r homology of $E(P^d(p^r))$ has the acyclic basis $x_\alpha, \beta^r x_\alpha$ as above and the canonical map $\epsilon : P^d(p^r) \to E(P^d(p^r))$ is the identity.

3) If the mod p^r Moore space $P^d(p^{r+s})$ with $s > 0$ occurs in the bouquet decomposition of ΣX, the geometric acyclic envelope is $E(P^d(p^{r+s})) = P^d(p^r) \vee P^{d+1}(p^r)$ where the mod p^r homology of $E(P^d(p^r))$ has the acyclic basis $y_\alpha, x_\alpha, \beta^r x_\alpha$ and $z_\alpha, \beta^{r+s} x_\alpha$ as above and the canonical map $\epsilon : P^d(p^{r+s}) \to E(P^d(p^{r+s}))$ is the map in Lemma 7.1..

For a fixed dimension d, let E^d be the union over all P^d of all these acyclic envelopes embedded in ΣY. Let W be a complementary bouquet to the E^d, that is, $\Sigma Y = E^d \vee W$ where both E^d and W are bouquets of mod p^r Moore spaces. Since the mod p^r homology of ΣY is β^r acyclic, so also is that of W and hence W can be chosen to be a bouquet of mod p^r Moore spaces.

Let P^d is any d-dimensional sphere or Moore space which occurs in the bouquet decomposition of ΣX and let $\epsilon : P^d \to E(P^d)$ be the canonical map.

THEOREM 10.1. *the map $\overline{f} : \Sigma X \to \Sigma Y$ restricts to*

$$P^d \xrightarrow{\epsilon+\delta+w} E^d \vee W$$

where

a) $\epsilon : P^d \to E(P^d) \subset E^d$ is the canonical map (up to addition of Whitehead products if $d = 3$.)

b) $\delta : P^d \to W$ factors through the $d-1$ skeleton of W (up to addition of Whitehead products if $d = 3$.)

c) $w : P^d \to E^d \vee W$ is a sum of Whitehead products involving the 3 skeleton.

PROOF. The Hilton-Milnor theorem says immediately that the restriction of $\overline{f}$ to P^d is a sum $\epsilon + \delta + w$ where ϵ is the projection onto E^d, δ is the projection onto W, and w is a sum of Whitehead products.

Since the smallest cells in W are dim 2 and 3, since E^d is union of moore spaces of dimension d, it follows that smallest Whitehead products in w are dimensions 2 + d-1 - 1 = d, and 3 + d-1 - 1 = d+1. All others are of higher dimension and must be zero.

Similarly, the projections of $\overline{f}$ onto E^d and onto W must decompose, up to the addition of Whitehead products involving the 3 skeleton, into the projections onto each bouquet factor. Recall that $\overline{f}$ is consistent with mod p^r homology. That is, in mod p^r homology, $\overline{f}$ agrees with ϵ onto E^d and is zero onto W.

LEMMA 10.2. [**18**] *If $d \geq 4$, then*

$$[P^d(p^{r+s}), P^d(p^r)] = Z/p^r Z = hom(H_{d-1}(P^d(p^{r+s}), H_{d-1}(P^d(p^r))).$$

$$[P^d(p^{r+s}), P^{d+1}(p^r)] = Z/p^r Z = hom(H_d(P^d(p^r)), H_d(P^{d+1}(p^r)))$$

$$[S^d, P^{d+1}(p^r)] = Z/p^r Z = hom(H_d(S^d), H_d(P^{d+1}(p^r))$$

$$[S^d, P^d(p^r)] = 0$$

If $d = 3$, the second and third equations are valid. If $d = 3$, then any map $P^3 \to P^3(p^r)$ which is zero in mod p^r homology factors through the Hopf map $S^3 \to S^2 \subset P^3(p^r)$.

The lemma asserts that any map $P^d \to E^d$ is uniquely determined by its effect in mod p^r homology, except when $d = 3$. But, even then, since the Hopf map is twice a Whitehead product, ϵ is as the theorem states.

Since $E(P^d)$ carries the mod p^r homology image of P^d, the map $\delta : P^d \to W$ is zero in mod p^r homology. It follows that the projections of δ onto any bouquet summands $S^d, P^d(p^r)$ or $P^{d-1}(p^r)$ are zero, unless $d = 3$ in which case it could be a Whitehead product. Thus δ is as the theorem states. $\square$

11. Proof of the semi-splitting

We conclude by proving Theorem 7.1, the adjoint form of the semi-splitting.

Let P^d be one of the $d-$dimensional spaces in the bouquet decomposition of ΣX. Hence, we know

LEMMA 11.1. *a) if $P^d = P^d(p^{r+s})$ with $s > 0$, $\epsilon : P^d \to E(P^d) = P^d(p^r) \vee P^{d+1}(p^r)$ is the geometric acyclic envelope,*
b) if $P^d = P^d(p^r)$, $\epsilon : P^d \to E(P^d) = P^d(p^r)$ is the geometric acyclic envelope,
c) if $P^d = S^d$, $\epsilon : P^d \to E(P^d) = P^{d+1}(p^r)$ is the geometric acyclic envelope.

The mod p^r homology monomorphism $\overline{f} : \Sigma X \to \Sigma Y$ restricts, up to addition of Whitehead products which involve the 3 skeleton of ΣY, to a map $\gamma : P^d \to E^d \vee W$ where $\gamma = \epsilon \vee \delta$ and the map $\epsilon : P^d \to E(P^d)$ is the map in the preceding lemma.. The map $\delta : P^d \to W$ induces zero in mod p^r homology, and up to addition of Whitehead products which involve the 3 skeleton, δ compresses into the $d-1$ skeleton of W.

The algebraic corollary above insures that the geometric acyclic envelopes are "disjoint" for all components of the bouquet decomposition, that is, over all dimensions d and over all the spheres and Moore spaces P^d, the geometric envelopes $E(P^d)$ fit together in a large bouquet of mod p^r Moore spaces inside the range ΣY.

We shall define $\overline{h} : \Sigma Y \to \Sigma G$ so that $\overline{h} \circ \overline{f} \equiv_w \overline{g}$ modulo Whitehead products.

We suppose that we have defined $\overline{h}$ on the union of all the mod p^r Moore spaces $P^\ell(p^r)$ in W with $\ell \leq d$. Let P^d be a sphere or Moore space of dimension d which occurs in the bouquet decomposition of ΣZ.

According to Theorem 9.1, $\overline{f}$ restricts to $\epsilon + \delta + w : P^d \to E^d \vee W = \Sigma Y$ where w is a sum of Whitehead products which involve the 3 skeleton.

If $d = 3$, the map $\overline{g}$ restricts to zero on P^d and we can define $\overline{h}$ to be zero on $E(P^d)$.

Suppose now that $d \geq 4$.

In all three cases of P^d, we can factor modulo Whitehead products $\overline{g} \equiv_w \overline{a} \circ \epsilon$ and $\overline{h} \circ \delta \equiv_w \overline{b} \circ \epsilon$. On $E(P^d)$, define $\overline{h} = \overline{a} - \overline{b}$.

Then on P^d we have the equation $\overline{f} = \epsilon + \delta + w$ where w is a sum of Whitehead products involving the 3 skeleton and hence is sent to zero by $\overline{h}$.

Hence, modulo Whitehead products

$$\overline{h} \circ \overline{f} = \overline{h} \circ (\epsilon + \delta) = \overline{h} \circ \epsilon + \overline{h} \circ \delta \equiv_w \overline{a} \circ \epsilon - \overline{b} \circ \epsilon + \overline{h} \circ \delta = \overline{a} \circ \epsilon = \overline{g}.$$

Thus, we have extended $\overline{h}$ to the geometric acyclic envelope $E(P^d)$. We can do this for all the $d-$dimensional pieces P^d of the bouquet decomposition of ΣX. And we can then do the same for all dimensions d.. That is, we have shown that, when restricted to any bouquet summand $P^d \subset \Sigma X$, the following diagram commutes modulo Whitehead products

$$\begin{array}{ccc} \Sigma Y & \xrightarrow{\overline{h}} & \Sigma G \\ \uparrow \overline{f} & \nearrow \overline{g} & \\ \Sigma X & & \end{array}$$

Write $\Sigma X = \bigvee P^d = \Sigma \bigvee P^{d-1}$ as spaces but not as suspensions. In other words, if

$$\Theta : \bigvee P^{d-1} \to \Omega\Sigma X$$

and

$$\Psi : \bigvee P^{d-1} \to \Omega\Sigma G$$

are the respective adjoints, the following commutes

$$\begin{array}{ccccc} \Omega\Sigma Y & \xrightarrow{\Omega\overline{h}} & \Omega\Sigma G & \xrightarrow{\pi} & G \\ \uparrow \Theta & & & & \uparrow = \\ \bigvee P^{d-1} & \xrightarrow{\Psi} & \Omega\Sigma G & \xrightarrow{\pi} & G \end{array}$$

Since G is homotopy associative, we can extend Θ and Ψ to multiplicative maps

$$\Omega\Sigma \bigvee P^{d-1} = \Omega\Sigma X \to \Omega\Sigma Y$$

and

$$\Omega\Sigma \bigvee P^{d-1} = \Omega\Sigma X \to \Omega\Sigma G.$$

Since these extensions are unique, the first map is $\Omega\Sigma f = \Omega\overline{f}$ and the second map is $\Omega\overline{g}$. Note that

$$\begin{array}{ccccc} \Omega\Sigma Y & \xrightarrow{\Omega\overline{h}} & \Omega\Sigma G & \xrightarrow{\pi} & G \\ \uparrow & & & & \uparrow = \\ \Omega\Sigma X & \xrightarrow{\Omega\overline{g}} & \Omega\Sigma G & \xrightarrow{\pi} & G \end{array}$$

commutes since the maps are multiplicative and $\bigvee P^{d-1}$ is a "generating module" for $\Omega\Sigma X$.

Hence, the following is commutative

$$\begin{array}{ccccccc} Y & \xrightarrow{\Sigma} & \Omega\Sigma Y & \xrightarrow{\Omega\overline{h}} & \Omega\Sigma G & \xrightarrow{\pi} & G \\ \uparrow f & & \uparrow \Omega\Sigma f & & & & \uparrow = \\ X & \xrightarrow{\Sigma} & \Omega\Sigma X & \xrightarrow{\Omega\overline{g}} & \Omega\Sigma G & \xrightarrow{\pi} & G \end{array}$$

Since the bottom composition is $\pi \circ \underline{g} : X \to \Omega\Sigma G \to G$, the top composition $\pi \circ \underline{h} : Y \to \Omega\Sigma G \to G$ is the required extension for the semi-splitting.

$\square$

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