Existence of localization

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Abstract

This is an expository paper on the existence of the localization or nullification of abelian groups. We show that there exists a nullification functor for any fixed abelian group M. The proof is a variation of a proof due to Gustavo Granja.

1 Definitions

The localization theory or nullification theory of abelian groups is the consequence of a fixed abelian group M being declared to be locally equivalent to zero.

An abelian group X is $M-$**null or** $0 \to M$ **local** if

$$0 = Ext^*(M, X), \quad * = 0, 1,$$

that is, $0 = hom(M, X) = Ext^1(M, X)$.

A homomorphism $f : A \to B$ of abelian groups is a **local equivalence** if, for all local X,

$$f^* : hom(B, X) \to hom(A, X)$$

is a bijection.

A homomorphism $\iota : X \to L_M X = LX$ is localization if

1) ι is a local equivalence.

2) LX is local.

Standard arguments show:

a) If localization exists, then any homomorphism $f : X \to Y$ into a local $R-$module Y has a unique extension to a map $g : LX \to Y$, that is, $g \cdot \iota = f$.

b) If it exists, localization is a functor and unique up to natural isomorphism.

2 Preliminaries on mapping cones and chain homotopy classes of maps

Let $f : A \to B$ be any chain map of complexes. The mapping cone $C_f = B \oplus sA$ is the chain complex with differential $\overline{d}$ given by

$$\overline{d}b = db \quad b \in B$$

$$\overline{d}(sa) = -s(da) + fa, \quad a \in A.$$

Let

$$A \xrightarrow{f} B \xrightarrow{\iota} C_f \xrightarrow{j} sA$$

be the natural sequence of maps of chain complexes. It is clear that

Lemma: There is a chain homotopy $\iota \circ f \simeq_D 0 : A \to C_f$ **given by**

$$Da = sa, \quad a \in A.$$

Let Y be any chain complex and let $[A, Y]$ denote chain homotopy classes of maps from A to Y. Then:

Lemma: If $g : B \to Y$ **is a chain map, then** $g \circ f$ **is chain homotopic to zero, that is,** $g \circ f \simeq_D 0$ **if and only if** g **extends to a chain map** $G : C_f \to Y$.

In the above lemma, the correspondence between D and the extension G is given by

$$G(sa) = D(a), \quad a \in A.$$

Lemma: If $g : C_f \to Y$ **is a chain map, then there is a chain homotopy** $g \circ \iota \simeq_D 0 : B \to Y$ **if and only if there exists a chain map** $G : sA \to Y$ **such that there is a chain homotopy** $G \circ j \simeq_E g : C_f \to Y$.

Given the homotopy D in the above lemma, we define a chain map $\overline{g} : C_f \to Y$ by

$$\overline{g}b = gb, \quad b \in B, \quad \overline{g}(sa) = D(fa), \quad a \in A.$$

Then $g - \overline{g} = G \circ j$ for a chain map $G : sA \to Y$ and $\overline{g} \simeq_F 0$ via the chain homotopy

$$Fb = Db, \quad b \in B, \quad F(sa) = 0, \quad a \in A.$$

The implication in the other direction is obvious.

Hence,

Corollary: The sequence of chain homotopy classes of maps is exact:

$$[A, Y] \xleftarrow{f^*} [B, Y] \xleftarrow{\iota^*} [C_f, Y] \xleftarrow{j^*} [sA, Y].$$

3 Localization exists

Suppose that an abelian group M has a free resolution:

$$0 \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0.$$

Then, if X is any other abelian group,

$$0 \leftarrow Ext(M,X) \leftarrow hom(P_1,X) \leftarrow hom(P_0,X) \leftarrow hom(M,X) \leftarrow 0$$

is exact.

Hence, if we regard M and X as complexes concentrated in degree 0, there is an isomorphism

$$hom(M,X) \xrightarrow{\simeq} hom(P_*,X) = [P_*,X]$$

where $hom(P_*,X)$ is the module of degree 0 chain maps and $[P_*,X]$ is the module of chain homotopy classes of chain maps.

If we denote by $s^{-1}P_*$ the desuspension of the complex P_*, then there is an epimorphism

$$hom(s^{-1}P_*,X) \to Ext^1(M,X)$$

and a resulting isomorphism

$$[s^{-1}P_*,X] \xrightarrow{\simeq} Ext^1(M,X)$$

where $[s^{-1}P_*,X]$ is the group of chain homotopy classes of chain maps.

Given an abelian group X, we define an abelian group X_H by requiring

$$\bigoplus_f M \xrightarrow{F} X \to X_H \to 0$$

to be exact where F is the sum of all the homomorphisms $f : M \to X$. It is clear that the map

$$hom(M,X) \to hom(M,X_H)$$

is zero and that $X \to X_H$ is a local equivalence.

Given an abelian group Y we define a positively graded complex Y_E by taking the mapping cone of the map of complexes

$$\bigoplus_g s^{-1}P_* \xrightarrow{G} Y$$

where G is the sum of all maps of complexes

$$g : s^{-1}P_* \to Y.$$

Thus,

$$Y_E = Y \oplus \bigoplus_g P_*$$

with differential $\overline{d}$:

$$\overline{d}y = 0, \quad \overline{d}p_0 = 0, \quad \overline{d}p_1 = gs^{-1}p_1 - dp_1, \quad \overline{d}p_n = -dp_n, n > 1.$$

We have a sequence of complexes

$$\bigoplus_g s^{-1}P_* \xrightarrow{G} Y \to Y_E \to \bigoplus_g P_*.$$

It is clear that the map

$$Ext^1(M, Y) = [s^{-1}P_*, Y] \to [s^{-1}P_*, Y_E]$$

is zero.

From the above sequence of complexes, we get for any $R-$module Z, the exact sequence of chain homotopy classes of maps

$$[\bigoplus_g s^{-1}P_*, Z] \leftarrow [Y, Z] \leftarrow [Y_E, Z] \leftarrow [\bigoplus_g P_*, Z].$$

In particular, if Z is local, then

$$hom(Y, Z) \leftarrow hom(Y_E, Z) \leftarrow 0$$

is an isomorphism.

Given a positively graded complex W, we have a map of complexes

$$W \to H_0W$$

where $H_0W = W_0/d_1(W_1)$ is the homology in dimension zero. The map

$$hom(W, Z) \overset{\simeq}{\longleftarrow} hom(H_0W, Z)$$

is an isomorphism for all abelian groups Z.

Finally, for any abelian group X, we define the abelian group

$$X_+ = H_0((X_H)_E)$$

so that we have homomorphisms

$$X \xrightarrow{\iota_1} X_H \xrightarrow{\iota_2} (X_H)_E \xrightarrow{\iota_3} H_0((X_H)_E) = X_+.$$

and we know that:

1)

$$X \to X_+$$

is a local equivalence.

2)

$$Ext^*(M, X) \to Ext^*(M, X_+)$$

is zero for $* = 0, 1$.

We now construct the localization LX by transfinite recursion.

First of all, let C be an infinite cardinal which is greater than or equal to the cardinality of a generating set of M. Then $C \cdot C = C$ implies that C is greater than or equal to the cardinality of the free resolution $P_* \to M$.

We record the fact that $C \cdot C = C$ implies that:

Proposition: If Γ is the first ordinal or cardinality greater than C and $\mathcal{B}$ is a set of cardinality less than C consisting of ordinals less than Γ, then

$$\sup \mathcal{B} = \bigcup_{\beta \in \mathcal{B}} \beta < \Gamma.$$

Remark: If $C = \aleph_0$ is the cardinality of the integers, then $\Gamma = \Omega$ the first uncountable ordinal and the above proposition says that Ω is not a sequential limit of countable ordinals.

Let α be an ordinal and set

$$X_0 = X$$

$$X_\beta = (X_\alpha)_+ \qquad \text{whenever } \beta = \alpha + 1 \text{ is a successor ordinal}$$

$$X_\beta = \lim_{\alpha < \beta} X_\alpha \qquad \text{whenever } \beta = \text{limit ordinal}$$

$$LX = X_\Gamma \qquad \lim_{\alpha < \Gamma} X_\alpha$$

That is, the localization sits at the end of the process

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_\omega \to X_{\omega+1} \to \ldots$$

$$\to X_\alpha \to \cdots \to \lim_{\alpha < \Gamma} X_\alpha = X_\Gamma = LX.$$

We claim that $\iota : X \to LX$ is localization. We need to check that ι is a local equivalence and that LX is local.

1) For all local Y,

$$hom(X_\Omega, Y) = hom(\lim X_\alpha, Y) = lim_{\leftarrow} hom(X_\alpha, Y) \simeq hom(X, Y)$$

since an inverse limit of isomorphisms is an isomorphism. Thus, ι is a local equivalence.

2) Let P_* is a free resolution of M with the cardinality of the resolution less than or equal to C.

Let $f : s^i P_* \to X_\Gamma$ be any map of chain complexes with i equal to 0 or -1.

For all elements $x \in P_*$, there exists an ordinal

$$\alpha(x) < \Gamma$$

such that $f(s^i x) \in X_{\alpha(x)}$ and thus f factors as

$$s^i P_* \to \bigcup_{x \in P_*} X_{\alpha(x)} = \varinjlim X_{\alpha(x)} = X_{\lim \alpha(x)} = X_\beta.$$

Since Γ is not a limit of lesser ordinals indexed by a set of cardinality less than or equal to C, it follows that $\beta < \Gamma$ and that $\beta + 1 < \Gamma$. Hence

$$s^i P_* \to X_\beta \to X_{\beta+1} \to X_\Gamma$$

induces zero on chain homotopy classes of maps. Thus, $LX = L_M X = X_\Gamma$ is $M-$null or local.