CHAPTER 4
PARTITIONS OF UNITY
AND SMOOTH FUNCTIONS

In this section, we construct a technical device for extending some local constructions to global constructions. It is called a partition of unity. We also use the opportunity to discuss C^∞ functions. We begin with examples of C^∞ functions on $\mathbf{R}$ and $\mathbf{R}^n$. Some of these are required for the construction of partitions of unity. At the end of the chapter, we return to C^∞ funcitons and apply our new techniques. The first examples we construct, Examples 4.1abc***, are standard and we follow Warner.

Example 4.1a*.** *A function on* $\mathbf{R}$ *which is* C^∞ *but not analytic. Let*

$$f(t) = \begin{cases} \dfrac{1}{e^{1/t}} & \text{if } t > 0 \\ 0 & \text{if } t \le 0. \end{cases}$$

We show that for each $n = 0, 1, 2, 3, \cdots$ there is a polynomial p_n such that

$$f^{(n)}(t) = \begin{cases} p_n(1/t)\dfrac{1}{e^{1/t}} & \text{if } t > 0 \\ 0 & \text{if } t \le 0. \end{cases}$$

We use mathematical induction to show the claim. The claim is true for $n = 0$ as $f^{(0)} = f$. Here $p_0(x) = 1$. For $k > 0$ we separately handle $t > 0$, $t < 0$ and $t = 0$. If $f^{(k)}(t) = p_k(1/t)\frac{1}{e^{1/t}}$ for $t \in (0, \infty)$, then

$$f^{(k+1)}(t) = p'_k(1/t)(-\frac{1}{t^2})\frac{1}{e^{1/t}} + p_k(1/t)(\frac{1}{t^2})\frac{1}{e^{1/t}}$$

and $p_{k+1}(x) = -x^2 p'_k(x) + x^2 p_k(x)$. For $t \in (-\infty, 0)$, f is zero and so are its derivatives of all orders. At $t = 0$ we use the definition of the derivative.

$$\begin{aligned} \lim_{h \to 0^+} \frac{f^{(k)}(h) - f^{(k)}(0)}{h} &= \lim_{h \to 0^+} \frac{p_k(1/h)\frac{1}{e^{1/h}}}{h} \\ &= \lim_{x \to \infty} \frac{x p_k(x)}{e^x} \\ &= 0. \end{aligned}$$

The substitution was $x = 1/h$, and the last limit was a polynomial divided by an exponential: a standard L'Hopitals Theorem example from Calculus. The limit, $\lim_{h \to 0^-} \frac{f^{(k)}(h) - f^{(k)}(0)}{h}$ is zero as the numerator is identically zero.

Example 4.1b*.** *Let*

$$g(t) = \frac{f(t)}{f(t)+f(1-t)} = \begin{cases} 0 & \text{if } t \le 0 \\ \dfrac{1}{1+\frac{e^{1/t}}{e^{1/(1-t)}}} & \text{if } 0 < t < 1 \\ 1 & \text{if } t \ge 1. \end{cases}$$

Then g is C^∞ on $\mathbf{R}$ and g is strictly increasing on $[0,1]$.

The function g is C^∞ since it is a quotient of C^∞ functions and the denominator is never 0. If $t \le 0$, then $f(t) = 0$ so $g(t) = 0$. If $t \ge 1$, then $f(1-t) = 0$ so $g(t) = \frac{f(t)}{f(t)} = 1$.

Recall that $f'(t) = \frac{1}{t^2} f(t)$ for $t > 0$, and so, $\frac{d\,f(t)}{dt} = -f'(1-t)$. Now, for $t \in (0,1)$,

$$\begin{aligned} g'(t) &= \frac{f'(t)(f(t)+f(1-t)) - f(t)(f'(t)-f'(1-t))}{(f(t)+f(1-t))^2} \\ &= \frac{f'(t)f(1-t)+f(t)f'(1-t)}{(f(t)+f(1-t))^2} \\ &= \frac{f(t)f(1-t)}{(f(t)+f(1-t))^2}\left(\frac{1}{t^2}+\frac{1}{(1-t)^2}\right) \end{aligned}$$

This expression is positive on $(0,1)$. Hence g is strictly increasing on $[0,1]$. The graph of g is as follows.

Example 4.1c*.** *The bump function on $(-2,2)$. Let*

$$h(t) = g(t+2)g(2-t) = \begin{cases} 0 & \text{if } t \le -2 \\ g(t+2) & \text{if } -2 < t < -1 \\ 1 & \text{if } -1 \le t \le 1 \\ g(2-t) & \text{if } 1 < t < 2 \\ 0 & \text{if } 2 \le t \end{cases}$$

The function h is C^∞ on all of $\mathbf{R}$, $h(x) = 0$ if $x \in (-\infty,-2] \cup [2,\infty)$, $h(x) = 1$ if $x \in [-1,1]$, h is strictly increasing on $[-2,-1]$, and strictly decreasing on $[1,2]$.

Its graph is shown below.

Let $C(r)$ denote the open cube in $\mathbf{R}^n$ which is $\{(x_1,\cdots,x_n) | x_i \in (-r,r) \text{ for } i = 1,2,3,\cdots,n\}$ and let $\bar{C}(r)$ be its closure.

Example 4.2*.** *The bump function on $C(2) \subset \mathbf{R}^n$. Let*

$$b(x_1,\cdots,x_n) = \Pi_{i=1}^n b(x_i).$$

Then $b(x_1,\cdots,x_n) = 0$ on the complement of $C(2)$, $b(x_1,\cdots,x_n) = 1$ for $(x_1,\cdots,x_n)$ in the closure of $C(1)$, and $0 < b(x_1,\cdots,x_n) < 1$ otherwise.

Definition 4.3*.** *If $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ is an open cover of a manifold M, then a subset of $\mathcal{U}$ which is also a cover is called a subcover.*

Definition 4.4*.** *If $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ is an open cover of a manifold M, then the open cover $\mathcal{V} = \{V_\gamma | \gamma \in \Gamma\}$ is a refinement if for all $\gamma \in \Gamma$ there is an $\alpha \in A$ such that $V_\gamma \subset U_\alpha$.*

Definition 4.5*.** *A collection of subsets $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ of a manifold M is called locally finite, if for all $m \in M$ there is an neighborhood O of m with $U_\alpha \cap O \neq \emptyset$ for only a finite subset of A.*

Definition 4.6*.** *A partition of unity on a manifold M is a collection of smooth functions $\{\phi_i : M \to \mathbf{R} \mid i \in I\}$ such that*

1. *$\{$ the support of $\phi_i \mid i \in I\}$ is locally finite*
2. *$\phi_i(p) \geq 0$ for all $p \in M$, $i \in I$, and,*
3. *$\sum_{i \in I} \phi_i(p) = 1$ for all $p \in M$.*

Note that the sum is finite for each p.

Definition 4.7*.** *The partition of unity on a manifold M $\{\phi_i \mid i \in I\}$ is subordinate to the open cover $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ if for all $i \in I$ there is an $\alpha \in A$ such that the support of ϕ_i is in U_α.*

Lemma 4.8*.** *Suppose M is a connected manifold. Then there is a sequence of open sets O_i such that*

1. *$\bar{O}_i$ is compact*
2. *$\bar{O}_i \subset O_{i+1}$*
3. *$\displaystyle\bigcup_{i=1}^{\infty} O_i = M$*

Proof. Take a countable basis for the topology of M (as M is second countable) and for each $x \in M$ pick a compact set K_x that contains x in its interior (as M is locally compact). Since M is Hausdorff, we obtain another basis for the topology of M by keeping only those basis elements which are in some K_x. We now have a countable basis $\mathcal{U} = \{U_i | i = 1, 2, 3, \cdots\}$ such that if $U_i \in \mathcal{U}$ then $\bar{U}_i$ is compact.

Let $O_1 = U_1$. Each of the other open sets will have the form $O_k = \bigcup_{i=1}^{j_k} U_i$. Suppose we have constructed O_k. We show how to construct O_{k+1}. Since $\bar{O}_k$ is compact let j_{k+1} be the smallest counting number with $\bar{O}_k \subset \bigcup_{i=1}^{j_{k+1}} U_i$. We establish the required properties. Since $\bar{O}_k = \bigcup_{i=1}^{j_k} \bar{U}_i$ is a finite union of compact spaces, it is compact. By construction, $\bar{O}_k \subset O_{k+1}$. If $\bar{O}_k \subset O_k$ then $O_k = M$ as O_k is open and closed, otherwise $j_{k+1} > j_k$. Therefore $\bigcup_{i=1}^{k} U_i \subset O_k$ so $\displaystyle\bigcup_{i=1}^{\infty} O_i = M$, and (3) follows. □

Proposition 4.9*.** *Suppose $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ is a basis for the manifold M with $\bar{U}_\alpha$ compact for all $\alpha \in A$. Suppose $\mathcal{W} = \{W_\beta | \beta \in B\}$ is any open cover. Then there is a countable locally finite refinement of $\mathcal{W}$, $\mathcal{V} = \{V_i | i \in I\}$, with $V_i \in \mathcal{U}$ for all $i \in I$.*

Proof. If the manifold has more than one component then we separately handle each component, hence we assume that M is connected. By the previous lemma, there is a collection of open sets $\{O_i | i = 1, 2, 3, \cdots\}$ which satisfy the conditions of the lemma. Now $\bar{O}_{i+1} \setminus O_i$ is compact and contained in the open set $O_{i+2} \setminus \bar{O}_{i-1}$. Note that $\mathcal{O} = \{O_{i+2} \setminus \bar{O}_{i-1}, O_4 \mid i = 3, 4, \cdots\}$ is an open cover of M.

We construct $\mathcal{V}$ by reducing $\mathcal{U}$ in two steps. Let $\mathcal{U}'$ be the set of all $U_\alpha \in \mathcal{U}$ such that there is a β with $U_\alpha \subset W_\beta$ and that $U_\alpha \subset O_{i+2} \setminus \bar{O}_{i-1}$ or O_4. The set $\mathcal{U}'$ is a basis since $\mathcal{O}$ and $\mathcal{W}$ are open covers and M is Hausdorff, i.e., for each $x \in M$, $\mathcal{U}'$ contains a nbhd basis. Take a finite subset of $\mathcal{U}'$ each of which is in O_4 and covers $\bar{O}_3$, a compact set. For each $i > 2$ take finite subsets of $\mathcal{U}'$ each of which is in $O_{i+2} \setminus \bar{O}_{i-1}$ and covers $\bar{O}_{i+1} \setminus O_i$, a compact set. The union of these various finite collections $\mathcal{V} = \{V_j | j \in I\}$ is locally finite since an open set in the i^{th} collection can only meet open sets from the $(i-2)^{nd}$ collection up through the $(i+2)^{nd}$. These are each finite collections. The set $\mathcal{V}$ is a countable union of finite sets and so countable. The cover $\mathcal{V}$ is subordinate to $\mathcal{W}$ since $\mathcal{U}'$ is subordinate to $\mathcal{W}$. □

Lemma 4.10*.** *Suppose M is a manifold. Then there is a basis $\{U_\alpha | \alpha \in A\}$ such that*

1. *$\bar{U}_\alpha$ is compact and*
2. *For each $\alpha \in A$ there is a smooth function $\varphi_\alpha : M \to \mathbf{R}$ such that $\varphi_\alpha(x) = 0$ if $x \notin U_\alpha$ and $\varphi_\alpha(x) > 0$ if $x \in U_\alpha$.*

Notice that the function guaranteed in the lemma cannot be analytic but must be C^∞. For example on $\mathbf{R}$ the support of φ_α is compact and so the function is zero on (m, ∞) for some m. Hence if $x > m$, then $\varphi_\alpha^{(n)}(x) = 0$, for $n = 1, 2, 3, \cdots$.

Proof. First note that $\mathcal{R}(m) = \{\phi^{-1}(C(2)) | (U, \phi) \text{is a chart centered at } m \text{ and } C(3) \subset \phi(U)\}$ is a neighborhood basis at m. If fact, if (U, ϕ) is any chart centered at m the charts $(U, k\phi)$ for k large suffice. Now, we produce the function. If $R \in \mathcal{R}(m)$, then there is a C^∞ function $\varphi_R : M \to \mathbf{R}$ with $\varphi_R(x) = 0$ if $x \notin R$ and $\varphi_R(x) > 0$ if $x \in R$. If (U, ϕ) is a chart with $C(3) \subset \phi(U)$ and $R = \phi^{-1}(C(2)) \in \mathcal{R}(m)$, then take

$$\varphi_R(x) = \begin{cases} b \circ \phi(x) & \text{if } x \in U \\ 0 & \text{if } x \notin \bar{R} \end{cases}$$

Here b is the C^∞ function we produced on $\mathbf{R}^n$ with $b(x) > 0$ for $x \in C(2)$ and $b(x) = 0$ for $x \notin C(2)$. The function φ_R is smooth since $b \circ \phi$ and 0 agree on the overlap of their domains, $U \setminus \bar{R}$ an open set. □

Theorem 4.12*.** *If M is a manifold and $\mathcal{W}$ is any open cover, then M admits a countable partition of unity subordinate to the cover $\mathcal{W}$ with the support of each function compact.*

Proof. Apply Proposition 4.9*** to the basis constructed in Lemma 4.10***. We obtain a locally finite collection $\{U_i | i = 1, 2, 3, \cdots\}$ with $\varphi_i : M \to \mathbf{R}$ as in the lemma. Let $\varphi(x) = \sum_{i=0}^{\infty} \varphi_i(x)$. Recall that for each x there is an open set O with $x \in O$ and $\varphi_i = 0$ on O for all but finitely many i. The sum is finite for each x. It is the fact that locally finite requires an open set about each x that meets only a finite number of U_i (say $i = 1, \cdots, m$)

that gives the C^∞ differentiablity, since on the open set O the function φ is a finite sum of smooth functions: the composition of

$$M \xrightarrow{(\varphi_1,\cdots,\varphi_m)} \mathbf{R}^m \xrightarrow{\Sigma} \mathbf{R}.$$

If only each x were in a finite number of U_i, then we could only guarantee well-defined but not smooth. Let $\psi_i = \dfrac{\varphi_i}{\varphi}$. Then $\{\psi_i | i = 1, 2, 3, \cdots\}$ form a partition of unity with the support of ψ_i being U_i. $\square$

It is interesting to note that there is no special field to study the zeros of C^∞ functions. Algebraic geometry studies the sets which are zeros of polynomials, and the nature of the zeros of analytic functions is also studied, but the sets which are zeros of smooth functions does not give a new area of study. The reason is:

Theorem 4.13*.** *If $X \subset M$ is closed, then there is a smooth function $f : M \to \mathbf{R}$ with $f(x) = 0$ if and only if $x \in X$.*

Proof. Let $N = M \setminus X$. N is an open subset of M and so a manifold. Let $\{U_\alpha | \alpha \in A\}$ be the basis of M produced in the lemma. Let $\mathcal{U} = \{U_\alpha | \alpha \in A \text{ and } U_\alpha \subset N\}$. The collection $\mathcal{U}$ is a basis for N with each $\bar{U}_\alpha$ compact for $U_\alpha \in \mathcal{U}$. We apply Proposition 4.9*** to the manifold N with $\mathcal{W} = \{N\}$ to get a countable locally finite subcollection of $\mathcal{U}$, $\{U_i | i = 1, 2, 3, \cdots\}$. Each U_i is equipped with $\phi_i : M \to \mathbf{R}$ whose support is exactly U_i, as in Lemma 4.10***. Let $f = \displaystyle\sum_{i=1}^{\infty} \phi_i$. The function f is C^∞ and is nonzero exacty on $\cup_{i=1}^{\infty} U_i = N$ $\square$

Theorem 4.13*** demonstrates a dramatic array of possible behavior of C^∞ functions compared to analytic functions. To observe the restricted nature of the zeros of analytic fuctions, we consider functions on the real line.

Theorem 4.14*.** *Suppose $f : \mathbf{R} \to \mathbf{R}$ and $X = \{x \mid f(x) = 0\}$.*

(1) *If f is a polynomial, then X is a finite set.*
(2) *If f is an analytic function, then X is a discrete set.*
(3) *If f is a C^∞ function, then X can be an arbitrary closed set.*

Proof. An n-th degree polynomial has at most n zeros, which shows the first item. The third item follows from Theorem 4.13***.

To show the second item we suppose that the zeros of f are not discrete and show that implies f is identically zero. Suppose that $p_m \in X$ for $m = 1, 2, 3, \cdots$ and $\lim_{m\to\infty} p_m = p$. Now,

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(p)(x-p)^n.$$

In order to show f is the zero function, it is enough to show that $f^{(n)}(p) = 0$ for all whole numbers n.

We show that $f^{(n)}(p) = 0$ for all whole numbers n by mathematical induction. We first observe it is true for $n = 0$, $f^{(0)} = f$. By continuity, $\lim_{m\to\infty} f(p_m) = f(p)$ and each

$f(p_m) = 0$, and therefore, $f(p) = 0$. We now assume that $f^{(k)}(p) = 0$ for $k < n$ and show this implies $f^{(n)}(p) = 0$. First note that

$$\lim_{m\to\infty} n!\frac{f(p_m)}{(p_m - p)^n} = \lim_{m\to\infty} n!\frac{0}{(p - p_m)^n} = 0.$$

Now compute the same limit using L'Hospital's rule,

$$\begin{aligned} 0 = \lim_{m\to\infty} n!\frac{f(p_m)}{(p_m - p)^n} &= \lim_{x\to p} n!\frac{f(x)}{((x-p)^n} \\ &= \lim_{x\to p} n!\frac{f'(x)}{n(x-p)^(n-1)} \\ &\vdots \\ &= \lim_{x\to p} n!\frac{f^{(n-1)}(x)}{n!(x-p)}. \end{aligned}$$

By the induction hypothesis, these limits are all of the indeterminate form $\frac{0}{0}$. One last application of L' Hopitals rule yields,

$$\lim_{m\to\infty} n!\frac{f(p_m)}{(p_m - p)^n} = \lim_{x\to p} n!\frac{f^n(x)}{n!} = f^{(n)}(p)$$

the last equality by continuity of the n-th dervative. Therefore $f^{(n)}(p) = 0$ for all whole numbers n, and f is identically zero. □

Exercises

Exercise 1*.** *Show that the bump function b on $\mathbf{R}^n$ has the following property: if $b(x) = 0$, then all of the derivatives of b at x are also zero. Show that if $X \subset \mathbf{R}^n$ is a closed set, then there is a function $f : \mathbf{R}^n \to \mathbf{R}$ such that $f(x) = 0$ if and only if $x \in X$ and all partials of f vanish on X.*

Exercise 2*.** *Assume the following version of the Stone-Weirerstrass Theorem: If $K \subset \mathbf{R}^n$ is compact and $f : K \to \mathbf{R}$ is continuous, then given any $\epsilon > 0$ there is a polynomial function g such that $|\ g|_K\ (x) - f(x)| < \epsilon$ for all $x \in K$.*

Prove that is M is a smooth manifold and $f : M \to \mathbf{R}$ is continuous, then given any $\epsilon > 0$ there is a C^∞ function $g : M \to \mathbf{R}^n$ such that $|f(x) - g(x)| < \epsilon$ for all $x \in M$.