LATTICE POINTS AND GENERALIZED DIOPHANTINE CONDITIONS

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Abstract. We prove that $(\int_0^{2\pi} (|N_\theta(t) - |D|t^2|)^2 d\theta)^{\frac{1}{2}} = O(t^{\frac{2}{3}})$, where D_θ is a rotation of a convex domain in $\mathbb{R}^2$ and $N_\theta(t) = \#\{\mathbb{Z}^2 \cap tD_\theta\}$. It follows that for any $\delta > 0$, there exists a set of θ's of measure $2\pi - \delta$, such that for $t \in \Lambda$, where Λ is any lacunary sequence, $|N_\theta(t) - |D|t^2| \le C_\Lambda t^{\frac{2}{3}} \log(t)$. Moreover, we prove, under some additional assumptions, that for almost every θ,

$$N_\theta(t) - |D|t^2 = O(t^{\frac{2}{3}}), \tag{*}$$

up to a small logarithmic transgression.

We also prove that if D is convex and finite type, and also in some infinite type situations, $N_\tau(t) = \#\{\mathbb{Z}^2 \cap tD + \tau\}$, $\tau \in \mathbb{T}^2$, the two-dimensional torus, and

$$\big|\big|N_\tau(t) - t^2|D|\big|\big|_{L^2(\mathbb{T}^2)} \le Ct^{\frac{1}{2}}, \tag{**}$$

the optimal bound, then $N_{(0,0)}(t) = t^2|D| + O(t^{\frac{2}{3}})$. We conclude that $(**)$ cannot in general hold if the boundary of D has order of contact ≥ 4 with one of its tangent lines.

Section 0: Introduction

Let D be a convex domain in $\mathbb{R}^2$. Let

$$N(t) = \#\{\mathbb{Z}^2 \cap tD\}, \tag{0.1}$$

where $\mathbb{Z}^2$ denotes the standard integer lattice. Let $R(t) = N(t) - |D|t^2$, where $|D|$ denotes the volume of D. It was shown by Gauss in the middle of the last century that $R(t) = O(t)$ since the measure of ∂D is $O(t)$. In the beginning of this century van der Corput, (see also [Ran]), used the Poisson summation formula and the method of stationary phase, and approximation of sums by integrals to show that $R(t) = O(t^{\frac{2}{3}})$ if one assumes, in addition, that ∂D has everywhere non-vanishing Gaussian curvature. In the case when D is a circle, or, more generally a convex domain with a sufficiently smooth boundary, there has been a

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series of improvements. See, for example, [BB]. However, in the case of a general domain where the boundary has non-vanishing Gaussian curvature, the estimate $R(t) = O(t^{\frac{2}{3}})$ is reasonable in the sense that given t large, one can construct a domain with curvature bounded below by $\approx \frac{1}{t^2}$, such that the boundary contains $\approx t^{\frac{2}{3}}$ lattice points. This is a result of Jarnik, see [L]. The author is unaware of an example of a fixed convex planar domain where it is know that the estimate $R(t) = O(t^{\frac{2}{3}})$ cannot be improved by replacing $\frac{2}{3}$ by a smaller power.

The case where the Gaussian curvature is allowed to vanish to finite order was treated by Randol. See [Ran]. He showed that if the Gaussian curvature vanishes to order $\leq m-2$, $m \geq 2$, then $R(t) = O(t^{1-\frac{1}{m}})$. Moreover, he showed that this result is the best possible. It is interesting to note that while sharpness of the $R(t) = O(t^{1-\frac{1}{m}})$, $m \geq 3$, is fairly easy to establish via a direct estimate (see also [CV]), the issue of sharpness of the $R(t) = O(t^{\frac{2}{3}})$ estimate in the case of non-vanishing Gaussian curvature appears to be much more subtle.

Colin de Verdiere, [CV], showed, under the same assumptions as Randol, that if the normal at the point where the Gaussian curvature vanishes has a sufficiently irrational slope, or, equivalently, if the domain is rotated by a sufficiently irrational angle θ satisfying a certain Diophantine condition, then the resulting error term $R_\theta(t) = O(t^{\frac{2}{3}})$. Moreover, he showed that this conclusion holds for almost every θ.

In this paper we shall consider the more general class of convex domains. In particular, the curvature is allowed to vanish to infinite order. Our main results are the following.

Theorem 0.1. *Let D be a convex domain in $\mathbb{R}^2$ and let D_θ denote its rotation by $\theta \in [0, 2\pi]$. Let $N_\theta(t) = \#\{\mathbb{Z}^2 \cap tD_\theta\}$, and let $R_\theta(t) = N_\theta(t) - |D|t^2$. Then*

$$\left(\int_0^{2\pi} |R_\theta(t)|^2 d\theta\right)^{\frac{1}{2}} = O(t^{\frac{2}{3}}). \tag{0.2}$$

It follows that for any $\delta > 0$, there exists a set of θ's of measure $2\pi - \delta$, such that for $t \in \Lambda$, where Λ is any lacunary sequence,

$$|R_\theta(t)| \leq C_\Lambda t^{\frac{2}{3}} \log(t). \tag{0.3}$$

Theorem 0.2. *Let D, $N_\theta(t)$, and $R_\theta(t)$ be as above. Suppose that the Gaussian curvature vanishes at isolated points.*

Suppose that in the neighborhood of each flat point, up to a rotation, ∂D is a graph of a smooth convex function γ, such that $\gamma'(0) = 0$.

Let g denote the inverse of γ'. At each point where the Gaussian curvature vanishes let

$$\Omega_\alpha(\theta) = \sup_{\beta \in [\alpha, 1]} [\gamma''(\beta g(\theta)]^{-\frac{1}{2}}, \tag{0.4}$$

$\alpha \in (0, 1)$, where $\theta = 0$ is the point where the Gaussian curvature vanishes. Suppose that there exists $0 < \alpha < 1$ and $p_0 > 1$, such that $\Omega_\alpha \in L^{p_0}(S^1)$, where S^1 denotes the unit circle.

Then, for almost every θ,

$$|R_\theta(t)| \leq Ct^{\frac{2}{3}} \log^{\delta(p_0)}(t), \tag{0.5}$$

with $\delta(p_0) > \frac{1}{p_0}$.

Remark. The reader can check that a convex domain with $\gamma'(s) = e^{-\frac{1}{s^a}}$, $a > 0$, satisfies the assumptions of Theorem 0.2 for any $0 < \alpha < 1$. Indeed, in this case

$$\Omega_\alpha(\theta) \approx \left(\theta^{\frac{1}{\alpha^\alpha}} \log^{\frac{\alpha-1}{\alpha}}(\theta)\right)^{-\frac{1}{2}}. \tag{0.6}$$

Since this quantity is in $L^p(S^1)$ for $p < 2\alpha^\alpha$, if we chose α sufficiently close to 1, the assumptions of Theorem 0.2 are satisfied for any $p < 2$.

On the other hand, if $\gamma'(s) = e^{-\frac{1}{e^{-\frac{1}{s}}}}$, the assumptions of Theorem 0.2 are not satisfied. The point here is that our proof exploits the idea that for some γ's that vanish to infinite order, the quantity $\gamma''(\beta(\gamma')^{-1}(\theta))$, only vanishes to finite order. If it vanishes to infinite order, our proof does not apply since $\Omega_\alpha \notin L^{p_0}(S^1)$ for any p_0.

The main stationary phase estimate we will need is the following result due to Podkorytov. See [Pod]. See also [BIT] for an alternate proof.

Lemma 0.3. *Let D be a convex domain in $\mathbb{R}^2$ and let $\hat{\chi}_D$ denote the Fourier transform of the characteristic function of D. Then $||\hat{\chi}_D(\lambda\cdot)||_{L^2(S^1)} \leq C|\lambda|^{-\frac{3}{2}}$. In other words,*

$$\left(\int_0^{2\pi} |\hat{\chi}_D(\lambda(cos(\theta), \sin(\theta))|^2 d\theta\right)^{\frac{1}{2}} \leq C|\lambda|^{-\frac{3}{2}}. \tag{0.7}$$

We shall also need the following oscillatory estimate.

Lemma 0.4. *Let $\gamma \in C^\infty(\mathbb{R})$ be convex and $\gamma(0) = \gamma'(0) = 0$. Then*

$$\left|\int_0^1 e^{it(\theta s \pm \gamma(s))} ds\right| \leq C_\alpha(1 + |t|)^{-\frac{1}{2}} \Omega_\alpha(\theta), \tag{0.8}$$

where $0 < \alpha < 1$.

Remark. Podkorytov's theorem (Lemma 0.3) and its earlier analogs have been used before to estimate L^2 averages of the remainder terms in lattice point expansions. See for example [Ran], [BCT], and [Var]. Indeed, let $R(t, \theta, \tau) = \sum_{k \in \mathbb{Z}^2} \chi_{t\theta^{-1}D-\tau}(k) - t^2|D|$, where $t\theta^{-1}D - \tau$ denotes the convex domain D dilated by t, rotated in the clockwise direction by the angle θ, and translated by τ. After writing the Fourier expansion for $R(t, \theta, \tau)$ on the torus in the τ variable, using Parseval's formula and Lemma 0.3, it is not hard to see that $\left(\int_{SO(2)} \int_{\mathcal{T}^2} |R(t, \theta, \tau)|^2 d\tau d\theta\right)^{\frac{1}{2}} \leq C|t|^{\frac{1}{2}}$. In other words, the average information about the Fourier transform of the characteristic function of D is used to obtain the average information about the remainder term. Our Theorem 0.1 is different in that we only average in rotation, which does not allow us to use the Fourier series approach. Moreover, in the proof of Theorem 0.2 we use average information in the form of Lemma 0.4 to obtain a point-wise result.

Theorem 0.5. *Let D be as in Theorem 0.2. Suppose that*

$$\Omega_\alpha(q\theta) \leq C(q)\Omega_\alpha(\theta), \tag{0.9}$$

for any $q > 1$. Let $R_\tau(t) = \#\{\mathbb{T}^2 \cap tD + \tau\} - t^2|D|$. Suppose that

$$\left(\int_{\mathbb{T}^2} |R_\tau(t)|^2 d\tau\right)^{\frac{1}{2}} \leq Ct^{\frac{1}{2}}. \tag{0.10}$$

Then

$$R_{(0,0)}(t) = O(t^{\frac{2}{3}}). \tag{0.11}$$

Remark. Again, if $\gamma'(s) = e^{-\frac{1}{s^\alpha}}$, the conditions of Theorem 0.5 are satisified. The integrability of $\Omega_\alpha(\theta)$, mandated by Theorem 0.2, has already been checked in the remark following Theorem 0.2. The condition (0.9) also holds since $(q\theta)^{\frac{1}{\alpha^\alpha}} = q^{\frac{1}{\alpha^\alpha}} \times \theta^{\frac{1}{\alpha^\alpha}}$.

Remark. Theorem 0.5 allows one to conclude that (0.10) cannot hold if (0.11) is already known not to hold. For example, using the methods in [Bleh] one can show, under the assumptions of Theorem 0.5, that if the curvature vanishes to order $m-2$, $m \geq 3$, and if the normals at those points have rational slopes, then the estimate $R_{(0,0)}(t) = O(t^{\frac{m-1}{m}})$ cannot be improved in the sense that $\frac{m-1}{m}$ cannot be replaced by a lower power. See also [Ran2]. If $m > 3$, $\frac{m-1}{m} > \frac{2}{3}$, so (0.10) cannot hold by Theorem 0.5.

Remark. As we shall see below, the condition (0.10) is equivalent to the condition

$$t^2 \left(\sum_{k \neq (0,0)} |\widehat{\chi_D}(tk)|^2\right)^{\frac{1}{2}} \leq Ct^{\frac{1}{2}}. \tag{0.12}$$

If we replace the sum by the corresponding integral, this estimate always holds if, for example, D is convex. See, for example, [BCT]. See also [IP], where similar ideas are used in the context of orthogonal frames and exponential bases. This phenomenon is reminiscent of the issues that come up in the study of irregularity of distribution.

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A note on notation. *Throughout this paper, $|x|$ denotes the l^∞ distance, i.e $\max_j\{|x_j|\}$. The letter C denotes a generic constant which may change from line to line.*

Section I: Proof of Theorem 0.1

Let $\rho \in C_0^\infty(\mathbb{R}^2)$ satisfy $\int_{\mathbb{R}^2} \rho(x)dx = 1$. Let $\rho_\epsilon(x) = \frac{1}{\epsilon^2}\rho\left(\frac{x}{\epsilon}\right)$. Let

$$N_{\epsilon,\theta}(t) = \sum_{k \in \mathbb{Z}^2} \chi_{tD_\theta} * \rho_\epsilon(k), \tag{1.1}$$

where χ_{tD_θ} denotes the characteristic function of D rotated by θ and dilated by t. By the Poisson summation formula, up to a constant,

$$N_{\epsilon,\theta}(t) = t^2 \sum_{k \in \mathbb{Z}^2} \hat{\chi}_{D_\theta}(tk)\hat{\rho}(\epsilon k) = |D|t^2 + R_{\epsilon,\theta}(t), \tag{1.2}$$

where

$$R_{\epsilon,\theta}(t) = t^2 \sum_{k \neq (0,0)} \hat{\chi}_{D_\theta}(tk)\hat{\rho}(\epsilon k). \tag{1.3}$$

It is not hard to see that

$$N_{\epsilon,\theta}(t - C\epsilon) \leq N_\theta(t) \leq N_{\epsilon,\theta}(t + C\epsilon), \tag{1.4}$$

for an appropriate $C > 0$, so the result will follow if we can show that $(\int_0^{2\pi} |R_{\epsilon,\theta}(t)|^2 d\theta)^{\frac{1}{2}} \leq Ct^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$ by taking $\epsilon = t^{-\frac{1}{3}}$.

We have

$$t^2 \left(\int_0^{2\pi} \left| \sum_{k \neq (0,0)} \widehat{\chi}_{D_\theta}(tk)\hat{\rho}(\epsilon k) \right|^2 d\theta \right)^{\frac{1}{2}} \leq \tag{1.5}$$

$$\leq t^2 \sum_{k \neq (0,0)} \left(\int_0^{2\pi} |\widehat{\chi}_{D_\theta}(tk)|^2 d\theta \right)^{\frac{1}{2}} < \epsilon k >^{-N}, \tag{1.6}$$

for any $N > 0$, where $< x >= \sqrt{1 + |x|^2}$.

By Lemma 0.3 the expression in (1.6) is bounded by

$$Ct^{\frac{1}{2}} \sum_{k \neq (0,0)} |k|^{-\frac{3}{2}} < \epsilon k >^{-N}. \tag{1.7}$$

The sum in (1.7) is bounded by the integral

$$Ct^{\frac{1}{2}} \int_{|x|\geq 1} |x|^{-\frac{3}{2}} < \epsilon x >^{-N} \leq Ct^{\frac{1}{2}} \epsilon^{-\frac{1}{2}}. \tag{1.8}$$

This completes the proof of the first part of Theorem 0.1. To prove the second part, let $S_k = \{\theta : |R_\theta(n_k)| \geq \alpha k n_k^{\frac{2}{3}}\}$, where $\Lambda = \{n_k\}$ is a lacunary sequence. By Chebyshev's inequality and the first part of Theorem 0.1, $|S_k| \leq \frac{C}{\alpha}\frac{1}{k^2}$, where C is a uniform constant. Summing up in k we see that

$$\left| \{\theta : |R_\theta(t)| \geq \alpha t^{\frac{2}{3}} \log(t)\} \right| \leq \frac{C}{\alpha} \tag{1.9}$$

if $t \in \{n_k\}$. The result follows. Obviously, the factor $\log(t)$ can be improved a bit, but that's not the point.

Section II: Proof of Theorem 0.5

Since $R_\tau(t)$ is periodic, its L^2 norm is given by the l^2 norm of its Fourier coefficients. We get

$$t^2 \left(\sum_{k\neq(0,0)} |\widehat{\chi_D}(tk)|^2 \right)^{\frac{1}{2}}. \tag{2.1}$$

By assumption, the expression in (2.1) is bounded by $Ct^{\frac{1}{2}}$. We let $R_\epsilon(t)$ be defined as in the proof of Theorem 0.1 and keep in mind (1.2), (1.3), and (1.4) above. Let W denote the normal at the point where the Gaussian curvature on ∂D vanishes. Let $\mathcal{A}$ denote the set of lattice points at most a distance $\frac{1}{2}$ away from $\mathbb{R}W$. Let $\mathcal{B}$ denote the complement of this set minus the origin. We have

$$R_\epsilon(t) = t^2 \sum_{\mathcal{A}} \widehat{\chi_D}(tk)\hat{\rho}(\epsilon k) + t^2 \sum_{\mathcal{B}} \widehat{\chi_D}(tk)\hat{\rho}(\epsilon k) = I + II. \tag{2.2}$$

We must prove that

$$|II| \leq Ct^{\frac{1}{2}} \epsilon^{-\frac{1}{2}}. \tag{2.3}$$

By Lemma 0.4,

$$|\widehat{\chi_D}(tk)| \leq Ct^{-\frac{3}{2}} |k|^{-\frac{3}{2}} \phi \left(dist \left(\frac{k}{|k|}, W \right) \right), \tag{2.4}$$

where $\phi\left(dist\left(\frac{k}{|k|}, W\right)\right) = \Omega_\alpha(Arg(k) - Arg(W))$. For $x, k \in \mathcal{B}$ with $|x-k| \leq \frac{1}{4}$, we have, by the assumption (0.9),

$$\phi\left(dist\left(\frac{k}{|k|}, W\right)\right) \leq 100\phi\left(dist\left(\frac{x}{|x|}, W\right)\right). \tag{2.5}$$

We have $|\widehat{\chi_D}(tk)||\hat{\rho}(tk)|$ is bounded by

$$C\int_{\{|x-k|\leq\frac{1}{4}\}} |tx|^{-\frac{3}{2}}\phi\left(dist\left(\frac{x}{|x|}, W\right)\right) < \epsilon x >^{-N} dx. \tag{2.6}$$

The balls in (2.6) are disjoint. It follows that the sum over $\mathcal{B}$ is bounded by

$$Ct^2\int_{\{|x|\geq\frac{1}{4}\}} |tx|^{-\frac{3}{2}}\phi\left(dist\left(\frac{x}{|x|}, W\right)\right) < \epsilon x >^{-N} dx. \tag{2.7}$$

Going into polar coordinates we see that the integral over the circle is bounded since

$$\int_{S^1}\left|\phi\left(dist\left(\frac{x}{|x|}, W\right)\right)\right|^p \tag{2.8}$$

$$\leq \int |\Omega_\alpha(\theta)|^p d\theta < \infty \tag{2.9}$$

for any $1 \leq p \leq p_0$, where p_0 is defined in the statement of the theorem. Here we have explicitly used our assumptions on the geometry of the boundary of the convex set in question.

So, (2.7) is bounded by

$$Ct^2\int_{\frac{1}{4}}^{\infty} t^{-\frac{3}{2}}r^{-\frac{3}{2}} < \epsilon r >^{-N} rdr \leq Ct^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}, \tag{2.10}$$

as desired.

On the other hand, by Schwartz's inequality,

$$|I| \leq t^2\left(\sum_{\mathcal{A}}|\widehat{\chi_D}(tk)|^2\right)^{\frac{1}{2}}\left(\sum_{\mathcal{A}}|\hat{\rho}(\epsilon k)|^2\right)^{\frac{1}{2}}. \tag{2.11}$$

Since, by assumption, the expression (2.1) is bounded by $Ct^{\frac{1}{2}}$, we see that the expression (2.11) is bounded by $Ct^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$ since

$$\left(\sum_{\mathcal{A}}|\hat{\rho}(\epsilon k)|^2\right)^{\frac{1}{2}} \leq C\Big(\int_{\mathcal{A}^*} < \epsilon x >^{-N} dx\Big)^{\frac{1}{2}} \leq C\epsilon^{-\frac{1}{2}}, \tag{2.12}$$

where $\mathcal{A}^*$ denotes the convex hull of $\mathcal{A}$

This completes the proof of Theorem 0.5.

Section III: Proof of Theorem 0.2

Lemma 3.1. *We have*

$$R_{\theta,\epsilon}(t-\epsilon)-(2t\epsilon-\epsilon^2)|D| \leq R_\theta(t) \leq R_{\theta,\epsilon}(t+\epsilon)-(2t\epsilon+\epsilon^2)|D|. \tag{3.1}$$

Proof. We may assume that D contains the origin. We have

$$\chi_{(t-\epsilon)\theta^{-1}D} * \rho_\epsilon(k) \leq \chi_{t\theta^{-1}D}(k) \leq \chi_{(t+\epsilon)\theta^{-1}D} * \rho_\epsilon(k), \tag{3.2}$$

with ρ as in the proof of Theorem 0.1, along with

$$N_{\theta,\epsilon}(t-\epsilon) \leq N_\theta(t) \leq N_{\theta,\epsilon}(t+\epsilon), \tag{3.3}$$

and the result follows.

Lemma 3.2. *We have*

$$\sup_{2^j \leq t \leq 2^{j+1}} t^{-\frac{2}{3}}|R_\theta(t)| \in L^{p_0}(S_1) \tag{3.4}$$

with constants independent of j.

Proof. We have

$$\sup_{2^j \leq t \leq 2^{j+1}} |R_{\theta,\epsilon}(t)| \leq$$

$$\epsilon^{3/2}2^{(j+1)/2} \sum_{k\neq(0,0)} |\epsilon k|^{-\frac{3}{2}}|\widehat{\rho}(\epsilon k)| \sup_{2^j \leq t \leq 2^{j+1}} |t\theta k|^{\frac{3}{2}}|\widehat{\chi}_D(t\theta k)|. \tag{3.5}$$

The function $\sup_{2^j \leq R \leq 2^{j+1}} |R\theta k|^{\frac{3}{2}}|\widehat{\chi}_\Omega(R\theta k)|$ is uniformly in $L^{p_0}(S^1)$, since

$$|\widehat{\chi}_\Omega(R\theta k)| \leq (R|k|)^{-\frac{3}{2}}\Omega_\alpha(\theta) \tag{3.6}$$

by the divergence theorem and Lemma 0.4, and, by assumption, $\Omega_\alpha \in L^{p_0}(S^1)$. It follows that the sum is also in $L^{p_0}(S^1)$, with the norm bounded by

$$\epsilon^{3/2}2^{j/2} \sum_{k\neq(0,0)} |\epsilon k|^{-\frac{3}{2}}|\widehat{\rho}(\epsilon k)| \leq C2^{j/2}\epsilon^{-1/2}. \tag{3.7}$$

The result now follows from Lemma 3.1 by taking $\epsilon = 2^{-j/3}$.

We are now ready to complete the proof of Theorem 0.2. Observe that

$$\sup_{t\geq 2} \log^{-\delta}(t)t^{-\frac{2}{3}}|R_\theta(t)| \leq C\left\{\sum_{j=1}^{\infty} j^{-p_0\delta} \sup_{2^j\leq t\leq 2^{j+1}} t^{-\frac{2p_0}{3}}|D(R,\theta)|^{p_0}\right\}^{\frac{1}{p_0}}. \tag{3.8}$$

The $L^1(S^1)$ norm of the left hand side is bounded by the $L^1(S^1)$ norm of the right hand side. Since $\sup_{2^j\leq t\leq 2^{j+1}} t^{-\frac{2p_0}{3}}|D(R,\theta)|^{p_0}$ is uniformly in $L^1(S^1)$,

$$\int \sup_{t\geq 2} \log^{-\delta}(t)t^{-\frac{2}{3}}|R_\theta(t)|d\theta < \infty, \tag{3.9}$$

if $p_0\delta > 1$. It follows that

$$|R_\theta(t)| \leq Ct^{\frac{2}{3}} \log^{\delta(p_0)}(t) \tag{3.10}$$

for almost every θ, as desired.

Section IV: Proof of Lemma 0.4

Let s_0 be defined by the equation

$$\gamma'(s_0) = \theta. \tag{4.1}$$

Let $\Phi(s) = \theta s - \gamma(s)$. Then $\Phi'(s_0) = 0$. Moreover, for $u \leq s_0$, we have

$$\Phi'(u) = \int_u^{s_0} |\Phi''(s)|ds = \int_u^{u+\alpha z} + \int_{u+\alpha z}^{s_0}, \tag{4.2}$$

where $z = s_0 - u$.

It follows that

$$|\Phi'(u)| \geq C(s_0-u) \inf_{s\in[s_0\alpha, s_0]} |\Phi''(s)|. \tag{4.3}$$

We are now ready to estimate the oscillatory integral. We have

$$\int_0^1 e^{it\Phi(s)}ds = \int_0^{s_0-\delta} + \int_{s_0-\delta}^{s_0+\delta} + \int_{s_0+\delta}^1 = I + II + III, \tag{4.4}$$

with δ to be determined. By straightforward domination,

$$|II| \leq 2\delta. \tag{4.5}$$

On the other hand,

$$I=\int_0^{s_0-\delta}\frac{1}{it\Phi'(s)}\frac{d}{ds}\left(e^{it\Phi(s)}\right)ds=\frac{e^{it\Phi(s)}}{it\Phi'(s)}\Big|_0^{s_0-\delta}-\int_0^{s_0-\delta}e^{it\Phi(s)}\frac{d}{ds}\left(\frac{1}{it\Phi'(s)}\right)ds. \tag{4.6}$$

The second term on the right hand side of (4.6) is bounded by

$$\frac{1}{|t|}\int_0^{s_0-\delta}\left|\frac{d}{ds}\left(\frac{1}{\Phi'(s)}\right)\right|ds=\frac{1}{|t|}\left|\int_0^{s_0-\delta}\frac{d}{ds}\left(\frac{1}{\Phi'(s)}\right)ds\right|, \tag{4.7}$$

where the equality is true by monotonicity.

Using the fundamental theorem of calculus and (4.3) we see that the right hand side of (4.7) is bounded by

$$C\frac{1}{|t|}\left|\frac{1}{\delta\inf_{s\in[s_0\alpha,s_0]}|\Phi''(s)|}\right|\le C\frac{1}{|t|}\left(\frac{1}{\delta\inf_{\{\beta\in[\alpha,1]\}}\gamma''(\beta(\gamma')^{-1}(\theta))}\right). \tag{4.8}$$

The first term on the right hand side of (4.6) and the term III satisfy the same estimate by the same argument. Therefore, I+III satisfies the estimates in (4.8). As we've seen in (4.5), the term II is bounded by 2δ. Setting the two estimates equal to each other we get

$$\left|\int_0^1 e^{it\Phi(s)}ds\right|\le C_\alpha\frac{1}{\sqrt{|t|}}\left(\frac{1}{\sqrt{\inf_{\{\beta\in[\alpha,1]\}}\gamma''(\beta(\gamma')^{-1}(\theta))}}\right) \tag{4.9}$$

as desired.

Section V: Remarks and open problems

Podkorytov. Lemma 0.3 says that the L^2-average decay of the Fourier transform of the characteristic function of a convex domain in $\mathbb{R}^2$ decays of order $-\frac{3}{2}$ at infinity. However, in view of the technique used to prove Theorem 0.2, what we really need to know is whether

$$\sup_{|\lambda|\ge 1}|\lambda|^{\frac{3}{2}}|\widehat{\chi}_D(\lambda(\cos(\theta),\sin(\theta)))|\in L^p(S^1) \tag{5.1}$$

for some $p>1$. There are no counterexamples known to the author to the conjecture that (5.1) always holds for $p<2$. Resolving this question would lead to the immediate extension of Theorem 0.2 to a much wider class of domains. This issue is taken up in [BCIPT].

The case of the unit square shows that (5.1) cannot in general hold for $p=2$. However, it is reasonable to conjecture that if the curvature of ∂D vanishes at isolated points, then (5.1) does hold with $p=2$.

In $\mathbb{R}^d$, the analog of (5.1) is

$$\sup_{|\lambda|\geq 1} |\lambda|^{\frac{d-1}{2}} |\widehat{\chi}_D(\lambda\omega)| \in L^p(S^{d-1}), \tag{5.2}$$

$\omega \in S^{d-1}$, for some $p > 1$. If ∂D is analytic, the result result is due to Randol (see [Ran2]) with $p = 2$. However, under the general convexity assumption, the problem remains open. Again, resolving this question would lead to a generalization of Theorem 0.2 to general convex domains in $\mathbb{R}^d$, with the error term of order $t^{\frac{2}{3}}$ replaced by one of order $t^{d-2+\frac{2}{d+1}}$.

Fractal dimension of the "bad" set. In Theorem 0.2 we prove under some assumptions that the error term $O(t^{\frac{2}{3}})$ holds for almost every rotation θ. It is a natural problem to find a bound for the Hausdorff dimension of the "bad" set of θ's, the set where the $O(t^{\frac{2}{3}})$ estimate does not hold.

If the curvature on ∂D only vanishes to finite order not exceeding $m \geq 1$, we can give a crude upper bound on the Hausdorff dimension of the "bad" set. In this case, Colin de Verdiere ([CV]) showed that "good" rotations satisfy the diophantine condition $\left|x - \frac{p}{q}\right| \geq Cq^{-2-\frac{2}{m-2}}$. It is a classical result due to Jarnik and Besicovich, see also [K], that the Hausdorff dimension of the numbers not satisfying this condition is $\frac{m-2}{m-1}$. This gives an upper bound on the Hausdorff dimension of the "bad" set, albeit a poor one, since this number is $\frac{1}{2}$ when $m = 3$, whereas Randol ([Ran]) proved that when $m = 3$, one gets the estimate $R(t) \leq Ct^{\frac{2}{3}}$ without any rotations, which means, in particular, that the actual dimension of the "bad" set in this case is 0. There should be a better way of carrying out this estimate and we hope to address this issue in a subsequent paper.

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