FALCONER'S $\frac{d+1}{2}$ BOUND VIA STEIN-TOMAS RESTRICTION THEOREM

Alex Iosevich

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Abstract. We use the Stein-Tomas restriction theorem to give an alternate proof of a result due to Falconer which says that if the Hausdorff dimension of a subset of $\mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d+1}{2}$, then the Lebesgue measure of the set of distances is positive. Using a "conversion mechanism" this implies that if A is a Delone set, then the number of Euclidean distances determined by $A \cap [0,R]^d$ is at least $CR^{2-\frac{2}{d+1}}$.

The purpose of this short expository note is to show that Stein-Tomas restriction theorem yields a proof of a result due to Falconer ([Falc86]) which says that if the Hausdorff dimension of a set is greater than $\frac{d+1}{2}$, then the Lebesgue measure of the set of distances is positive. This result has been improved significantly since then in a series of papers by Bourgain ([Bour94]), Wolff ([W99]), and Erdogan ([Erd04]). The purpose of this note is to give a proof of Falconer's result via the Stein-Tomas restriction theorem. While the proof itself only yields the $\frac{d+1}{2}$, the approach is quite promising and was in fact used by the aforementioned individuals to obtain sharper estimates.

Theorem 0.1. *Let $E \subset [0,1]^d$ of Hausdorff dimension $> \frac{d+1}{2}$. Then the Lebesgue measure of $\Delta(E) = \{|x-y| : x, y \in E\}$ is positive, where $|\cdot|$ denotes the standard Euclidean distance.*

Proof of Theorem 0.1

We shall make use of the result due to Mattila ([Mat95]; see also [W03] for nice proof and a description of related topics) which says that in order to show that the Lebesgue measure of $\Delta(E)$ is positive, it suffices to prove that there exists a Borel measure μ supported on E such that

$$M(\mu) = \int_1^\infty \left(\int_{S^{d-1}} |\widehat{\mu}(t\omega)|^2 d\omega \right)^2 t^{d-1} dt < \infty. \tag{1.1}$$

Let $\tau_t f(x) = t^{-d} f(x/t)$. Observe that $\widehat{\tau_t f}(\xi) = \widehat{f}(t\xi)$, and $||\tau_t f||_p = t^{-\frac{d}{p'}} ||f||_p$.

Let f be a Schwartz class function such that $\int f(x)dx = 1$ and

$$\int\int |x-y|^{-\alpha} f(x)f(y)dxdy < \infty \tag{1.2}$$

for $\alpha > \frac{d+1}{2}$. It suffices to prove that $M(f) < \infty$.

Let f_j be defined by the relation $\widehat{f_j}(\xi) = \beta(2^{-j}\xi)\widehat{f}(\xi)$, where β is the usual Littlewood-Paley cut-off function. By Stein-Tomas restriction theorem (see e.g. [W02] for a proof),

$$\int_{S^{d-1}} \left|\widehat{f}(t\omega)\right|^2 d\omega \lesssim ||\tau_t f_j||_p^2 = t^{-\frac{2d}{p'}} ||f_j||_p^2, \tag{1.3}$$

where $p = \frac{2(d+1)}{d+3}$.

Now, $||f_j||_1 \leq 1$, and $||f_j||_2 \approx 2^{\frac{d-\alpha}{2}j}$. It follows that

$$||f_j||_p \lesssim 2^{\frac{d-\alpha}{p'}j}. \tag{1.4}$$

It follows that the right hand side of (1.2) is

$$\lesssim 2^{-j\frac{2d}{p'}} 2^{j\frac{2(d-\alpha)}{p'}} = 2^{-j\frac{2\alpha}{p'}}. \tag{1.5}$$

Let

$$M_j(f) = \int_{2^j}^{2^{j+1}} \left(\int_{S^{d-1}} \left|\widehat{f}(t\omega)\right|^2 d\omega \right)^2 t^{d-1} dt. \tag{1.6}$$

It follows from (1.5) that

$$M_j(f) \lesssim 2^{j(d-\alpha)} 2^{-\frac{2\alpha j}{p'}}, \tag{1.7}$$

and the series converges if $\alpha > \frac{d+1}{2}$.

Applications to the Erdos distance problem

Using the "conversion mechanism" from [IL04], [HI05], or by rewriting the argument above in a discrete setting, we can prove the following.

Theorem 2.1. *Let A be a Delone set, i.e there exist $0 < c < C$ such that points of A are c-separated and every cube of side-length C contains at least one point of A. Then*

$$\#\Delta(A \cap [0,q]^d) \gtrsim q^{2-\frac{2}{d+1}}. \tag{2.1}$$

References

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