LECTURE #4: WOLFF'S $\frac{n+2}{2}$ RESULT: WE ARE IN THE 90'S!

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Abstract. Building on the discrete model of the previous lecture, we prove that the Hausdorff dimension of the Kakeya set is at least $\frac{n+2}{2}$.

In this lecture we shall prove a theorem due to Wolff, which says that the Hausdorff dimension of a Kakeya set in $\mathbb{R}^n$ is at least $\frac{n+2}{2}$. In the previous lectures we have used Theorem 1.1 to deduce lower bounds on the Hausdorff dimension of a Kakeya set. This theorem is based on the properties of the Kakeya maximal operator where a function is averaged over tubes pointing in various directions. We shall see in a moment that one can also deduce information about the Hausdorff dimension of a Kakeya set by averaging characteristic functions of tubes instead. More precisely, we shall prove the following.

Theorem 9.1. *Let Ω be a δ-separated subset of S^{n-1}. Suppose that*

$$\left|\left|\sum_{e\in\Omega}\chi_{T_e^\delta}\right|\right|_{L^p(\mathbb{R}^n)} \lesssim \delta^{\frac{n}{p}-(n-1)-\epsilon}, \tag{9.1}$$

for any $\epsilon>0$. Then the Hausdorff dimension of a Kakeya set is at least $\frac{p}{p-1}$.

The proof is basically the same as before, so we'll be a bit sketchy. Let $\{B_j=B(x_j,r_j)\}$ denote the cover of a Kakeya set E by balls of radius r_j centered at x_j. As usual, we may assume that $r_j << 1$. By extracting a 2^{-k}-separated subset of the set Ω_k constructed in the proof of Theorem 1.1, we construct a subset of the sphere which we also call Ω (abuse of notation is so much fun), such that $\#\Omega \gtrsim 2^{k(n-1)}$, and

$$\int_{\cup_{j\in\Sigma_k}B_j}\sum_{e\in\Omega}\chi_{T_e^{2^{-k}}} \gtrsim 1 \tag{9.2}$$

up to logarithmic factors, where, as before, $\Sigma_k=\{j: 2^{-k}\le r_j\le 2^{-k+1}\}$.

By Holder, the left hand side of (9.2) is bounded above by

$$\left|\left|\sum_{e\in\Omega}\chi_{T_e^\delta}\right|\right|_{L^p(\mathbb{R}^n)} \times |\cup_{j\in\Sigma_k}B_j|^{\frac{1}{p'}}, \tag{9.3}$$

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which, by (9.1), is bounded above by

$$(2^{-k})^{\frac{n}{p}-(n-1)-\epsilon}|\cup_{j\in\Sigma_k} B_j|^{\frac{1}{p'}}. \tag{9.4}$$

It follows that

$$|\Sigma_k| \gtrsim (2^{\frac{nk}{p}}2^{-k(n-1)}2^{-k\epsilon})^{p'}2^{nk}, \tag{9.5}$$

which means that

$$\sum_{\Sigma_k} r_j^s \geq 2^{-ks}2^{\frac{kp}{p-1}}2^{-k\epsilon p'} \gtrsim 1, \tag{9.6}$$

since $s < \frac{p}{p-1}$. This completes the proof.

Theorem 9.2. *The estimate* (9.1) *holds with* $p = \frac{n+2}{n}$.

By Theorem 9.1 it follows that the Hausdorff dimension of a Kakeya set is at least $\frac{n+2}{2}$.

The heuristic. All theorems are true for a reason. Wolff's result is no exception. Cover a Kakeya set with balls of radius δ. We shall refer to these balls as "points". If the dimension of this set is d, we need around $\left(\frac{1}{\delta}\right)^d$ points. Since there is a line segment in every direction, we have around $\left(\frac{1}{\delta}\right)^{n-1}$ lines with $\frac{1}{\delta}$ points each. This means that we have roughly $\left(\frac{1}{\delta}\right)^{n-d}$ lines per point. This, in turn, implies that roughly $\left(\frac{1}{\delta}\right)^{n-d+1}$ lines intersect a given line. The key observation is that the lines that intersect a given a line are essentially disjoint. We shall refer to the collection of lines intersecting a given line as a hairbrush. The disjointness property implies that there are at least $\left(\frac{1}{\delta}\right)^{n-d+2}$ points in a hairbrush. Since a hairbrush lives inside our Kakeya set, we must have $\left(\frac{1}{\delta}\right)^{n-d+2} \lesssim \left(\frac{1}{\delta}\right)^d$, which means that $d \geq \frac{n+2}{2}$ as desired.

Making the above heuristic into a proof will hurt a little bit, but it will be worth it...

Bilinearization. The estimate (9.1) is equivalent to the estimate

$$\left|\left|\sum_{e\in\Omega}\sum_{e'\in\Omega}\chi_{T_e^\delta}\chi_{T_{e'}^\delta}\right|\right|_{\frac{p}{2}}^{\frac{p}{2}} \lesssim \delta^{n-p(n-1)} \tag{9.7}$$

up to ϵ (which we shall ignore from now on). We now play the separation game of Lecture #2. We have

$$\sum_{e\in\Omega}\sum_{e'\in\Omega} = \sum_{k=0}^{\log(1/\delta)}\sum_{|e-e'|\approx 2^{-k}} + \sum_{e=e'}. \tag{9.8}$$

We shall handle the first sum since the estimate for the second follows by the same argument. We no longer have Holder's inequality at our disposal since $p/2 < 1$, but we do have the deep fact which says that $a+b \leq (a^q+b^q)^{\frac{1}{q}}$ if $0<q<1$. This means that we just have to prove that

$$\sum_{k=0}^{\log(1/\delta)} \left\| \sum_{|e-e'|\approx 2^{-k}} \chi_{T_e^\delta} \chi_{T_{e'}^\delta} \right\|_{\frac{p}{2}}^{\frac{p}{2}} \lesssim \delta^{n-p(n-1)}. \tag{9.9}$$

As usual, we don't care about logarithmic quantities, so we just need to prove that

$$\left\| \sum_{|e-e'|\approx 2^{-k}} \chi_{T_e^\delta} \chi_{T_{e'}^\delta} \right\|_{\frac{p}{2}}^{\frac{p}{2}} \lesssim \delta^{n-p(n-1)} \tag{9.10}$$

for each k.

We now cover Ω by $\approx 2^{k(n-1)}$ finitely overlapping spherical caps of width $\approx 2^{-k}$ in such a way that given e, e' with $|e-e'| \approx 2^{-k}$ we can find a cap C containing both of them. Applying the pseudo-triangle inequality again, and using the fact that there are $\approx 2^{k(n-1)}$ caps, we see that it is enough to show that

$$\left\| \sum_{e,e' \in C\cap\Omega: |e-e'|\approx 2^{-k}} \chi_{T_e^\delta} \chi_{T_{e'}^\delta} \right\|_{\frac{p}{2}}^{\frac{p}{2}} \lesssim 2^{-k(n-1)} \delta^{n-p(n-1)} \tag{9.11}$$

for each cap C.

Exercise. *It is enough to establish* (9.11) *for* $k=0$. *Rescale...*

This reduces matters to showing that

$$\left\| \sum_{e,e' \in C\cap\Omega: |e-e'|\approx 1} \chi_{T_e^\delta} \chi_{T_{e'}^\delta} \right\|_{\frac{p}{2}}^{\frac{p}{2}} \lesssim \delta^{n-p(n-1)} \tag{9.12}$$

for each cap C.

Applying the pseudo-triangle inequality again, we reduce matters to showing that

$$\left\| \sum_{e\in\Omega_1} \sum_{e'\in\Omega_2} \chi_{T_e^\delta} \chi_{T_{e'}^\delta} \right\|_{\frac{p}{2}}^{\frac{p}{2}} \lesssim \delta^{n-p(n-1)}, \tag{9.13}$$

where Ω_1, Ω_2 are subsets of Ω separated by ≈ 1.

The pigeon is back. Let

$$E_{\eta,\eta'} = \{x : \sum_{e \in \Omega_1} \chi_{T_e^\delta}(x) \approx \eta, \sum_{e' \in \Omega_2} \chi_{T_{e'}^\delta}(x) \approx \eta'\}. \tag{9.14}$$

Now, (9.13) says that

$$\int \left(\sum_{e \in \Omega_1} \chi_{T_e^\delta}(x) \right)^{\frac{p}{2}} \left(\sum_{e' \in \Omega_2} \chi_{T_{e'}^\delta}(x) \right)^{\frac{p}{2}} dx \lesssim \delta^{n-p(n-1)}. \tag{9.15}$$

The left hand side of (9.15) is bounded by

$$\sum_{\eta,\eta'} \eta^{\frac{p}{2}} {\eta'}^{\frac{p}{2}} |E_{\eta,\eta'}|, \tag{9.16}$$

where η, η' are dyadic parameters. Since the number of η's and η''s needed is logarithmic in $\frac{1}{\delta}$, we see that it is enough to show that

$$\eta^{\frac{p}{2}} {\eta'}^{\frac{p}{2}} |E_{\eta,\eta'}| \lesssim \delta^{n-p(n-1)}, \tag{9.17}$$

for all η, η' that live between $\approx \delta^{-n}$ and ≈ 1.

At this point, we insert $p = \frac{n+2}{n}$, so (9.17) takes the form

$$(\eta\eta')^{\frac{n+2}{2n}} |E_{\eta,\eta'}| \lesssim \delta^{-\frac{n-2}{n}}. \tag{9.18}$$

We just did a bunch of things, but what do they mean? The numbers η and η' give us the number of 1-separated tubes a point x belongs to. Recall that our goal is to work with a "hairbrush", a collection of tubes intersecting a given tube. We are now ready to move in that direction.

Let's build a hairbrush. Let

$$\Omega_1^\lambda = \{e \in \Omega_1 : |T_e^\delta \cap E_{\eta,\eta'}| \approx \lambda |T_e^\delta|\}, \tag{9.19}$$

and

$$\Omega_2^{\lambda'} = \{e' \in \Omega_2 : |T_{e'}^\delta \cap E_{\eta,\eta'}| \approx \lambda' |T_{e'}^\delta|\}. \tag{9.20}$$

We need to get some kind of control on the size of λ and λ'. By definition,

$$\int_{E_{\eta,\eta'}} \sum_{e \in \Omega_1} \sum_{e' \in \Omega_2} \chi_{T_e^\delta} \chi_{T_{e'}^\delta} \approx \eta\eta' |E_{\eta,\eta'}|. \tag{9.21}$$

If we consider $\lambda, \lambda' \gtrsim \delta^{100000n}$, the number of $\lambda's$ and λ's is logarithmic in $\frac{1}{\delta}$, so the pigeonhole principle now tells us (up to logarithmic factors which we may ignore) that there exist λ, λ' dyadic such that

$$\int_{E_{\eta,\eta'}} \sum_{e \in \Omega_1^{\lambda}} \sum_{e' \in \Omega_2^{\lambda'}} \chi_{T_e^\delta} \chi_{T_{e'}^\delta} \gtrsim \eta\eta' |E_{\eta,\eta'}|. \tag{9.22}$$

We can see that λ may be chosen to be $\gtrsim \delta^{100000n}$ as follows. Consider $\int_{E_{\eta,\eta'}} \sum_{\{e \in \Omega_1 : |T_e^\delta \cap E_{\eta,\eta'}| \approx \lambda |T_e^\delta|; \lambda << \delta^{1000n}\}} \sum_{e' \in \Omega_2} \chi_{T_e^\delta} \chi_{T_{e'}^\delta}$. If this expression is $\gtrsim \eta\eta' |E_{\eta,\eta'}|$, it follows that $\lambda \gtrsim \eta\eta' |E_{\eta,\eta'}|$. Since $\eta\eta' < \delta^{-10000n}$, and $\lambda << \delta^{100000n}$, (9.18) follows. Thus, the estimate (9.18) is only non-trivial if $\lambda \gtrsim \delta^{100000n}$.

It follows that

$$\int_{E_{\eta,\eta'}} \sum_{e \in \Omega_1^{\lambda}} \chi_{T_e^\delta} \gtrsim \eta |E_{\eta,\eta'}|. \tag{9.23}$$

Ignoring logarithmic factors yet again, we see that

$$\lambda \geq \eta |E_{\eta,\eta'}|. \tag{9.24}$$

Invoking (9.22) again, we see that

$$\sum_{e' \in \Omega_2^{\lambda'}} \int_{T_{e'}^\delta} \sum_{e \in \Omega_1^{\lambda}} \chi_{T_e^\delta} \gtrsim \eta\eta' |E_{\eta,\eta'}|. \tag{9.25}$$

The number of directions in $\Omega_2^{\lambda'}$ is $\approx \delta^{-(n-1)}$, so there exists e' such that

$$\int_{T_{e'}^\delta} \sum_{e \in \Omega_1^{\lambda}} \chi_{T_e^\delta} \gtrsim \delta^{n-1} \eta\eta' |E_{\eta,\eta'}|, \tag{9.26}$$

which means that

$$\sum_{e \in \Omega_1^{\delta}} |T_e^\delta \cap T_{e'}^\delta| \gtrsim \delta^{n-1} \eta\eta' |E_{\eta,\eta'}|. \tag{9.27}$$

By (4.4), or, rather, its higher dimensional analog,

$$|T_e^\lambda \cap T_{e'}^\delta| \lesssim \delta^n. \tag{9.28}$$

It follows that

$$\delta^n \#\{e \in \Omega_1^\lambda : T_e^\delta \cap T_{e'}^\delta \neq \emptyset\} \gtrsim \delta^{n-1} \eta\eta' |E_{\eta,\eta'}|. \tag{9.29}$$

What have we just done? We found a tube $T^\delta_{e'}$ which intersects at least $\delta^{-1}\eta\eta'|E_{\eta,\eta'}|$ tubes T^δ_e, each at an angle ≈ 1 to $T^\delta_{e'}$, and are filled with density λ by $E_{\eta,\eta'}$. Notice how much harder this is than the heuristic above, or the finite field case for that matter! We are in for more pain...

Let $\mathcal{T}$ denote the collection of all tubes T^δ_e which intersect $T^\delta_{e'}$. We have, by the above,

$$\#\mathcal{T} \gtrsim \delta^{-1}\eta\eta'|E_{\eta,\eta'}|. \tag{9.30}$$

By definition,

$$\int_{E_{\eta,\eta'}} \chi_{T^\delta_e} \approx \lambda\delta^{n-1}. \tag{9.31}$$

For technical reasons that will become clear in a moment, we shall monkey this into

$$\int_{E_{\eta,\eta'}} \chi_{T^\delta_e \cap \Sigma} \approx \lambda\delta^{n-1}, \tag{9.32}$$

where

$$\Sigma = \{x : dist(x, T^\delta_{e'}) > C^{-1}\lambda\}, \tag{9.33}$$

where C is a very large constant. Summing over $\mathcal{T}$ we get

$$\int_{E_{\eta,\eta'}} \sum_{\mathcal{T}} \chi_{T^\delta_e \cap \Sigma} \gtrsim \lambda\delta^{n-1}\#\mathcal{T}. \tag{9.34}$$

Cordoba is back. We apply Cauchy-Schwarz (you know, $2ab \leq a^2 + b^2$...) to the left hand side of (9.34) to see that it is bounded by

$$|E_{\eta,\eta'}|^{\frac{1}{2}} \left\| \sum_{\mathcal{T}} \chi_{T^\delta_e \cap \Sigma} \right\|_2, \tag{9.35}$$

which implies that

$$\left\| \sum_{\mathcal{T}} \chi_{T^\delta_e \cap \Sigma} \right\|_2 \gtrsim \lambda\delta^{n-1}\#\mathcal{T}|E_{\eta,\eta'}|^{-\frac{1}{2}}. \tag{9.36}$$

L^2 estimate to be proved. We shall see that

$$\left\| \sum_{\mathcal{T}} \chi_{T^\delta_e \cap \Sigma} \right\|_2 \lesssim (\#\mathcal{T}\lambda^{-(n-2)}\delta^{n-1})^{\frac{1}{2}}. \tag{9.37}$$

Along with (9.36) this implies that

$$\lambda^n \delta^{n-1} \# \mathcal{T} \lesssim |E_{\eta,\eta'}|. \tag{9.38}$$

Taking (9.30) into account we get

$$\lambda^n \delta^{n-2} \eta\eta' \lesssim 1. \tag{9.39}$$

In view of (9.24) we get

$$\eta^{n+1}\eta' |E_{\eta,\eta'}| \lesssim \delta^{-(n-2)}, \tag{9.40}$$

and, by symmetry,

$$\eta'^{n+1}\eta |E_{\eta,\eta'}| \lesssim \delta^{-(n-2)}. \tag{9.41}$$

Taking the geometric mean we get (9.18) and the proof is complete. No, wait! I still have to prove (9.37). Oh well...

Squaring and applying Fubini, we see that it is enough to show that

$$\sum_{T^\delta_{e_1} \in \mathcal{T}} \sum_{T^\delta_{e_2} \in \mathcal{T}} |T^\delta_{e_1} \cap T^\delta_{e_2} \cap \Sigma| \lesssim \# \mathcal{T} \lambda^{-(n-2)} \delta^{n-1}. \tag{9.42}$$

Dividing both sides by $\#\mathcal{T}$ we see that it suffices to show that

$$\sum_{T^\delta_{e_2} \in \mathcal{T} : T^\delta_{e_1} \cap T^\delta_{e_2} \cap \Sigma \neq \emptyset} |T^\delta_{e_1} \cap T^\delta_{e_2} \cap \Sigma| \lesssim \lambda^{-(n-2)} \delta^{n-1} \tag{9.43}$$

for all $T^\delta_{e_1} \in \mathcal{T}$.

If $e_1 = e_2$, the estimate is trivial, so it is enough to show that

$$\sum_{k=0}^{\log(1/\delta)} \sum_{T^\delta_{e_2} \in \mathcal{T} : \cos^{-1}(e_1 \cdot e_2) \approx 2^{-k},\ T^\delta_{e_1} \cap T^\delta_{e_2} \cap \Sigma \neq \emptyset} |T^\delta_{e_1} \cap T^\delta_{e_2} \cap \Sigma| \lesssim \lambda^{-(n-2)} \delta^{n-1}, \tag{9.44}$$

and we can again ignore the sum in k since the number of terms is logarithmic.

Key observation. Since the angle between e_1 and e_2 is about 2^{-k}, $T^\delta_{e_1} \cap T^\delta_{e_2}$ is essentially contained in a $\delta \times \cdots \times \delta \times 2^k \delta$ tube, so the measure of the intersection cannot exceed $2^k \delta^n$.

A consequence of this observation is that we just need to show (for a fixed k) that

$$\sum_{T^\delta_{e_2} \in \mathcal{T} : \cos^{-1}(e_1 \cdot e_2) \approx 2^{-k},\ T^\delta_{e_1} \cap T^\delta_{e_2} \cap \Sigma \neq \emptyset} 2^k \delta^n \lesssim \lambda^{-(n-2)} \delta^{n-1}, \tag{9.45}$$

which can be rephrased as

$$\#\{T^\delta_{e_2} : \cos^{-1}(e_1 \cdot e_2) \approx 2^{-k},\ T^\delta_{e_1} \cap T^\delta_{e_2} \cap \Sigma \neq \emptyset\} \lesssim 2^{-k} \delta^{-1} \lambda^{-(n-2)}. \tag{9.46}$$

This follows from the fact that the tubes in $\mathcal{T}$ are δ-separated, and the following geometric fact.

Lemma 9.3. *If $T^\delta_{e_1}$ and $T^\delta_{e_2}$ both intersect $T^\delta_{e'}$ at an angle ≈ 1, and intersect each other in Σ at an angle $\approx 2^{-k}$, then $T^\delta_{e_2}$ lies within a $O(\delta/\lambda)$ neighborhood of the plane generated by the long axis of $T^\delta_{e'}$ and $T^\delta_{e_1}$, and when projected to that plane, makes an angle of $\approx 2^{-k}$ with $T^\delta_{e_1}$.*