LECTURE #3: HIGHER DIMENSIONAL ADVENTURES: $\frac{n+1}{2}$ AND DISCRETE $\frac{n+2}{2}$

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ABSTRACT. In this installment of our weekly recreation, we shall see that the dimensions of a Kakeya set in $\mathbb{R}^n$ is at least $\frac{n+1}{2}$. We shall then see that if we restate the problem in the context of finite fields, the dimension of a coresponding "Kakeya" set in $\mathbb{F}^n$ is at least $\frac{n-2}{2}$. In the next installment of the notes we shall see that these ideas lead to the proof that a Hausdorff dimension of the Kakeya set in $\mathbb{R}^n$ is at least $\frac{n-2}{2}$.

SECTION 7: WE MUST TAKE A STEP BACK BEFORE WE GO FORWARD

Th following estimate implies, via Theorem 1.1 that the Hausdorff dimension of a Kakeya set is at least $\frac{n+1}{2}$. In a sense, this is a step back, since we already know that the Hausdorff dimension of a Kakeya set in $\mathbb{R}^2$ is exactly 2. On the other hand, the $\frac{n+1}{2}$ estimate applied to all dimensions, and the method of proof will introduce some of the necessary machinery and ideas needed to move forwards. The result in question is due to Drury.

Theorem 7.1. *With the same notation as before, we have the following restricted weak-type result.*

$$||f_\delta^*||_{n+1,\infty} \leq C_n \delta^{-\frac{n-1}{n+1}} ||f||_{\frac{n+1}{2},1}. \tag{7.1}$$

Recalling Theorem 1.1 we see that Theorem 7.1 implies that the dimension of a Kakeya set in $\mathbb{R}^n$ is at least $\frac{n+1}{2}$. Let's now prove the theorem. We must show that if E is a measurable subset of $\mathbb{R}^2$, $f = \chi_E$, and $\Omega = \{e \in S^{n-1} : f_\delta^*(e) > \lambda\}$, then

$$|\Omega| \lesssim \delta^{-(n-1)} \lambda^{-(n+1)} |E|^2. \tag{7.2}$$

This would follow from showing that

$$|E| \gtrsim \delta^{\frac{n-1}{2}} \lambda^{-\frac{n+1}{2}} |\Omega|. \tag{7.3}$$

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Let $\{T^\delta_{e_j}\}_{j=1}^M$ be a set of tubes with δ-separated directions with

$$|E \cap T^\delta_{e_j}| \geq \lambda |T^\delta_{e_j}|. \tag{7.4}$$

If we recall that (2.16) says that $\mathcal{N}_\delta(\Omega) \geq \frac{|\Omega|}{\delta^{n-1}}$, we see that (7.3) would follow from the estimate

$$|E| \gtrsim \delta^{n-1}\lambda^{-\frac{n+1}{2}} M^{\frac{1}{2}}. \tag{7.5}$$

Let's first suppose that no point of E belongs to no more than μ tubes $T^\delta_{e_j}$. This means that

$$\sum_j \chi_{E\cap T^\delta_{e_j}}(x) \leq \mu, \tag{7.6}$$

and integrating both sides over E shows that

$$|E| \gtrsim \mu^{-1} \sum_j |E \cap T^\delta_{e_j}| \gtrsim \mu^{-1} M \lambda \delta^{n-1} \tag{7.7}$$

by (7.4).

We must now deal with the case that some point $x_0 \in E$ belongs to more than μ tubes $T^\delta_{e_j}$. Without loss of generality assume that $x_0 \in T^\delta_{e_j}$, $j = 1, \dots, \mu+1$. Chose ϵ small enough so that

$$|T^\delta_{e_j} \cap B_{\epsilon\lambda}(x_0)| \leq \frac{\lambda}{2} |T^\delta_{e_j}|, \tag{7.8}$$

where $B_{\epsilon\lambda}(x_0)$ denotes the ball of radius $\epsilon\lambda$ centered at x_0.

It follows that for $j \leq \mu+1$,

$$|E \cap T^\delta_{e_j} \cap B^c_{\epsilon\lambda}(x_0)| \geq \frac{\lambda}{2} |T^\delta_{e_j}| \gtrsim \lambda\delta^{n-1}, \tag{7.9}$$

where $B^c_{\epsilon\lambda}(x_0)$ denotes the complement of the ball defined above.

We must now recall some definitions easy derivations from the previous lectures. We have $\theta(e,e') = Arccos(e \cdot e')$ by (4.2). The formula (4.3) easily generalizes in the form

$$diameter(T^\delta_{e_j}(x_0) \cap T^\delta_{e_k}(x_0)) \leq \frac{\delta}{\theta(e_j,e_k)+\delta} \leq \frac{\delta}{\theta(e_j,e_k)}. \tag{7.10}$$

Recall that $x_0 \in T^\delta_{e_j} \cap T^\delta_{e'_j}$ of $j, j' \leq \mu+1$. It follows that if

$$\theta(e_j, e_k) \geq \frac{100}{\epsilon}\frac{\delta}{\lambda}, \tag{7.11}$$

then the sets

$$E \cap T^\delta_{e_j} \cap B^c_{\epsilon\lambda}(x_0) \text{ and } E \cap T^\delta_{e'_j} \cap B^c_{\epsilon\lambda}(x_0) \tag{7.12}$$

are disjoint.

Home stretch. By (7.12) and (7.9), we see that

$$|E| \gtrsim N\lambda\delta^{n-1}, \tag{7.13}$$

where N is the maximum possible cardinality of the $\frac{100}{\epsilon}\frac{\delta}{\lambda}$-separated subset of $\{e_j\}_{j=1}^{\mu+1}$. Since e_j's are δ-separated,

$$N\left(\frac{\delta}{\lambda}\right)^{n-1} \gtrsim \mu\delta^{n-1}, \tag{7.14}$$

which implies that

$$N \geq \lambda^{n-1}\mu, \tag{7.15}$$

and, consequently,

$$|E| \gtrsim \lambda^n\delta^{n-1}\mu. \tag{7.16}$$

For a given μ, either (7.16) or (7.7) holds. This implies (7.5) since by setting (7.16) and (7.7) equal to each other, we get $\mu = \frac{M^{\frac{1}{2}}}{\lambda^{\frac{n-1}{2}}}$. This completes the proof of Theorem 7.1. Note that technicalities aside, the proof of this result used only one essential fact- that if many δ tubes pointing in δ-separated directions intersect at a point, then outside a small ball these tubes are disjoint. Later, we shall see that one can improve on this argument by considering a bunch of tubes intersecting a given tube, instead of a given point.

Section 8: Back to finite fields

Before we grapple with the technical difficulties of the higher dimensional improvements, we revisit the finite field set up encountered in Section VI. We prove the following generalization of Theorem 6.1.

Theorem 8.1. *Let $\mathbb{F}_q^n$ denote an n-dimensional vector space over a field of q elements. Let E be a subset of $\mathbb{F}_q^n$ with the property that for all $e \in \mathbb{F}_q^n \setminus (0, \dots, 0)$ there exists $x \in \mathbb{F}_q^n$ such that $x + te \in E$ for all $t \in \mathbb{F}_q$. Then there exists $C_n > 0$ such that*

$$|E| \geq C_n^{-1} q^{\frac{n+2}{2}}. \tag{8.1}$$

We have already proved the two-dimensional case, so let $n \geq 3$. Since E contains a line in every direction, it contains

$$\frac{q^n - 1}{q - 1} \approx q^{n-1} \tag{8.2}$$

lines. We say that l_k is a high multiplicity line if for at least $\frac{q}{2}$ points of l_k, the number of lines that pass through each of those points is at least $\mu+1$, where μ is a number to be determined later. Given $\mu>0$, there are two possibilities- the one where no high multiplicity lines exist, and the one where at least one does.

We first treat the case where no high multiplicity lines exist. Let

$$E' = \{x \in E : x \in \mu \ l_j's\}. \tag{8.3}$$

Since each point of E' belongs to at most μ l_j's,

$$|E| \geq |E'| \tag{8.4}$$

$$\geq \mu^{-1} \sum_j |E' \cap l_j| \tag{8.5}$$

$$\geq \mu^{-1} q \cdot q^{n-1} = \mu^{-1} q^n. \tag{8.6}$$

We must now deal with the case where at least one line of high multiplicity exists. Denote this line by l_k, and let $\{\Pi_i\}$ be the set of two-planes containing l_k. By definition of high multiplicity, there exist at least

$$\mu \frac{q}{2} \tag{8.7}$$

lines l_j, $j \neq k$, that intersect l_k. Each of these lines is contained in the unique Π_i since two intersecting lines determine a plane. We now invoke Theorem 6.1 to see that

$$|E \cap \Pi_i \cap (\mathbb{F}_q^n \setminus l_k)| \geq q|\mathcal{L}_i|, \tag{8.8}$$

where $\mathcal{L}_i$ denotes the set of lines which are contained in a given Π_i. Since the sets corresponding to the left hand side of (8.8) are pairwise disjoint, we can sum in i to see that

$$|E| \gtrsim q \sum_i |\mathcal{L}_i| \geq q \frac{\mu q}{2} = \frac{q^2 \mu}{2} \tag{8.9}$$

by (8.7).

If we set $\mu \approx q^{\frac{n-2}{2}}$, the lower bounds in (8.9) and (8.6) agree, and the conclusion of the theorem follows. As clever as this setup is, the only geometric property we really used was that two non-intersecting lines determine a plane. This gives us some hope that the conclusion of Theorem 8.1 can be improved.

In the next lecture, we shall suffer through the technicalities of the $\mathbb{R}^n$ version of Theorem 8.1. We shall discover that we need no suffer all that much and, in fact, we shall actually prove the corresponding statement for the maximal operator f^*_δ.