

## LECTURE #2: ADVENTURES IN THE PLANE

ALEX IOSEVICH

September 20, 2000

ABSTRACT. The purpose of this lecture is to prove sharp estimates for the restriction operator and the Keakeya maximal function in the plane, and use those to deduce that the Hausdorff dimension of a Keakeya set in the plane is 2. We also discuss a discrete analog of the Keakeya problem.

### SECTION 4: THE KEAKEYA MAXIMAL FUNCTION IN THE PLANE

**Theorem 4.1.** *The restricted weak type  $(2, 2)$  norm of the Keakeya maximal operator in  $\mathbb{R}^2$  is  $\lesssim (\log(\frac{1}{\delta}))^{\frac{1}{2}}$ . In other words, let  $E \subset \mathbb{R}^2$ ,  $\lambda \in (0, 1]$ ,  $f = \chi_E$ , and  $\Omega = \{e \in S^1 : f_\delta^*(e) > \lambda\}$ . Then*

$$(4.1) \quad |\Omega| \leq \left( \log \left( \frac{1}{\delta} \right) \right) \frac{|E|}{\lambda^2}.$$

In view of Lemma 1.1, this implies that the Hausdorff dimension of a Keakeya set in the plane is exactly 2. To prove the lemma, we need a bit of trigonometry. Let

$$(4.2) \quad \theta(e, e') = \text{Arccos}(e \cdot e'),$$

$e, e' \in S^1$ . It is not hard to see that

$$(4.3) \quad \text{diameter}(T_e^\delta(a) \cap T_{e'}^\delta(b)) \lesssim \frac{\delta}{\theta(e, e') + \delta}$$

for any  $a, b \in \mathbb{R}^2$ , and, consequently,

$$(4.4) \quad |T_e^\delta(a) \cap T_{e'}^\delta(b)| \lesssim \frac{\delta^2}{\theta(e, e') + \delta}.$$

Fix a  $\delta$ -separated subset  $\{e_j\}_{j=1}^M$  of  $\Omega$  with  $M \geq \frac{|\Omega|}{\delta}$ . (We can do that by (2.16)). For each  $j$ , there is a tube  $T_j = T_{e_j}^\delta(a_j)$  with  $|T_j \cap E| \geq \lambda|T_j| \approx \lambda\delta$ .

---

Research supported in part by the NSF grant DMS00-87339

**Sprint for the finish line.** We have

$$(4.5) \quad M\lambda\delta \lesssim \sum_j |T_j \cap E|$$

$$(4.6) \quad = \int_E \sum_j \chi_{T_j}$$

$$(4.7) \quad \leq |E|^{\frac{1}{2}} \left\| \sum_j \chi_{T_j} \right\|_2$$

$$(4.8) \quad = |E|^{\frac{1}{2}} \left( \sum_{j,k} |T_i \cap T_j| \right)^{\frac{1}{2}}$$

$$(4.9) \quad \lesssim |E|^{\frac{1}{2}} \left( \sum_{j,k} \frac{\delta^2}{\theta(e_j, e_k) + \delta} \right)^{\frac{1}{2}}$$

$$(4.10) \quad \lesssim |E|^{\frac{1}{2}} \left( \sum_k \sum_{\{j: |j-k| \leq \frac{2\pi}{\delta}\}} \frac{\delta^2}{|j-k|\delta + \delta} \right)^{\frac{1}{2}}$$

$$(4.11) \quad \leq |E|^{\frac{1}{2}} \left( M\delta \log \left( \frac{1}{\delta} \right) \right)^{\frac{1}{2}}.$$

It follows that

$$(4.12) \quad M\lambda\delta \lesssim |E|^{\frac{1}{2}} \left( M\delta \log \left( \frac{1}{\delta} \right) \right)^{\frac{1}{2}},$$

which implies (4.1) since  $M \geq \frac{|\Omega|}{\delta}$ . This completes the proof of the lemma.

For our next trick, we will prove that the estimate (0.5) indeed holds in  $\mathbb{R}^2$ .

## SECTION 5: RESTRICTION CONJECTURE IN THE PLANE

**Theorem 5.1.** *Let  $\sigma$  denote the measure on the unit circle. Then*

$$(5.1) \quad \|\widehat{fd\sigma}\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(d\sigma)}$$

for  $p > 4$ .

We follow Terry Tao's proof given in [TerryTao2]. We start with the following bilinear setup.

**Lemma 5.2.** *Suppose that  $f$  and  $g$  are supported on distinct  $\theta$ -arcs of the circle whose separation is also comparable to  $\theta$ . Let  $d\sigma$  denote the Lebesgue measure on the circle. Then*

$$(5.2) \quad \|\widehat{fd\sigma}\widehat{gd\sigma}\|_2 \lesssim \theta^{-\frac{1}{2}} \|f\|_{L^2(d\sigma)} \|g\|_{L^2(d\sigma)}.$$

By Plancherel's theorem, the estimate is equivalent to

$$(5.3) \quad \|(fd\sigma) * (gd\sigma)\|_2 \lesssim \theta^{-\frac{1}{2}} \|f\|_{L^2(d\sigma)} \|g\|_{L^2(d\sigma)}.$$

This would follow from bilinear interpolation if we could show that

$$(5.4) \quad \|(fd\sigma) * (gd\sigma)\|_1 \lesssim \|f\|_{L^1(d\sigma)} \|g\|_{L^1(d\sigma)},$$

and

$$(5.5) \quad \|(fd\sigma) * (gd\sigma)\|_\infty \lesssim \theta^{-1} \|f\|_{L^\infty(d\sigma)} \|g\|_{L^\infty(d\sigma)}.$$

While (5.4) follows courtesy of our Italian friend Fubini, (5.5) is the key estimate. Since

$$(5.6) \quad fd\sigma \leq \|f\|_{L^\infty(d\sigma)} d\sigma_I \quad \text{and} \quad gd\sigma \leq \|g\|_{L^\infty(d\sigma)} d\sigma_J,$$

where  $I$  and  $J$  denote the  $\theta$ -arcs where  $f$  and  $g$  are supported, and  $d\sigma_I, d\sigma_J$  are the corresponding measures, it suffices to show that

$$(5.7) \quad \|d\sigma_I * d\sigma_J\|_\infty \lesssim \theta^{-1}.$$

Approximate  $d\sigma_I$  by  $\frac{1}{2\epsilon}\chi_{I_\epsilon}$ , where  $I_\epsilon$  is an  $\epsilon$  neighborhood of  $I$ . It suffices to show that

$$(5.8) \quad \left\| \frac{1}{2\epsilon}\chi_{I_\epsilon} * d\sigma_J \right\|_\infty \lesssim \theta^{-1}$$

for all sufficiently small  $\theta$ 's. This follows since a translate of  $J$  intersects  $I_\epsilon$  on an arc of length at most  $\frac{\epsilon}{\sin(\theta)} \approx \epsilon\theta^{-1}$ . This completes the proof of the lemma.

**From bilinear back to linear.** By Marcinkiewicz's interpolation theorem, it is enough to prove that

$$(5.9) \quad \|\widehat{\chi_E d\sigma}\|_p \lesssim |E|^{\frac{1}{p}},$$

$p > 4$ , where  $E$  is any measurable set. Equivalently, it is enough to show that

$$(5.10) \quad \|\widehat{\chi_E d\sigma} \widehat{\chi_E d\sigma}\|_{\frac{p}{2}} \lesssim |E|^{\frac{2}{p}},$$

under the same restrictions.

For every  $n \geq 0$ , divide the circle into  $2^n$  equal arcs, so that each arc at each stage has exactly two children at stage  $n+1$ . We denote the set of all arcs at stage  $n$  by  $A_n$ . Define  $I \approx J$ ,  $I, J \in A_n$  if  $I$  and  $J$  are not adjacent, but their parents are. Note that for a given  $I$  there are only finitely many  $J$ 's such that  $I \approx J$ .

For every  $x \neq y$  on the circle, there is exactly one pair of arcs  $I, J$  containing  $x$  and  $y$  respectively, such that  $I \approx J$ . It follows that

$$(5.11) \quad \widehat{\chi_E d\sigma} \widehat{\chi_E d\sigma} = \sum_{I \approx J} \widehat{\chi_E d\sigma}_I \widehat{\chi_E d\sigma}_J = \sum_{n=1}^{\infty} \sum_{\{I, J \in A_n : I \approx J\}} \widehat{\chi_E d\sigma}_I \widehat{\chi_E d\sigma}_J.$$

The first reduction is easy:

$$(5.12) \quad \|\widehat{\chi_E d\sigma} \widehat{\chi_E d\sigma}\|_{\frac{p}{2}} \lesssim \sum_{n=1}^{\infty} \left\| \sum_{\{I, J \in A_n : I \approx J\}} \widehat{\chi_E d\sigma}_I \widehat{\chi_E d\sigma}_J \right\|_{\frac{p}{2}}.$$

What do we do with the  $\frac{p}{2}$  norm? We are analysts, so we interpolate. We have

$$(5.13) \quad \left\| \sum_{\{I, J \in A_n : I \approx J\}} \widehat{\chi_E d\sigma}_I \widehat{\chi_E d\sigma}_J \right\|_{\infty} \lesssim \sum_{\{I, J \in A_n : I \approx J\}} |E \cap I| |E \cap J| \leq \sum_{\{I \in A_n\}} 2^{-n} |E \cap I| = 2^{-n} |E|.$$

On the other hand, we may ignore the  $I \approx J$  restriction to obtain

$$(5.14) \quad \left\| \sum_{\{I, J \in A_n : I \approx J\}} \widehat{\chi_E d\sigma}_I \widehat{\chi_E d\sigma}_J \right\|_{\infty} \lesssim \left( \sum_{\{I \in A_n\}} |E \cap I| \right) \left( \sum_{\{I \in A_n\}} |E \cap J| \right) = |E|^2.$$

The two estimates combine to say that

$$(5.15) \quad \left\| \sum_{\{I, J \in A_n : I \approx J\}} \widehat{\chi_E d\sigma}_I \widehat{\chi_E d\sigma}_J \right\|_{\infty} \lesssim |E| \min\{|E|, 2^{-n}\}.$$

**Fefferman's observation.** As  $I \approx J$  vary, the supports of the functions  $\widehat{\chi_E d\sigma_I} \widehat{\chi_E d\sigma_J}$  are essentially disjoint. The point here is that if  $I \approx J$  and  $I' \approx J'$ , on the same scale, then the set theoretic sums  $I + J$  and  $I' + J'$  are essentially always disjoint. One can actually achieve perfect orthogonality by not using all the pairs at once, and then applying the triangle inequality.

An immediate consequence of Fefferman's observation followed by Lemma 2.3 is that

$$(5.16) \quad \left\| \sum_{\{I, J \in A_n: I \approx J\}} \widehat{\chi_E d\sigma_I} \widehat{\chi_E d\sigma_J} \right\|_2 \lesssim \left( \sum_{\{I, J \in A_n: I \approx J\}} \|\widehat{\chi_E d\sigma_I} \widehat{\chi_E d\sigma_J}\|_2^2 \right)^{\frac{1}{2}}$$

$$(5.17) \quad \lesssim 2^{\frac{n}{2}} \left( \sum_{\{I, J \in A_n: I \approx J\}} |E \cap I| |E \cap J| \right)^{\frac{1}{2}} \lesssim 2^{\frac{n}{2}} (|E| \min\{2^{-n}, |E|\})^{\frac{1}{2}}.$$

Interpolating (5.13) and (5.14)-(5.15) using Holder's inequality, and summing over  $n$ , we get

$$(5.18) \quad \|\widehat{\chi_E d\sigma} \widehat{\chi_E d\sigma}\|_{\frac{p}{2}} \lesssim \sum_{n=1}^{\infty} 2^{\frac{2n}{p}} (|E| \min\{2^{-n}, |E|\})^{1-\frac{2}{p}} \lesssim |E|^{\frac{2}{p}}.$$

This completes the proof. We shall now follow Tom Wolff in giving ourselves a technicality free way of looking at the Kakeya problem.

## SECTION 6: LONG LIVE FINITE FIELDS!

Let  $\mathbb{F}_q^n$  denote an  $n$ -dimensional vector space over a field of  $q$  elements. A natural variant of the Kakeya problem in  $\mathbb{R}^n$  is the following. Let  $E$  be a subset of  $\mathbb{F}_q^n$  with the property that for all  $e \in \mathbb{F}_q^n \setminus (0, \dots, 0)$  there exists  $x \in \mathbb{F}_q^n$  such that  $x + te \in E$  for all  $t \in \mathbb{F}_q$ . The question we ask is, does there exist  $C_n > 0$  such that

$$(6.1) \quad |E| \geq C_n^{-1} q^n?$$

We conclude this lecture with the following observation.

**Theorem 6.1.** *Let  $n = 2$ , and suppose that  $E$  contains at least  $\frac{q}{2}$  points on each of  $m$  lines with different slopes. Then*

$$(6.2) \quad |E| \gtrsim mq.$$

Setting  $m = q + 1$  gives us (6.1) with  $n = 2$ . To prove Theorem 6.1, let  $\{l_j\}_{j=1}^m$  be the lines in question. Any two of these lines intersect at a point. It follows that

$$(6.3) \quad \frac{1}{2}qm \leq \sum_j |E \cap l_j| = \sum_{x \in E} \left( \sum_j \chi_{l_j}(x) \right)$$

$$(6.4) \quad \leq |E|^{\frac{1}{2}} \left( \sum_{x \in E} \left( \sum_j \chi_{l_j}(x) \right)^2 \right)^{\frac{1}{2}}$$

$$(6.5) \quad = |E|^{\frac{1}{2}} \left( \sum_j \sum_k \sum_{x \in E} \chi_{l_j}(x) \chi_{l_k}(x) \right)^{\frac{1}{2}} = |E|^{\frac{1}{2}} \left( \sum_j \sum_k |l_j \cap l_k| \right)^{\frac{1}{2}}$$

$$(6.6) \quad = |E|^{\frac{1}{2}} (m(m-1+q))^{\frac{1}{2}} \leq |E|^{\frac{1}{2}} (2mq)^{\frac{1}{2}}$$

since  $m \leq q + 1$ . The results follows.

In the next section, we shall start a long, exciting and painful discussion of the higher dimensional situation.