# FUGLEDE CONJECTURE FOR LATTICES

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Abstract. The purpose of this expository article is to prove that the statement that a set in Euclidean space tiles by a lattice is equivalent to the statement that the dual lattice is a spectrum for the $L^2$ space of this set. This result is originally due to Bent Fuglede.

In 1974 Fuglede ([Fug74]) proved that the statement that a domain $D \subset \mathbb{R}^d$ is tiled by a lattice $A$ is equivalent to the statement that $\{e^{2\pi i b \cdot}\}_{b \in B}$, $B$ the dual lattice of $A$, is an orthogonal basis for $L^2(D)$. The purpose of this note is to give an almost completely elementary proof of this result.

**Tiling.** We say that $A$ tiles a domain $D \subset \mathbb{R}^d$ if

$$\sum_A \chi_D(x+a) \equiv 1 \tag{1}$$

for almost every $x \in \mathbb{R}^d$.

**Dual Lattice.** We say that $A$ is a lattice if there exists an invertible $d$ by $d$ matrix $M$ such that $A = M\mathbb{Z}^2$. We say that $B$ is a dual lattice of the lattice $A$, if

$$B = \{b : a \cdot b \in \mathbb{Z} \ \forall \ a \in A\}. \tag{2}$$

In other words, $B = (M^t)^{-1}\mathbb{Z}^2$.

The statement that $\{e^{2\pi i b \cdot}\}_{b \in B}$ is an orthogonal basis for $L^2(D)$ means that

$$||f||^2_{L^2(D)} = \sum_B |\widehat{f\chi_D}(b)|^2. \tag{3}$$

Since the exponentials are dense, (0.3) is equivalent to the statement that

$$\sum_B |\widehat{\chi_D}(\xi+b)|^2 = |D|^2 \tag{4}$$

for almost every $\xi \in \mathbb{R}^d$.

**Theorem.** *Let $D$ be a domain in $\mathbb{R}^d$. Suppose that $A$ is a lattice and $B$ is the dual lattice. Then* (1) *holds for almost every $x \in \mathbb{R}^d$ if and only if* (3) *holds for almost every $\xi \in \mathbb{R}^d$.*

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Let $f(x)$ equal to the expression in (1). Let $F(\xi)$ equal to the expression in (4).

**Part I: (1) implies (4).** Suppose that (1) holds. The function $F(\xi)$ is periodic with respect to $B$. Let $Q_B$ denote the fundamental domain of $B$, i.e the image under $M$ of $[0,1]^d$, where $B = M\mathbb{Z}^d$. It is clear that $B$ tiles $Q_B$, and it is an elementary fact from Fourier analysis that $\{e^{2\pi i a \cdot}\}_{a \in A}$ is an orthogonal exponential basis for $L^2(Q_B)$, where $A$ is the dual lattice of $B$. Thus $F \in L^2(Q_B)$. Let's compute the Fourier coefficients of $F$. We have, for each $a \in A$,

$$\hat{F}(a) = |Q_B|^{-1} \int_{Q_B} e^{-2\pi i \xi \cdot a} \sum_B |\widehat{\chi_D}(\xi + b)|^2 d\xi \tag{5}$$

$$= |Q_B|^{-1} \sum_B \int_{Q_B + b} e^{-2\pi i \xi \cdot a} |\widehat{\chi_D}(\xi)|^2 d\xi \tag{6}$$

$$= |Q_B|^{-1} \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot a} |\widehat{\chi_D}(\xi)|^2 d\xi \tag{7}$$

$$= \frac{|D \cap D + a|}{|Q_B|}. \tag{8}$$

Since $A$ tiles $D$, the expression in (8) is 0 if $a \neq (0, \dots, 0)$. If $a = (0, \dots, 0)$, we get

$$\frac{|D|}{|Q_B|}. \tag{9}$$

In order to analyze (9) we compute the Fourier coefficients of $f$. This function is periodic with respect to $A$ and lives in $L^2(Q_A)$, where $Q_A$ is the fundamental domain of $A$. Observe that it follows immediately from the definition of $Q_A$ and $Q_B$ that

$$|Q_A||Q_B| = 1. \tag{10}$$

Let's compute the Fourier coefficients of $f$. Let $b \in B$. Then

$$\hat{f}(b) = |Q_A|^{-1} \int_{Q_A} e^{-2\pi i x \cdot b} \sum_A \chi_D(x + a) dx \tag{11}$$

$$= \frac{\widehat{\chi}_D(b)}{|Q_A|}. \tag{12}$$

Since, by assumption, $f(x) = 1$, for almost every $x$, it follows that

$$\hat{f}(b) = 1 \text{ if } b = (0, \ldots, 0), \text{ and } 0 \text{ otherwise.} \tag{13}$$

Plugging (13) in (12) we see that

$$|D| = |Q_A|. \tag{14}$$

In view of (10), (14) implies that (9) equals $|D|^2$. Thus we have shown that if $a = (0, \ldots, 0)$,

$$\hat{F}(a) = |D|^2, \tag{15}$$

and 0 otherwise. It follows that

$$F(\xi) = |D|^2 \tag{16}$$

for almost every $\xi$, i.e (1) implies (4).

**Part II:** (4) **implies** (1)**.** Since $F(\xi) = 1$ and $\hat{f}(b) = |Q_A|^{-1}\widehat{\chi}_D(b)$, we see that

$$\hat{f}(b) = 0 \tag{17}$$

if $b \neq (0, \ldots, 0)$.
If $b = (0, \ldots, 0)$, we get

$$\hat{f}(0, \ldots, 0) = \frac{|D|}{|Q_A|}. \tag{18}$$

On the other hand, by (9) and (4),

$$\hat{F}(0, \ldots, 0) = \frac{|D|}{|Q_B|} = |D|^2, \tag{19}$$

which implies that

$$|D| = |Q_B|^{-1} = |Q_A|. \tag{20}$$

Plugging (20) into (18) proves that

$$\hat{f}(0, \ldots, 0) = 1. \tag{21}$$

In conjunction with (17) this implies that

$$f(x) = 1 \tag{22}$$

for almost every $x$, i.e (4) implies (1).
The proof of Theorem is complete.

[Fug74] B. Fuglede, *Commuting self-adjoint partial differential operators and a group theoretic problem*, J. Funct. Anal. **16** (1974), 101-121.