Polygons, Curved Spaces, and the Gauss-Bonnet Theorem

Part 1: Polygons in the plane and the sphere

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Northwestern University

May 2020

Anatomy of a polygon

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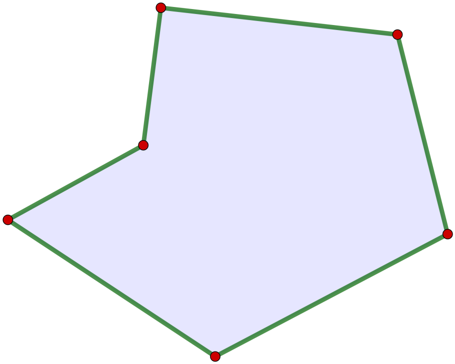

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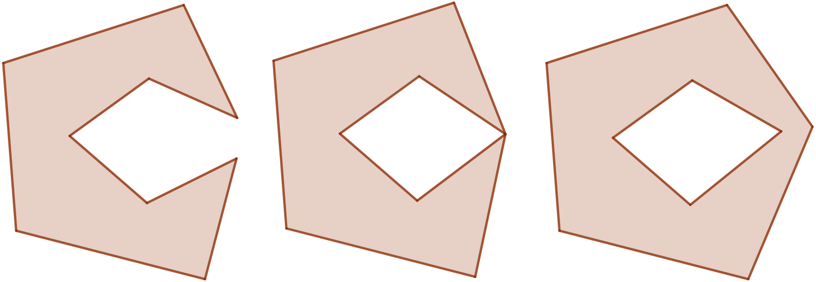

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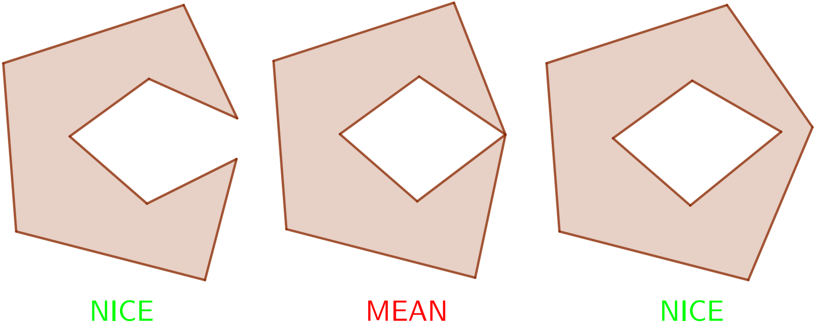

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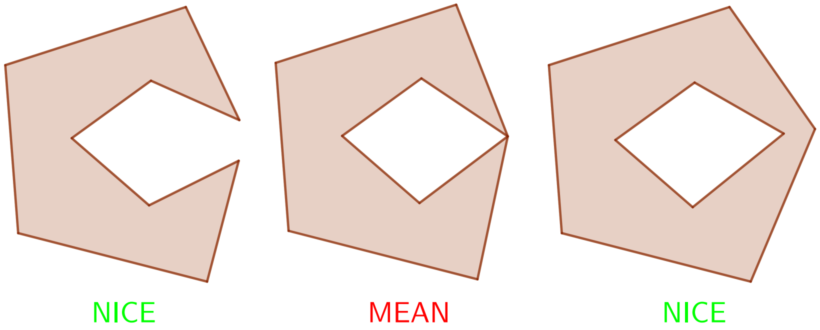

From here on, all polygons will be nice.

Cutting polygons into triangles

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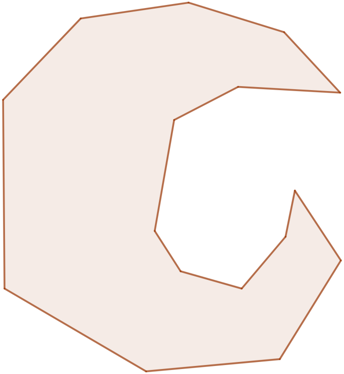
Take a polygon...



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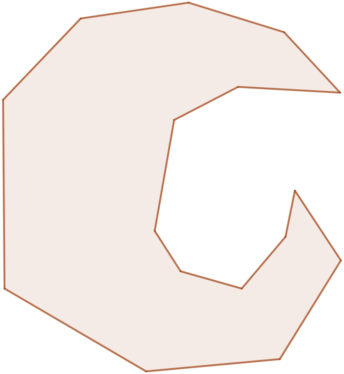
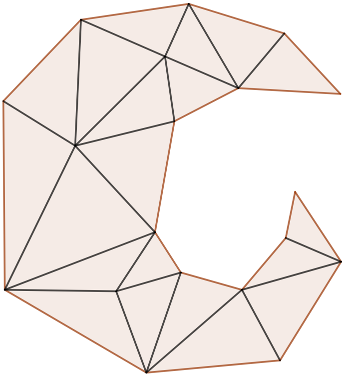


...and cut it into triangles.



Rules for cutting polygons into triangles

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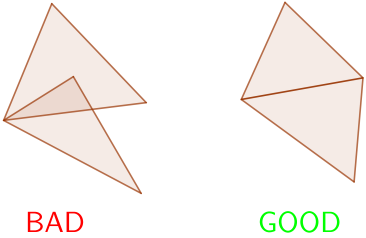
1. The faces of any two triangles must not overlap.



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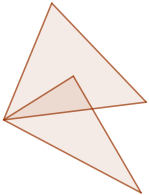
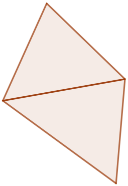
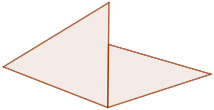


BAD



GOOD

2. No vertex can lie along an edge except at one of the endpoints.



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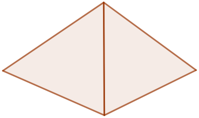

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Counting vertices, edges, and faces (example 1)

Let's cut a polygon into triangles in a few different ways.

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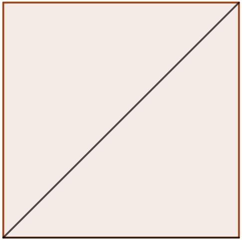

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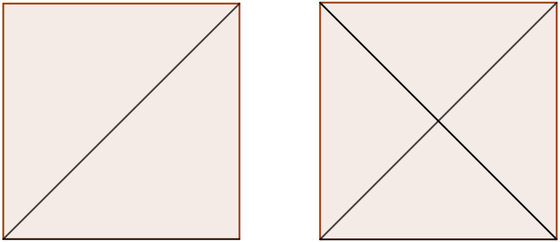

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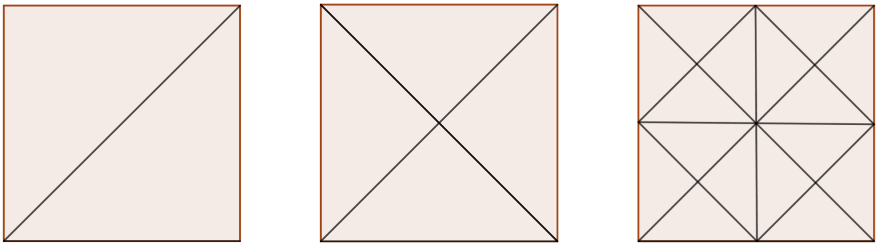

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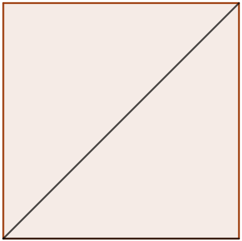

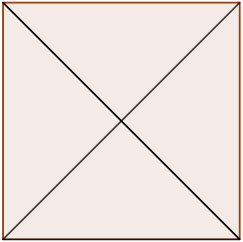

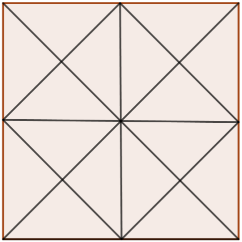

V	E	F
4	5	2

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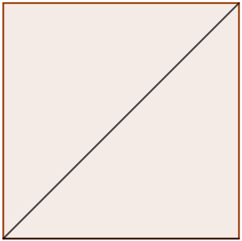

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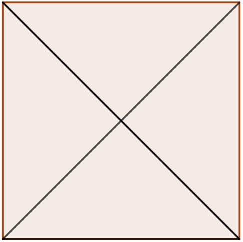

V	E	F
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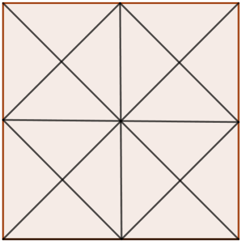

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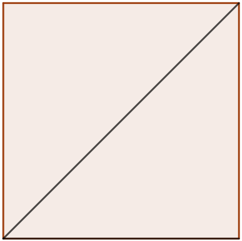

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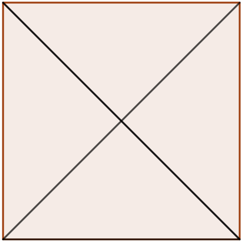

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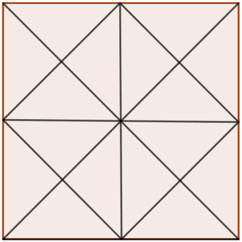

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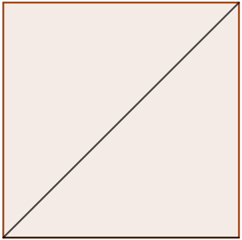

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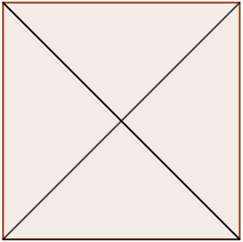

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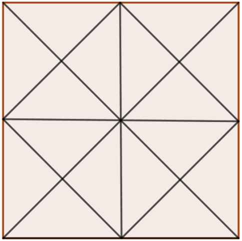

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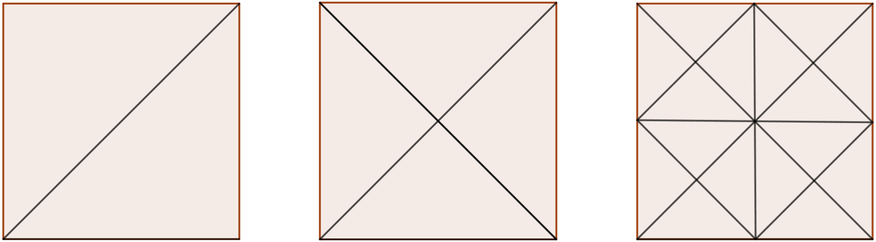

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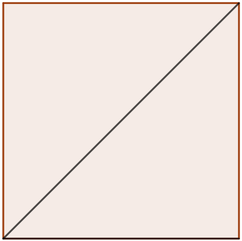

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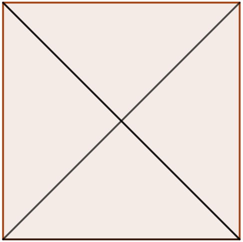

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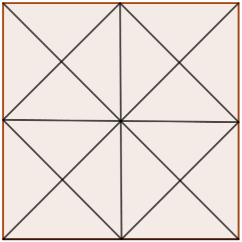

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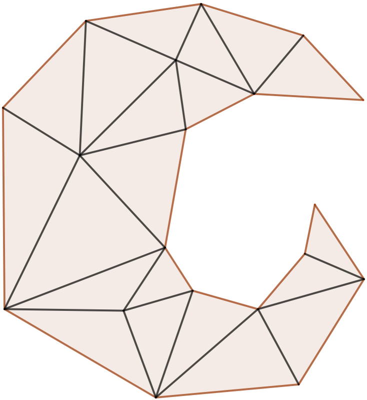

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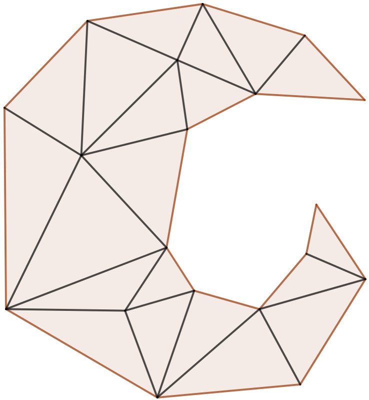

V	E	F
19	38	20

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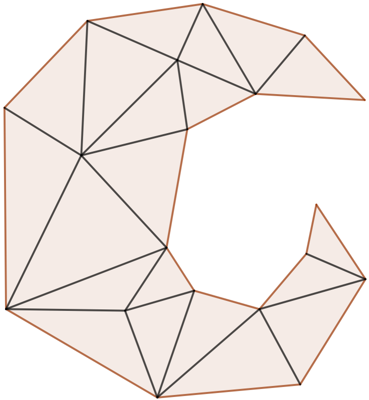

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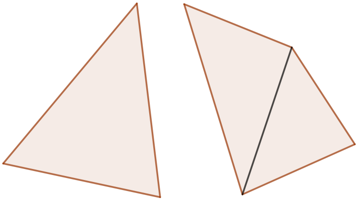

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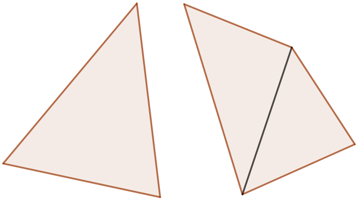

V	E	F
7	8	3

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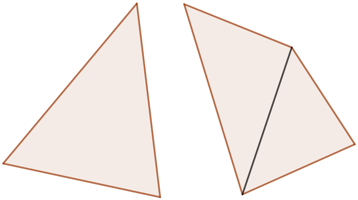

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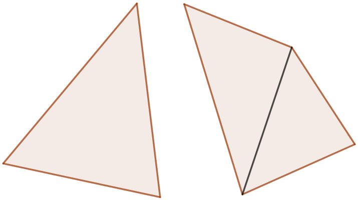

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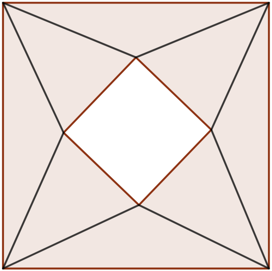
$$V - E + F = 2$$

- $V - E + F$ is affected by the number of components.

What about polygons with holes?



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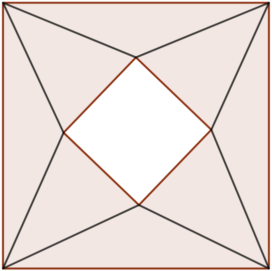

V	E	F
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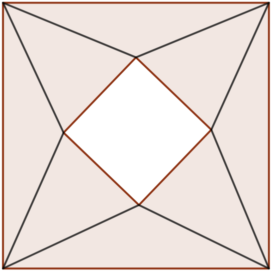

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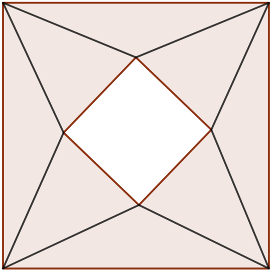

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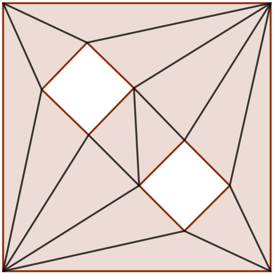

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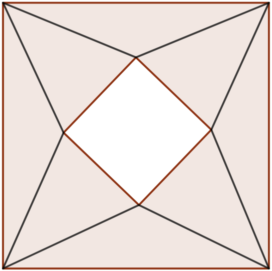

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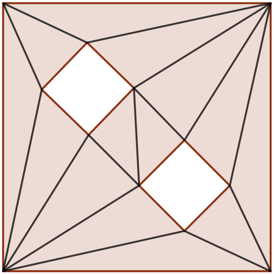

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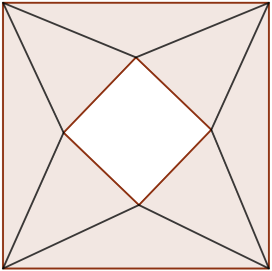

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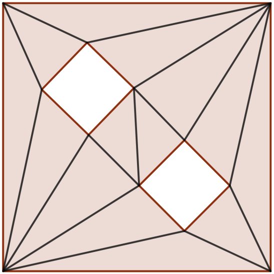

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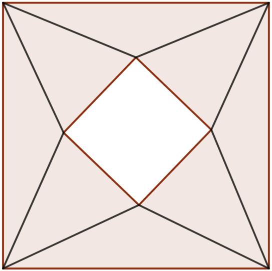

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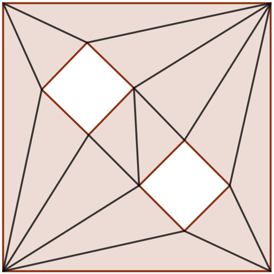

V	E	F
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$$V - E + F = -1$$

$V - E + F$ seems to be affected by the number of holes a polygon has.

A conjecture

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Conjecture

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- We will look for another way of quantifying "the number of components minus the number of holes."
- Let's look at angles.

Interior and exterior angles

- At each vertex, the *interior angle* is the angle between the two adjacent edges as measured from one edge to the next along the face of the polygon. (Polygon must be nice.)

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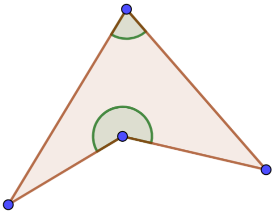

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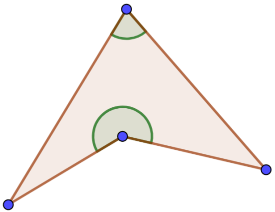

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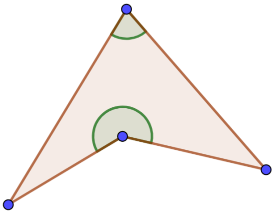

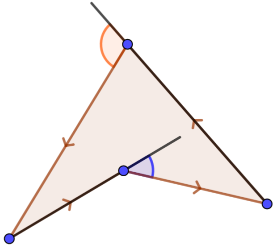

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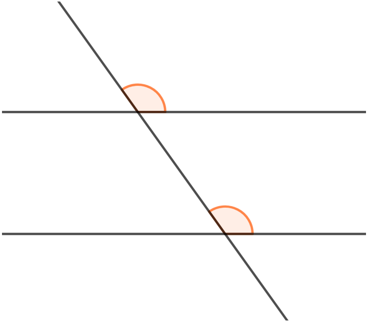

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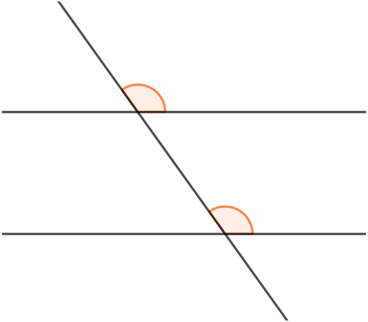

- Requires a notion of "parallel."

Sum of exterior angles of a triangle

- Recall, the sum of interior angles of a triangle is π. Why?

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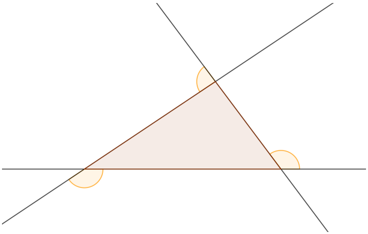

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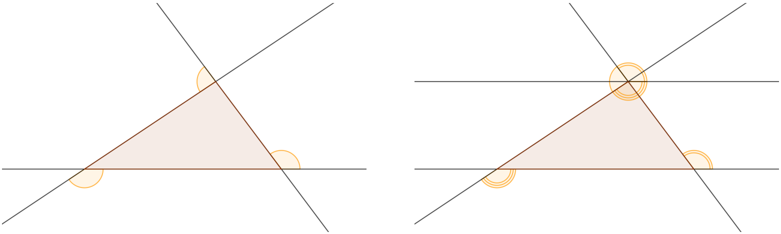

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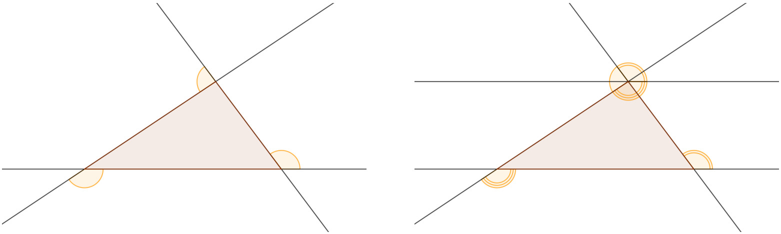

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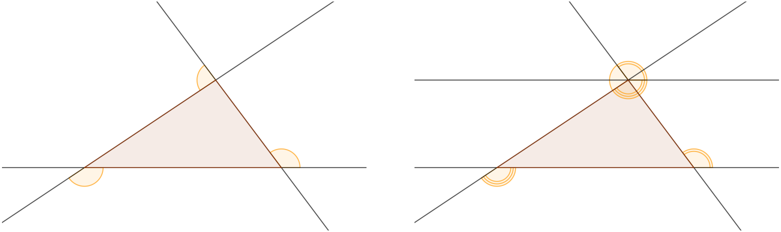

- The sum of exterior angles of a triangle is 2π.
- Recall interior angle $= \pi -$ exterior angle. So,

$$\sum \text{interior angles} = 3\pi - \sum \text{exterior angles} = \pi.$$

How to sum the exterior angles of a nice polygon

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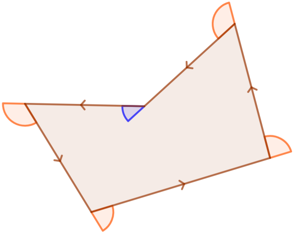

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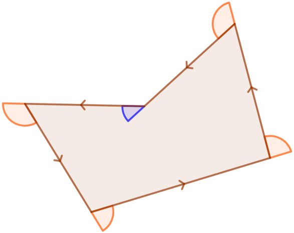

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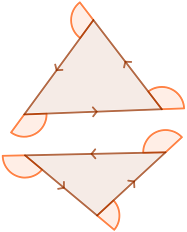

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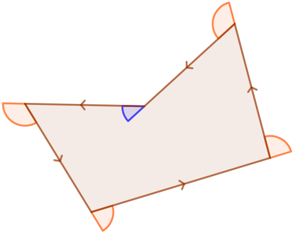

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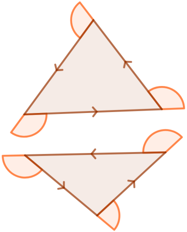

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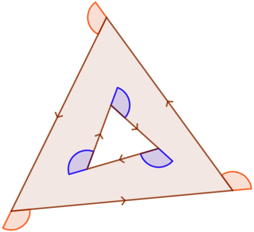

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Conjecture

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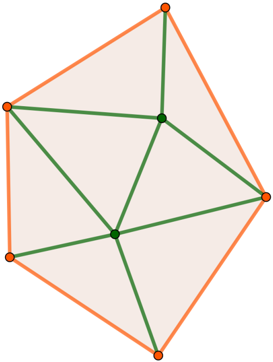
Theorem

$$\sum \textit{exterior angles} = 2\pi(V - E + F).$$

How are we going to prove this?



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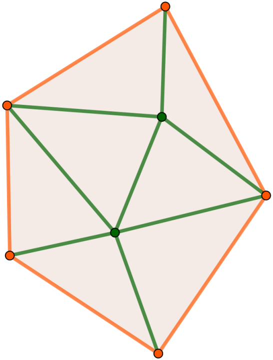

- Cut our nice polygon into triangles.
- We have V vertices, E edges, and F faces.

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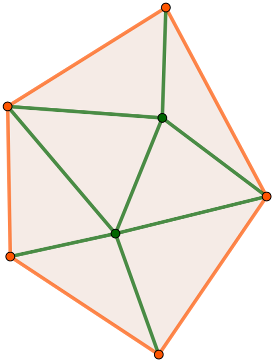

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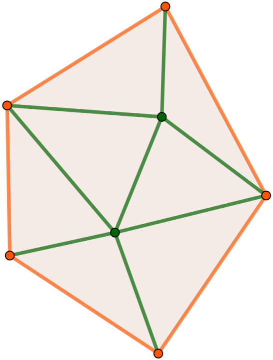

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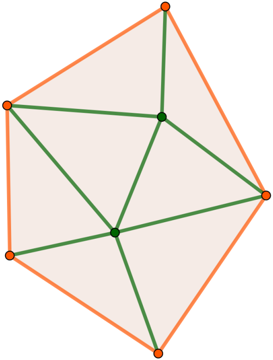

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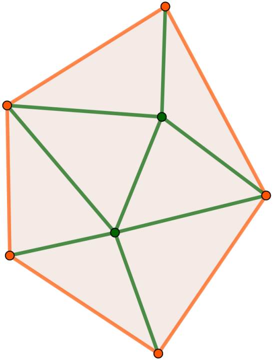

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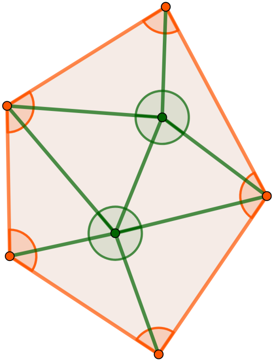

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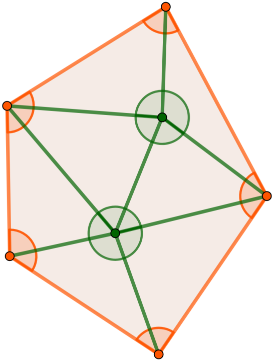

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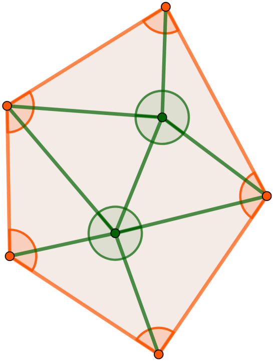

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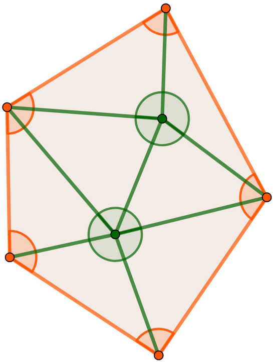

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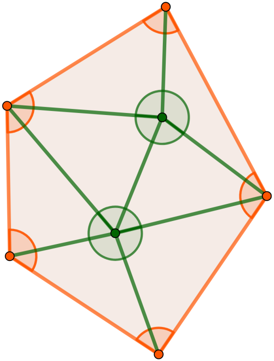

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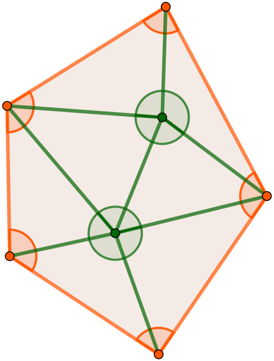

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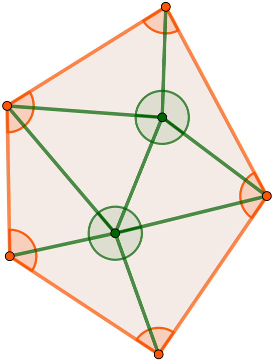

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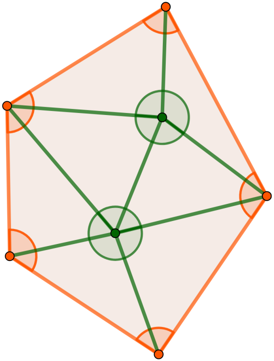

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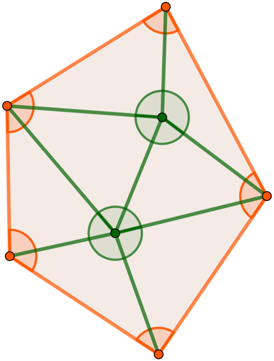

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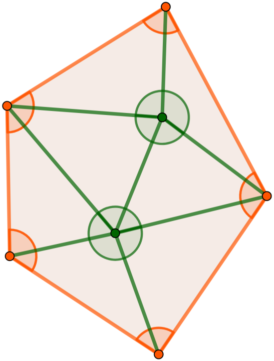

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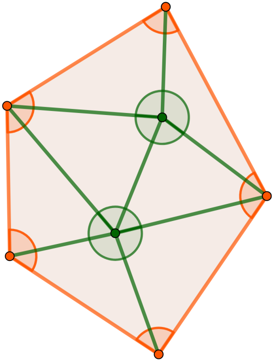

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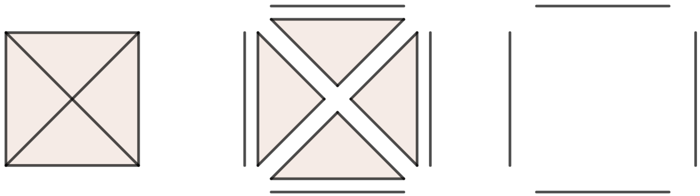

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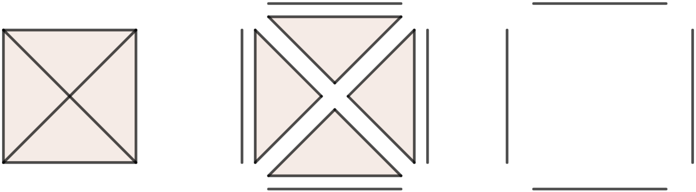

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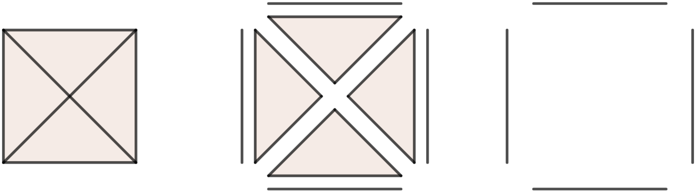

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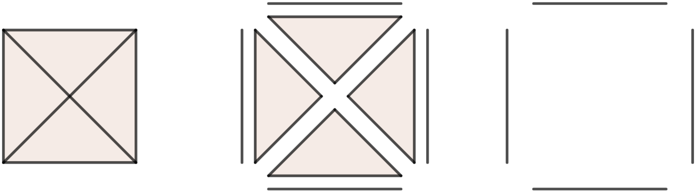

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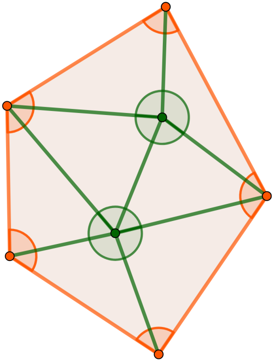

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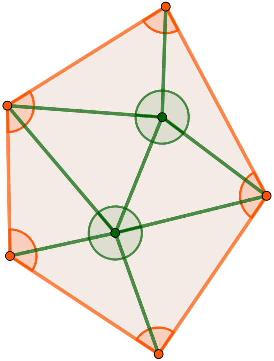

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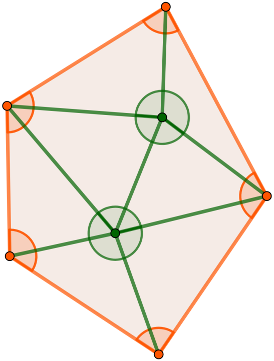

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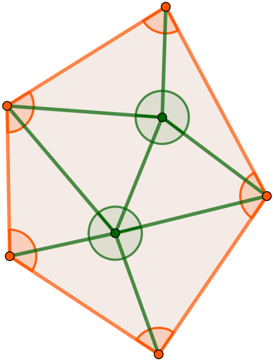

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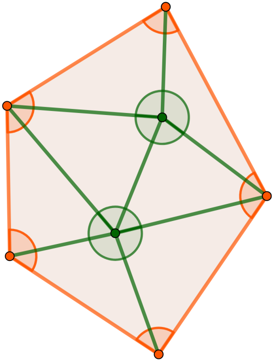

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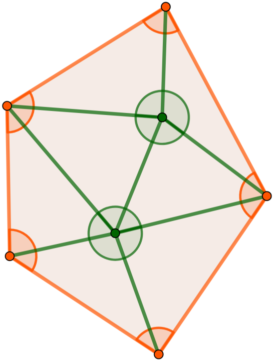

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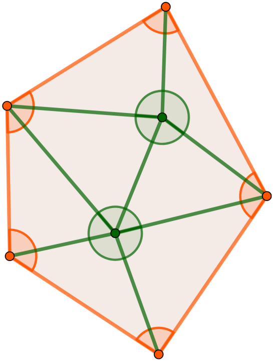

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The Euler Characteristic

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Definition

Let P be a nice polygon that has been cut into triangles so that in the end we see F triangles, E edges, and V vertices. Then

$$V - E + F$$

is called the *Euler characteristic* of P, and is denoted $\chi(P)$.

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Corollary

The Euler characteristic of a nice polygon does not depend on how we cut it into triangles.

Down the rabbit hole

Planar geometry is nice, but why stop there?

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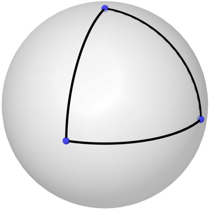
Spherical geometry



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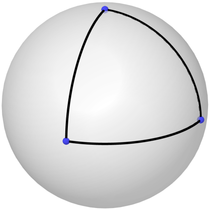

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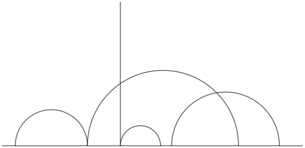

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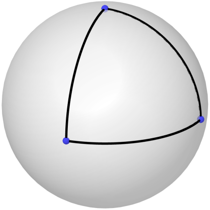

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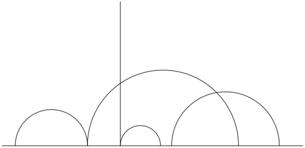
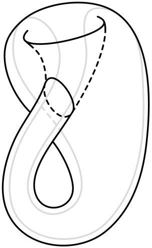


Other geometries



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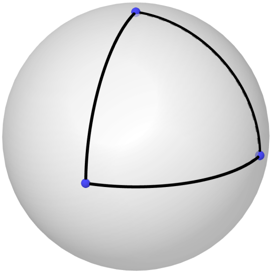

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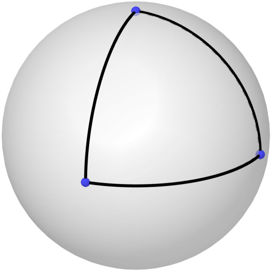

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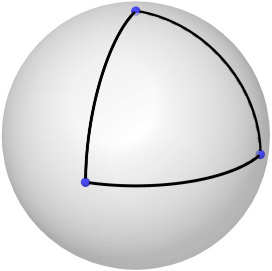

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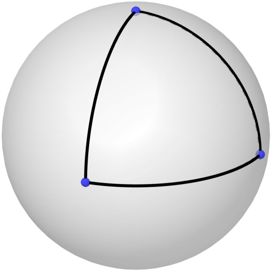

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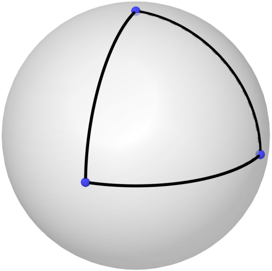

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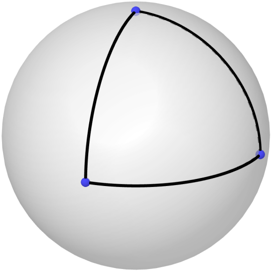

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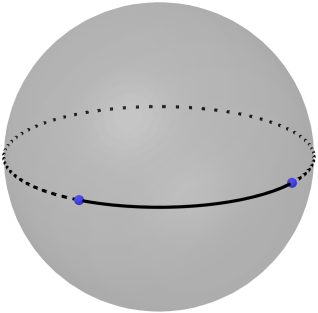

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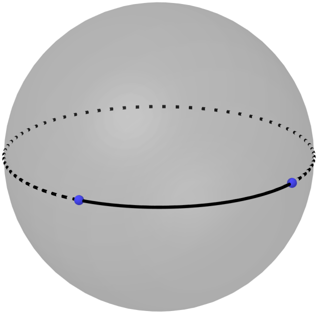

- Great circles will be our 'lines' and arcs of great circles will be our 'line segments.'

Triangles and polygons on the sphere

Our definitions transfer from the plane to the sphere.

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- A vertex of a polygon is where two edges meet.
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- A polygon is 'nice' if each of its vertices is adjacent to exactly two edges.

The Euler characteristic of a polygon in the sphere

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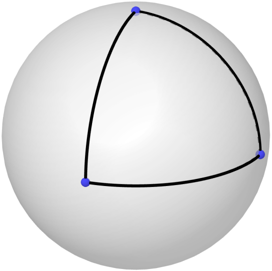
- Is $\chi(P)$ still equal to $2\pi \sum \text{ext. angles}$?

Example: Euler characteristic on the sphere



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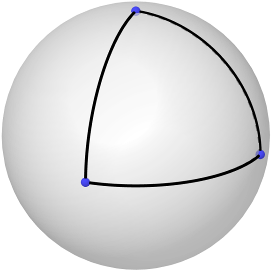

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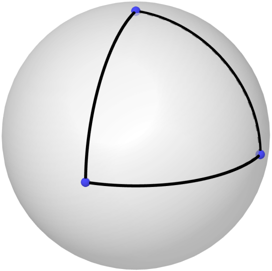

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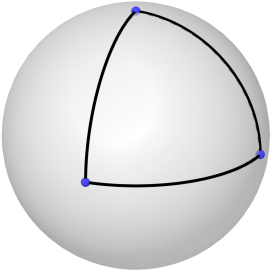

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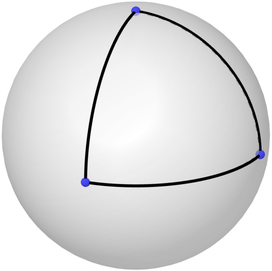

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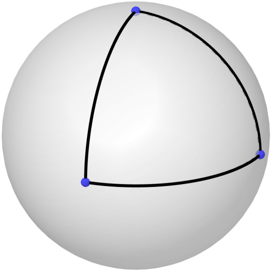

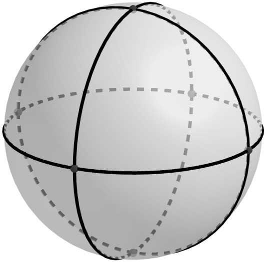

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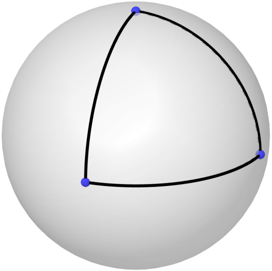

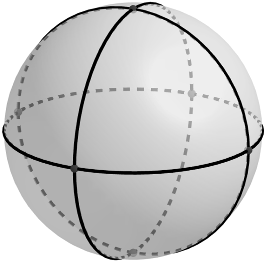

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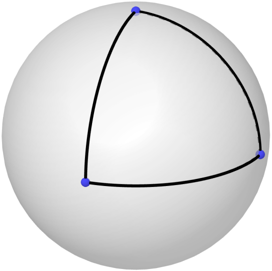

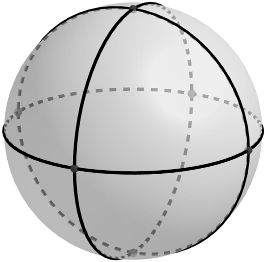

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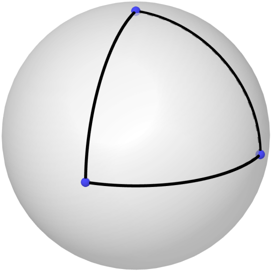

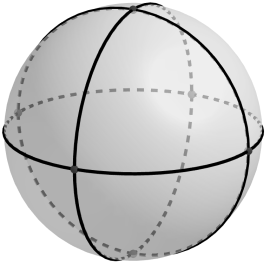

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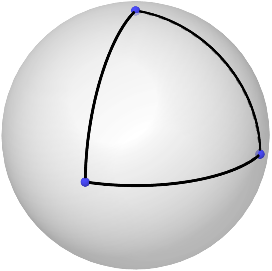

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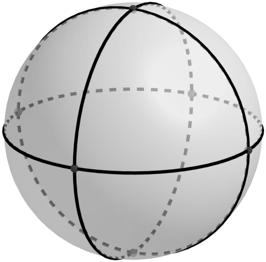

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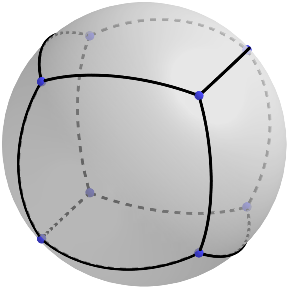

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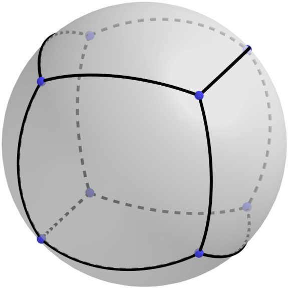

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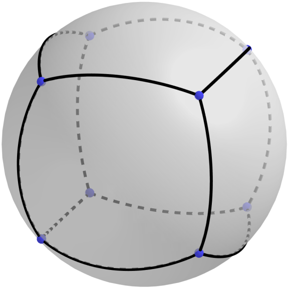

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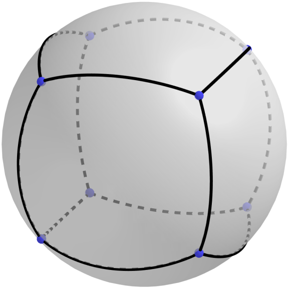

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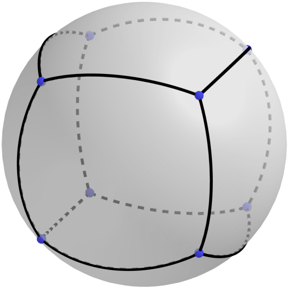

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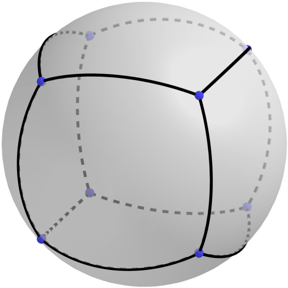

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Let P be a polygon which is cut into triangles $\Delta_1, \ldots, \Delta_F$ (following the rules). Then,

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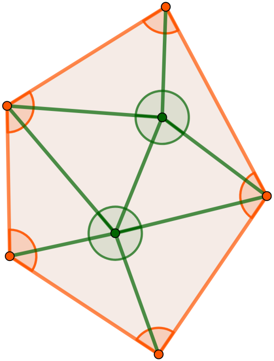
Proof of lemma



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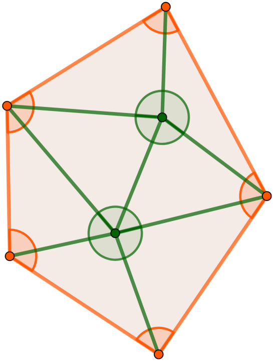

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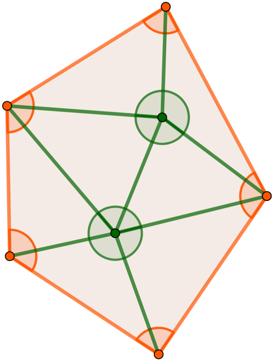

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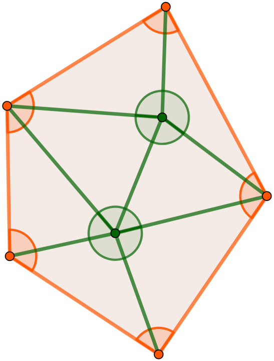

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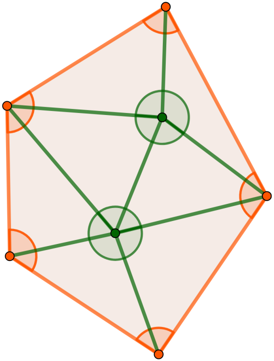

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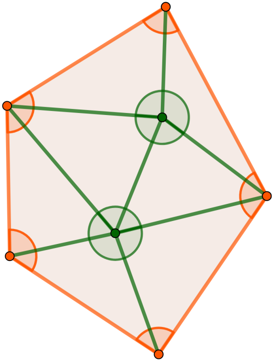
Proof of lemma (continued)



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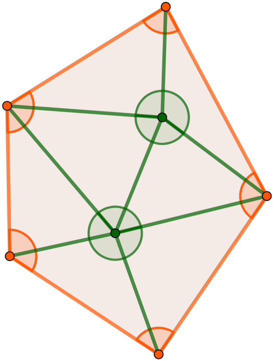

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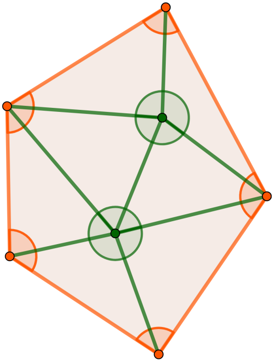
Proof of lemma (continued)



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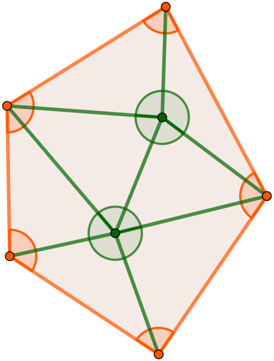

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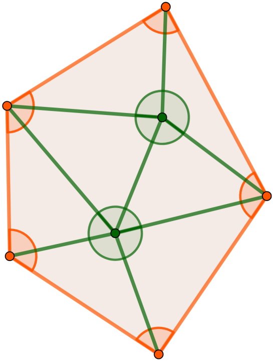

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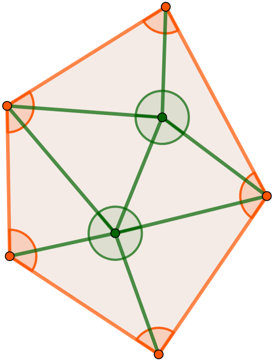

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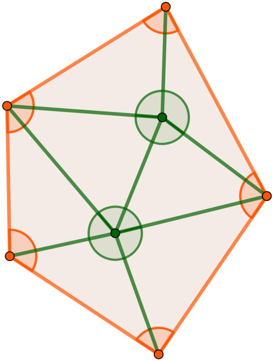

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Defect and area

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- Since defect and area follow the same rules, perhaps we can compare them.

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- Remark: This is kind of like taking a derivative.

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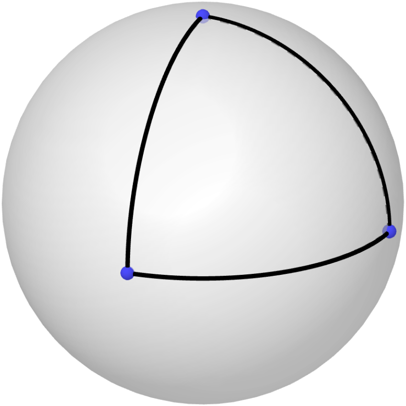

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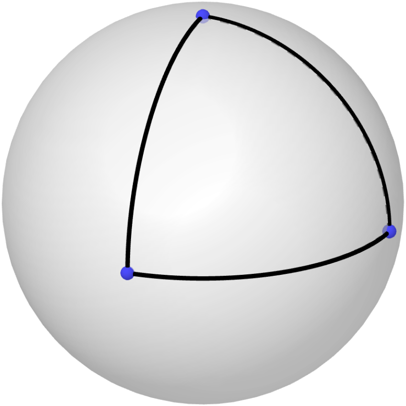

Recall:

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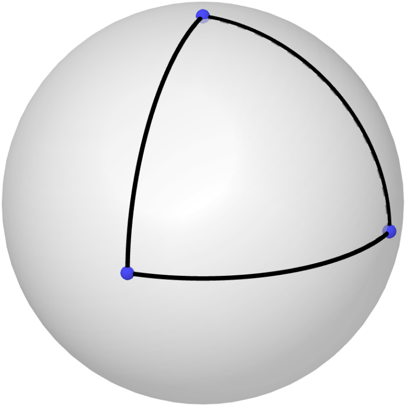

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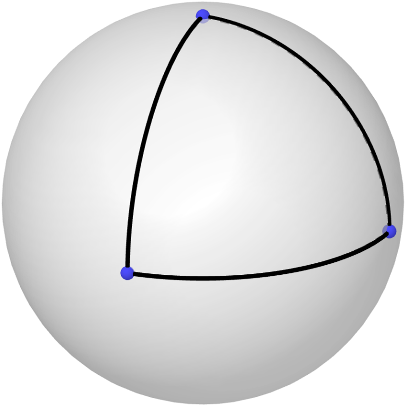

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- Hence, $\text{defect}(\Delta) = K\,\text{area}(\Delta)$ is solved by $K = 1$.

Angle Defect and Area on the Sphere

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Theorem

For a polygon P on the sphere, $\text{defect}(P) = \text{area}(P)$.

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