Vol. 82, No. 1 DUKE MATHEMATICAL JOURNAL January 1996

OSCILLATORY INTEGRALS AND MAXIMAL AVERAGES OVER HOMOGENEOUS SURFACES

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CONTENTS

1. Introduction. Let S be a smooth hypersurface in R^{n+1}, let $d\sigma$ denote Lebesgue measure on S, and let ψ denote a smooth cutoff function in R^{n+1}. Let δ_t denote the dilation $\delta_t h(x, x_{n+1}) = t^{-n}h(t^{-1}x, t^{-1}x_{n+1})$. We consider the convolution operators

$$M_t f(x, x_{n+1}) = f * \delta_t(\psi d\sigma)(x, x_{n+1})$$

and their associated maximal operator

$$\mathcal{M}f(x, x_{n+1}) = \sup_{t>0} M_t f(x, x_{n+1}). \tag{1}$$

It is not obvious that such convolutions are well defined for f in L^p spaces since S has measure zero in R^{n+1}. Nevertheless, a priori L^p estimates are possible when S has suitable curvature properties. A basic problem is thus to determine the optimal range of indices p such that

$$\|\mathcal{M}f\|_{L^p(R^{n+1})} \leqslant C_p\|f\|_{L^p(R^{n+1})}, \tag{2}$$

where f is initially taken to be in the class of rapidly decreasing functions.

Received 12 August 1994. Revision received 14 March 1995.
Sawyer supported in part by NSERC grant OGP0005149.

The study of such a maximal operator over dilations of a fixed hypersurface $S \subset R^{n+1}$ has its beginnings in the spherical maximal theorem of E. M. Stein [St2]. Stein showed that when $S = S^n$, the unit n-dimensional sphere, the inequality (2) holds for $p > (n+1)/n$, $n > 1$. The 2-dimensional version of this result ($n = 1$) was proved by Bourgain [B]. The key feature of the spherical maximal operator is the nonvanishing Gaussian curvature of the sphere. Indeed, one obtains the same L^p bounds if the sphere is replaced by a piece of any hypersurface in R^{n+1} with everywhere nonvanishing Gaussian curvature [Gr]. More generally, one can treat the case where the surfaces vary in the presence of nonvanishing rotational curvature; see, e.g., [St3, p. 494].

A fundamental unsolved problem is to characterize the L^p boundedness properties of the maximal operator associated to hypersurfaces where the Gaussian curvature is allowed to vanish. Of course, the maximal operator will not be bounded on any L^p, $p < \infty$, if a part of the hypersurface in question lies in a hyperplane not containing the origin, and even if the curvature is allowed to vanish to infinite order at just one point where the tangent plane doesn't pass through the origin; see, e.g., [St3, p. 512]. On the other hand, if the hypersurface is of finite type, then C. D. Sogge and E. M. Stein showed that there exists a $p_0 < \infty$ such that inequality (2) holds for $p > p_0$ [SoSt].

The purpose of this paper is to determine the best possible value for p_0 when the hypersurface S is the graph of $\Phi + c$ where Φ is a homogeneous function and c is a nonzero constant. In Theorems 1 and 2 below, we show that in most cases, $1/p_0 = \max\{\rho\colon \Phi(\omega)^{-1} \in L^\rho(S^{n-1})\}$ (at least when the maximum is at most 1/2). We note that some related cases of this problem have been handled. For example, under the assumption that Φ is convex, and that the determinant of the Hessian of Φ vanishes only at isolated points, sharp L^p bounds for the maximal operator have been obtained by Cowling and Mauceri [CM2]. Some sharp estimates have recently been obtained by Nagel, Seeger, and Wainger (see [NSW] which includes the case $p \leqslant 2$). The case when Φ is a homogeneous polynomial on R^2 where the gradient is nonvanishing away from the origin is obtained in [I3].

Our approach will follow the original square-function estimates of Stein used in the spherical case, and in so doing, we obtain sharp estimates for the Fourier transform of measures carried on the graphs of homogeneous functions. The basic ideas of the paper are as follows. We begin by showing in Theorem 2 below that if $\mathcal{M}$ is bounded on $L^p(R^{n+1})$ with S as above, then $p > m/n$ and $\Phi(\omega)^{-1} \in L^{1/p}(S^{n-1})$ are necessary conditions.

Conversely, we assume that $\Phi(\omega)^{-1} \in L^\rho(S^{n-1})$, $0 < \rho \leqslant \min\{n/m, 1/2\}$, and we consider measures given by weighting S with powers of $|\Phi|$. More precisely, let

$$d\beta_\alpha(y, y_{n+1}) = |\Phi(y)|^\alpha (\psi\, d\sigma)(y, y_{n+1}).$$

Given an additional finite type assumption on the level set $\Sigma = \{x\colon \Phi(x) = 1\}$, we will prove that $|\widehat{\beta_\alpha}(\xi, \xi_{n+1})| \leqslant C|(\xi, \xi_{n+1})|^{-(1/2)-\varepsilon}$ for some $\varepsilon > 0$ if $\alpha > (1/2) - \rho$. We obtain the decay $|(\xi, \xi_{n+1})|^{-(1/2)}$ merely from the curvature of Φ in the radial direction together with the assumptions that $\Phi^{-1} \in L^{\rho}(S^{n-1})$ and $\alpha > (1/2) - \rho$. To obtain the crucial stronger decay of $|(\xi, \xi_{n+1})|^{-(1/2)-\varepsilon}$, we need the finite type hypothesis on the level set Σ, and in fact this stronger decay estimate can fail if part of Σ lies in a hyperplane (see Example 10 below). Once this has been established, a theorem of Cowling and Mauceri [CM2] or Sogge and Stein [SoSt] uses square-function techniques to establish the L^2 boundedness of

$$\mathcal{M}_\alpha f(x, x_{n+1}) = \sup_{t>0} \left| \int_S f(x - ty, x_{n+1} - ty_{n+1})\, d\beta_\alpha(y, y_{n+1}) \right|, \tag{3}$$

when $1/2 > \alpha > (1/2) - \rho$. A simple application of Hölder's inequality, together with the local integrability of $|\Phi(y)|^{\varepsilon-\rho}$, shows that $\mathcal{M}_0 = \mathcal{M}$ is bounded on L^p for $p > 1/\rho$.

1.1. Maximal averages

THEOREM 1. *Let $S = \{(x, x_{n+1}) : x_{n+1} = \Phi(x) + c\}$ and let $\mathcal{M}f(x)$ be defined as in (1) above. Suppose that $\Phi \in C^\infty(R^n \backslash \{0\})$ is homogeneous of degree $m \geqslant n$, and that there is $0 < \rho \leqslant \min\{n/m, 1/2\}$ such that*

$$\Phi(\omega)^{-1} \in L^{\rho}(S^{n-1}) \tag{4}$$

and $\Sigma = \{x : \Phi(x) = 1\}$ is of finite type with polynomial bounds, i.e.,

$$\sum_{2 \leqslant |\beta| \leqslant l} \left| \frac{\partial^{|\beta|}}{\partial y^\beta} \Phi(x) \right| \geqslant c|x|^{-M}, |x| > 1, \tag{5}$$

for some $M \geqslant 0$, $\ell \geqslant 2$, and where $\beta = (\beta_1, \ldots, \beta_{n-1})$ is a multi-index, and $(y_1, \ldots, y_{n-1})$ is a coordinate system orthogonal to $\nabla\Phi(x)$ at x. Then (2) holds for $p > 1/\rho$, where f is initially taken to be in the class of rapidly decreasing functions.

Remark. The finite type condition (5) has the following implication for graphing functions. If $\nabla\Phi(x) \neq 0$, the implicit function theorem implies that we can find a normalized coordinate system $(y_1, \ldots, y_n)$ at x such that $\nabla\Phi(x)$ is parallel to the vector $(0, 0, \ldots, 1)$, and a smooth function $\Psi(y_1, \ldots, y_{n-1})$ such that for a sufficiently small radius R,

$$\Sigma \cap B(x, R) = \{x + (y_1, \ldots, y_n) : y_n = \Psi(y_1, \ldots, y_{n-1})\},$$

and

$$\sum_{2 \leqslant |\beta| \leqslant l} \left| \frac{\partial^{|\beta|}}{\partial y_1^{\beta_1} \cdots \partial y_{n-1}^{\beta_{n-1}}} \Psi(y) \right| \geqslant c|x|^{-M},$$

for $|(y_1, \ldots, y_{n-1})| < R$.

Note that the finite-type condition (5) is satisfied with $\ell = M$ if Φ is a polynomial of degree M. In the converse direction, if (2) holds for a given p, where S is the graph of $\Phi + c$, Φ is homogeneous of degree m, and $c \neq 0$, then the following theorem shows that we must have both $p > m/n$ and $\Phi(\omega)^{-1} \in L^{1/p}(S^{n-1})$. As a simple example, note that when $\Phi(x)$ is the monomial $x_1^{m_1} \cdots x_n^{m_n}$, then $\Phi(\omega)^{-1} \in L^{\rho}(S^{n-1})$ if and only if $\rho < 1/\max\{m_1, \ldots, m_n\}$, and thus (2) holds if and only if $p > \max\{m_1, \ldots, m_n\}$.

THEOREM 2. *Let S be a smooth hypersurface, ψ a smooth cutoff function, and let $\mathcal{M}f(x, x_{n+1})$ be defined as in (1) above. Suppose $\mathcal{H}$ is a hyperplane not passing through the origin and set*

$$d(x, x_{n+1}) = \text{distance}\,((x, x_{n+1}), \mathcal{H}).$$

If $\mathcal{M}$ is bounded on $L^p(R^{n+1})$ for some $p > 1$, then $d(x, x_{n+1})^{-1/p}$ is locally integrable on S. In particular, if $S = \{(x, x_{n+1}) : x_{n+1} = \Phi(x) + c\}$, where Φ is homogeneous of degree m and $c \neq 0$, and if $\mathcal{M}$ is bounded on $L^p(R^{n+1})$ for some $p > 1$, then $p > m/n$ and $\Phi(\omega)^{-1} \in L^{1/p}(S^{n-1})$.

Proof. The general case is easily reduced to the case where

$$S = \{(x, x_{n+1}) : x_{n+1} = \Phi(x) + c\},$$

$$\Phi(0) = 0,$$

$$c \neq 0,$$

$$\mathcal{H} = \{(x, x_{n+1}) : x_{n+1} = c\}.$$

Then $d(x, x_{n+1}) = |\Phi(x)|$ for $(x, x_{n+1}) \in S$, and it remains to show that $|\Phi(x)|^{-1/p}$ is locally integrable on R^n. Clearly, $|\{y \in R^{n+1} : \Phi(y) = 0\}|$ vanishes, since otherwise $\mathcal{M}f$ is identically infinite for any $f \geqslant 0$ that is infinite on the hyperplane $\{x_{n+1} = 0\}$. So let

$$\varphi(t) = \int_{\{y \in B : |\Phi(y)| > t\}} |\Phi(y)|^{-1/p} dy,$$

where B denotes the unit ball in R^n, and suppose, in order to contradict (2), that φ is unbounded on $(0, 1)$. Then we can find $\gamma(t)$ increasing on $(0, 1)$ such that

$$\begin{cases} \displaystyle\int_0^1 \gamma(t)^p (dt/t) = 2 \int_0^1 \gamma(t^2)^p (dt/t) < \infty, \\ \displaystyle\int_0^1 \left(\gamma(t^2)\varphi(t)\right)^p (dt/t) = \infty. \end{cases} \tag{6}$$

Define $f(x, x_{n+1}) = |x_{n+1}|^{-1/p}\gamma(|x_{n+1}|)\psi(x, x_{n+1})$, where ψ is as above. Then $f \in L^p$ for $p > 1$ by (6). On the other hand, if $t = x_{n+1}/c$, then for (x, x_{n+1}) near the origin, we have

$$M_t f(x, x_{n+1}) = \int_{R^n} f(x - ty, x_{n+1} - t(\Phi(y) + c))\psi(y, \Phi(y) + c)\, dy$$

$$= \int_{R^n} |t\,\Phi(y)|^{-1/p}\gamma(t|\Phi(y)|)\psi(x - ty, -t\,\Phi(y))\psi(y, \Phi(y) + c)\, dy$$

$$\geqslant t^{-1/p} \int_B |\Phi(y)|^{-1/p}\gamma(t|\Phi(y)|)\, dy \quad \text{(provided supp } \psi \text{ large enough)}$$

$$\geqslant t^{-1/p}\gamma(t^2) \int_{\{y \in B : |\Phi(y)| > t\}} |\Phi(y)|^{-1/p} dy \quad \text{(since } \gamma \text{ is increasing)}$$

$$= t^{-1/p}\gamma(t^2)\varphi(t) = |x_{n+1}/c|^{-1/p}\, \gamma(|x_{n+1}/c|^2)\varphi(|x_{n+1}/c|).$$

Thus $\mathcal{M}f(x, x_{n+1}) \geqslant |x_{n+1}/c|^{-1/p}\gamma(|x_{n+1}/c|^2)\varphi(|x_{n+1}/c|)$ for (x, x_{n+1}) near the origin, and so $\|\mathcal{M}f\|_{L^p(S^{n-1})} = \infty$ by (6), contradicting (2).

In the case where Φ is homogeneous of degree m, we take $\mathcal{H} = \{x_{n+1} = c\}$ and compute that

$$\int_B |\Phi(y)|^{-1/p}\, dy = \left(\int_0^1 r^{n-1} r^{-m/p}\, dr\right)\left(\int_{S^{n-1}} |\Phi(\omega)|^{-1/p}\, d\omega\right).$$

The finiteness of the integrals implies both $p > m/n$ and $\Phi(\omega)^{-1} \in L^{1/p}(S^{n-1})$.

Remark 1. In the case where $c = 0$ above, i.e., $S = \{(x, x_{n+1}) : x_{n+1} = \Phi(x)\}$ and Φ is homogeneous of degree m, we can often do better, namely show that $\mathcal{M}$ is bounded on $L^p(R^{n+1})$ for $p > (n+1)/n$. For example, this holds if, in addition, S has nonvanishing Gaussian curvature away from the origin. To see this, simply apply the proof of Theorem 3 in [I3], but with $c = 0$. See also Chapter 6 of [So1] where this type of argument using Fourier integral operators originates.

Remark 2. The index $(n+1)/n$ in the previous remark is sharp. More generally, if S is a smooth hypersurface such that the projection onto the sphere S^n has positive measure, then $\mathcal{M}$ cannot be bounded on $L^p(R^{n+1})$ for $p \leqslant (n+1)/n$. To see this, simply use the example $f(x) = |(x, x_{n+1})|^{-n}(\log(1/|(x, x_{n+1})|))^{-1} \cdot \psi(x, x_{n+1})$ as on p. 472 of [St3]. Then $f \in L^{(n+1)/n}(R^{n+1})$, but $\mathcal{M}f$ is easily seen to be infinite on a set of positive measure.

Remark 3. On the other hand, if the projection of S onto the sphere S^n has measure zero, and S is of finite type, then $\mathcal{M}$ is bounded on $L^p(R^{n+1})$ for $p > 1$.

In this case, $\mathcal{M}$ is essentially a supremum of averages over a fixed submanifold of finite type, and Theorem 1 on p. 476 of [St3] applies.

There is also a more general version of our maximal theorem which is essentially a perturbation of Theorem 1.

THEOREM 3. *Let* $S = \{(x, x_{n+1}) : x_{n+1} = \Phi(x) + c\}$. *Let* $\mathcal{M}f(x)$ *be defined as in (1) above. Suppose that* $\Phi(tx) = \Gamma(t)\Phi(x)$ *for all* $t > 0$, *where*

$$\Gamma(0) = \Gamma'(0) = \cdots = \Gamma^{(m-1)}(0) = 0 \ \textit{with}\ \Gamma^{(m)}(0) > 0 \ \textit{for some}\ m \geqslant n,$$

$$\Gamma'' \textit{ is nonvanishing away from zero,} \tag{7}$$

and that there is $0 < \rho \leqslant \min\{n/m, 1/2\}$ *such that both (4) and (5) hold. Then (2) holds for* $p > 1/\rho$, *where* f *is initially taken to be in the class of rapidly decreasing functions.*

1.2. Oscillatory integrals. Our main tool in proving Theorem 1 is part (B) of the following estimates for weighted oscillatory integrals. These estimates are motivated by the following simple observations. Let Φ be a smooth function homogeneous of degree $m \geqslant n$. Then,

$$\begin{aligned}\int_{R^n} e^{-|\Phi(x)|}\,dx &= c_n \int_{S^{n-1}} \int_0^\infty e^{-r^m|\Phi(\omega)|} r^{n-1}\,dr\,d\omega \\ &= c_n \int_{S^{n-1}} \left(\int_0^\infty e^{-t} t^{(n/m)-1}(1/m)\,dt \right) |\Phi(\omega)|^{-n/m}\,d\omega \\ &= c_n(1/m)\Gamma\left(\frac{n}{m}\right) \int_{S^{n-1}} |\Phi(\omega)|^{-n/m}\,d\omega,\end{aligned} \tag{8}$$

where $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt$ is the usual Gamma function. Formally, we can carry out this calculation for the oscillatory integral of the first kind $\int e^{i\lambda\Phi(x)}dx$. If, as above, we go into polar coordinates and make a change of variables sending $r \to r(\lambda\Phi(\omega))^{-1/m}$, we see that

$$\begin{aligned}\int e^{i\lambda\Phi(x)}dx &= \lambda^{-n/m} c_n \int_{S^{n-1}} \int_0^\infty e^{ir^m} r^{n-1} \Phi(\omega)^{-n/m}\,dr\,d\omega \\ &= \lambda^{-n/m} c_n \left(\int_0^\infty e^{ir^m} r^{n-1} dr \right) \left(\int_{S^{n-1}} \Phi(\omega)^{-n/m}\,d\omega \right) \\ &= \lambda^{-n/m} c_n e^{2\pi i/m}(1/m)\Gamma(n/m) \left(\int_{S^{n-1}} \Phi(\omega)^{-n/m}\,d\omega \right),\end{aligned} \tag{9}$$

for $m \geqslant n$. This suggests that the optimal decay for the Fourier transform of a smooth measure carried by the graph of a homogeneous function will hold if and only if $\Phi^{-1} \in L^{n/m}(S^{n-1})$ (that this is indeed the case is in Corollary 5 below).

THEOREM 4. *Suppose that $\Phi \in C^{\infty}(R^n \backslash \{0\})$ is homogeneous of degree $m > n$, and that $0 < \rho \leqslant \min\{n/m, 1/2\}$. Let*

$$F_\alpha(\xi, \lambda) = \int e^{i(x \cdot \xi + \lambda \Phi(x))} |\Phi(x)|^\alpha \psi(x)\, dx,$$

where ψ is as above and $n/m \leqslant \alpha + (n/m) < 1$.

(A) *If $0 < \alpha + \rho \leqslant 1/2$ and $\Phi(\omega)^{-1} \in L^\rho(S^{n-1})$, then*

$$|F_\alpha(\xi, \lambda)| \leqslant C(|\lambda| + |\xi|)^{-(\alpha+\rho)}. \tag{10}$$

If, however, $0 < \rho < n/m$ and $0 < \alpha + \rho < 1/2$, then the weaker assumption $\Phi(\omega)^{-1} \in$ weak *$L^\rho(S^{n-1})$ implies (10). Conversely, suppose $\Phi > 0$ almost everywhere, (10) holds with $\xi = 0$, $0 < \alpha + (n/m) < 1$, and that ψ is smooth and radially decreasing. Then $\Phi(\omega)^{-1} \in$* weak *$L^\rho(S^{n-1})$ if $\rho > 0$, while $\Phi(\omega)^{-1} \in L^{n/m}(S^{n-1})$ if $\rho = n/m$.*

(B) *If $\alpha + \rho > 1/2 > \alpha$, and in addition $\Sigma = \{x\colon \Phi(x) = 1\}$ is of finite type with polynomial bounds, i.e., (5) holds, then there exists an $\varepsilon > 0$ such that*

$$|F_\alpha(\xi, \lambda)| \leqslant C(|\lambda| + |\xi|)^{-(1/2)-\varepsilon}. \tag{11}$$

Remark 1. In the converse assertion in (A) of Theorem 4, the restriction $\Phi > 0$ almost everywhere is not essential. In fact, if $\left|\{\omega \in S^{n-1}\colon \Phi(\omega) = 0\}\right| = 0$ and

$$\tilde{F}_\alpha(\xi, \lambda) = \int e^{i(x \cdot \xi + \lambda \Phi(x))} \operatorname{sgn}(\Phi(x)) |\Phi(x)|^\alpha \psi(x)\, dx$$

satisfies (10) in place of F_α, then the same conclusions hold for Φ. Moreover, if $\alpha = 0$ and $\tilde{F}_\alpha$ satisfies (10) in place of F_α, then we must have $|\{\Phi = 0\}| = 0$ (see Subsection 2.1 below).

Remark 2. In the case $\alpha + (n/m) < 1/2$ (the proof of Theorem 1 requires only the remaining case), (A) of Theorem 4 has a particularly simple proof using only the fact that the Fourier transform of a nondegenerate plane curve has decay $C|\xi|^{-1/2}$. Indeed, begin by letting $\Omega = \{\omega \in S^{n-1}\colon |\Phi(\omega)| > 1/\lambda\}$ and writing

$$F_\alpha(\xi, \lambda) = \int e^{i(x \cdot \xi + \lambda \Phi(x))} |\Phi(x)|^\alpha \psi(x)\, dx$$

$$= \int_{x/|x| \in S^{n-1} \backslash \Omega} + \int_{x/|x| \in \Omega} = F_\alpha^{S^{n-1} \backslash \Omega} + F_\alpha^{\Omega}.$$

Now if we take absolute values inside the first integral, we see immediately that

$$\begin{aligned}|F_\alpha^{S^{n-1}\backslash\Omega}(\xi,\lambda)| &\leqslant C\int_{\{\omega\in S^{n-1}:\,|\Phi(\omega)|\leqslant 1/\lambda\}}|\Phi(\omega)|^\alpha\,d\omega\\ &\leqslant C\lambda^{-\alpha}|\{\omega\in S^{n-1}:\,|\Phi(\omega)|^{-1}>\lambda\}|\\ &\leqslant C\lambda^{-\alpha-\rho},\end{aligned}$$

since $\Phi(\omega)^{-1}\in$ weak $L^\rho(S^{n-1})$. To handle the second integral, let φ be supported in the annulus $\mathscr{A}=\{x\in R^n:\,1/2\leqslant|x|\leqslant 2\}$ so that

$$\sum_{k=0}^\infty \varphi(2^k x)=1,\qquad |x|\leqslant 1.$$

Setting

$$F_{\alpha,k}^\Omega(\xi,\lambda)=\int_{x/|x|\in\Omega}e^{i(x\cdot\xi+\lambda\Phi(x))}\varphi(2^kx)\psi(x)|\Phi(x)|^\alpha\,dx,$$

we obtain

$$\begin{aligned}F_\alpha^\Omega(\xi,\lambda)&=\sum_{k=0}^\infty F_{\alpha,k}^\Omega(\xi,\lambda)=\sum_{k=0}^\infty 2^{-kn}2^{-km\alpha}\tilde F_{\alpha,k}^\Omega(2^{-k}\xi,2^{-km}\lambda),\\ &=\left(\sum_{2^{km}\geqslant\lambda|\Phi(\omega)|}+\sum_{2^{km}<\lambda|\Phi(\omega)|}\right)2^{-kn}2^{-km\alpha}\tilde F_{\alpha,k}^\Omega(2^{-k}\xi,2^{-km}\lambda)\\ &=I+II,\end{aligned}$$

where

$$\begin{aligned}&\left|\tilde F_{\alpha,k}^\Omega(2^{-k}\xi,2^{-km}\lambda)\right|\\ &\quad=\left|\int_{x/|x|\in\Omega}e^{i(x\cdot 2^{-k}\xi-2^{-km}\lambda\Phi(x))}\varphi(x)\psi(2^{-k}x)|\Phi(x)|^\alpha\,dx\right|\\ &\quad\leqslant\int_\Omega\left|\int e^{i\{r(2^{-k}\omega\cdot\xi)-r^m(2^{-km}\lambda\Phi(\omega))\}}\varphi(r)\psi(2^{-k}r)r^{m\alpha+n-1}\,dr\right||\Phi(\omega)|^\alpha\,d\omega\\ &\quad\leqslant C\int_\Omega(1+2^{-km}\lambda\Phi(\omega))^{-1/2}|\Phi(\omega)|^\alpha\,d\omega,\end{aligned}$$

since the curve (r, r^m) is nondegenerate for $r \in (1/2, 2)$. Thus we have

$$|I| \leqslant C \int_\Omega \left(\sum_{2^{km} \geqslant \lambda|\Phi(\omega)|} 2^{-kn} 2^{-km\alpha} \right) |\Phi(\omega)|^\alpha \, d\omega$$

$$\leqslant C \int_\Omega (\lambda |\Phi(\omega)|)^{-(n/m)-\alpha} |\Phi(\omega)|^\alpha \, d\omega$$

$$\leqslant C\lambda^{-\alpha-(n/m)} \int_{\{\omega \in S^{n-1}: |\Phi(\omega)| > 1/\lambda\}} |\Phi(\omega)|^{-n/m} \, d\omega$$

$$= C\lambda^{-\alpha-(n/m)} \int_0^\lambda \frac{n}{m} t^{(n/m)-1} |\{\omega \in S^{n-1}: t < |\Phi(\omega)|^{-1} < \lambda\}| \, dt$$

$$\leqslant C\lambda^{-\alpha-(n/m)} \int_0^\lambda \frac{n}{m} t^{(n/m)-1} C(1+t)^{-\rho} \, dt \leqslant C\lambda^{-\alpha-\rho},$$

since $|\Phi|^{-1} \in$ *weak* $L^\rho(S^{n-1})$ (in the case $\rho = n/m$, simply use the hypothesis $|\Phi|^{-1} \in L^\rho(S^{n-1})$) and

$$|II| \leqslant C \int_\Omega \sum_{2^{km} < \lambda|\Phi(\omega)|} 2^{-kn} 2^{-km\alpha} \left(2^{-km} \lambda |\Phi(\omega)|\right)^{-1/2} |\Phi(\omega)|^\alpha \, d\omega$$

$$\leqslant C\lambda^{-1/2} \int_\Omega \left(\sum_{2^{km} < \lambda|\Phi(\omega)|} (2^{km})^{(1/2)-\alpha-(n/m)} \right) |\Phi(\omega)|^{\alpha-(1/2)} \, d\omega$$

$$\leqslant C\lambda^{-\alpha-(n/m)} \int_{\{\omega \in S^{n-1}: |\Phi(\omega)| > 1/\lambda\}} |\Phi(\omega)|^{-n/m} \, d\omega,$$

(since $\alpha + (n/m) < 1/2$) which is dominated by $C\lambda^{-\alpha-\rho}$ just as in the estimate for I. This completes the proof since if $|\xi| \geqslant C\lambda$, then integration by parts yields rapid decay (see Proposition 1, p. 331 in [St3]).

The case $\alpha = 0$ in (A) of Theorem 4 implies the following characterization of the decay of the Fourier transform of measures carried by the graphs of homogeneous functions.

COROLLARY 5. *Suppose* $\Phi \geqslant 0$, ψ *is smooth and radially decreasing and that* $0 < (n/m) \leqslant 1/2$. *Then*

$$\left| \int e^{i(\langle x,\xi \rangle + \lambda\Phi(x))} \psi(x) \, dx \right| \leqslant C(|\lambda| + |\xi|)^{-n/m}$$

if and only if $\Phi(\omega)^{-1} \in L^{n/m}(S^{n-1})$. *Now suppose* $m \geqslant n$ *and* $0 < \rho < \min\{n/m, 1/2\}$. *Then*

$$\left| \int e^{i(\langle x,\xi\rangle + \lambda\Phi(x))} \psi(x)\, dx \right| \leqslant C(|\lambda| + |\xi|)^{-\rho}$$

if and only if $\Phi(\omega)^{-1} \in$ weak $L^{\rho}(S^{n-1})$.

In Example 10 at the end of the next section, we show that (11) can fail for all $\alpha > 0$ if part of Σ lies in a hyperplane. In order to prove the perturbation result, Theorem 3, we need part (B) of the above estimates to hold when $\Phi(tx) = \Gamma(t)\Phi(x)$ for all $t > 0$, where Γ satisfies (7). This is the content of the next theorem.

THEOREM 6. *Suppose that* $\Phi(tx) = \Gamma(t)\Phi(x)$ *for all* $t > 0$ *where* Γ *satisfies (7), and that there is* ρ *satisfying* $0 < \rho \leqslant \min\{n/m, 1/2\}$ *such that both (4) and (5) hold. Then for* supp ψ *sufficiently small, there exists an* $\varepsilon > 0$ *such that (11) holds.*

We will also need the following result due to Cowling and Mauceri [CM2]. Alternatively, Theorem 4 could easily be strengthened so that the analogous result of Sogge and Stein [SoSt] could be used instead.

THEOREM 7. *Let* ϕ *be a compactly supported distribution on* R^n. *Suppose that for some* $\varepsilon > 0$ *we have*

$$|\hat{\phi}(x)| \leqslant C(1+|x|)^{-(1/2)-\varepsilon}, \quad x \in R^n, \quad x \neq 0. \tag{12}$$

Let $\widehat{\delta_t\phi}(\xi) = \hat{\phi}(t\xi)$ *and* $\phi^{\#}f(x) = \sup_{t>0} |\delta_t\phi * f(x)|$. *Then*

$$\|\phi^{\#} f\|_2 \leqslant C\|f\|_2, \qquad f \in \mathcal{S}(R^n). \tag{13}$$

We can now prove Theorems 1 and 3.

Proof of Theorems 1 and 3. Hölder's inequality shows that

$$|\mathcal{M}f| \leqslant (\mathcal{M}_\alpha |f|^r)^{1/r} \left(\int |\Phi(y)|^{-\alpha(r'/r)} \psi(y)\, dy \right)^{1/r'} \leqslant C_{\alpha,r} (\mathcal{M}_\alpha |f|^r)^{1/r},$$

where $\mathcal{M}_\alpha$ is as in (3), provided $\alpha(r'/r) < \rho$, i.e., $r > (\alpha + \rho)/\rho$, since

$$\int |\Phi(y)|^{-q} \psi(y) dy = c_n \int_{S^{n-1}} \left(\int_0^\infty r^{-qm+n-1} \psi(r)\, dr \right) |\Phi(\omega)|^{-q}\, d\omega < \infty$$

for $q \leqslant \rho$, $q < n/m$. By Theorems 4(B) and 7, $\mathcal{M}_\alpha$ is bounded on L^2 if $\alpha + \rho > 1/2$. Now fix $p > 1/\rho \geqslant 2$ and set $r = p/2$. Then $(1/2) - \rho < \rho(r-1)$ and thus we can choose α in $((1/2)-\rho, \rho(r-1))$, which yields both $\alpha+\rho > 1/2$ and $r > (\alpha+\rho)/\rho$. Then

$$\left(\int |\mathcal{M}f|^p\right)^{1/p} \leqslant C_{\alpha,r}\left(\int (\mathcal{M}_\alpha|f|^r)^2\right)^{1/p} \leqslant CC_{\alpha,r}\left(\int |f|^p\right)^{1/p}.$$

To prove Theorem 3, we decompose the maximal operator $\mathcal{M}$ as $\mathcal{M}_1+\mathcal{M}_2$ where the measure $\psi_1 d\sigma$ used in the definition of $\mathcal{M}_1$ has support so small that the conclusions of Theorem 6 hold. The proof in the previous paragraph shows that $\mathcal{M}_1$ is bounded on L^p for $p > 1/\rho$. For $\mathcal{M}_2$, we can proceed in two ways. First, since there is at least one nonvanishing principal curvature on the support of the measure $\psi_2 d\sigma$, a result of C. D. Sogge [So2] shows that $\mathcal{M}_2$ is bounded on L^p for $p > 2$. Alternatively, with some additional work, we could obtain Theorem 6 without any restriction on the support of ψ, and then the previous proof applies.

1.3. Mixed homogeneous surfaces. In the case $n = 2$, we can obtain more precise information about the weighted oscillatory integrals described above, and moreover, we can treat both oscillatory integrals and maximal functions when Φ satisfies a more general notion of homogeneity. We say that $\Phi\colon R^2 \to R$ is mixed homogeneous of degree (a_1, a_2) if $\Phi(t^{1/a_1}x_1, t^{1/a_2}x_2) = t\Phi(x_1, x_2)$ for all $t > 0$. Note that the degree (a_1, a_2) is not always uniquely determined. In the case Φ is homogeneous of degree m in the usual sense, we always choose the mixed homogeneity to be (m, m). This choice maximizes the decay estimates below. Let $Z_\Phi = \{x\colon \Phi(x) = 0\}$, and $Z_{H\Phi} = \{x\colon H\Phi(x) = 0\}$, where $H\Phi$ denotes the determinant of the Hessian matrix of Φ. We prove the following estimates. The case when Φ is homogeneous in the usual sense is in [I3].

THEOREM 8. *Let $S = \{(x, x_3) \in R^3\colon x_3 = \Phi(x) + c\}$, where Φ is mixed homogeneous of degree (a_1, a_2), $a_1 \geqslant a_2 \geqslant 2$. Suppose $Z_\Phi \cap Z_{H\Phi} = \{(0,0)\}$ and that $\nabla\Phi$ vanishes only at $(0,0)$.*

(A) *If $\alpha \geqslant 0$, $(1/a_1)+(1/a_2)+\alpha < 1$ and $F_\alpha(\xi,\lambda) = \int e^{i(x\cdot\xi+\lambda\Phi(x))}|\Phi(x)|^\alpha \cdot \psi(x)\,dx$, then*

$$|F_\alpha(\xi,\lambda)| \leqslant C(|\xi|+|\lambda|)^{-(1/a_1)-(1/a_2)-\alpha}, \tag{14}$$

for $\alpha \leqslant (1/2)-(1/a_2)$.

(B) *If $(1/a_1)+(1/a_2) \leqslant 1/2$ and $\mathcal{M}f(x)$ is defined as in (1) above, then (2) holds for $p > 1/((1/a_1)+(1/a_2))$, where f is initially taken to be in the class of rapidly decreasing functions.*

Note that (B) follows from (A) in the same way that Theorem 1 follows from Theorem 4.

Remark. The parameter α cannot exceed $(1/2)-(1/a_2)$. Indeed, this is easy to see if $\Phi(x) = x_1^{a_1} + x_2^{a_2}$, since the variables almost separate.

It would be of interest to extend Theorem 8 to higher dimensions with the analogous notion of mixed homogeneity. However, the techniques used here rely on detailed asymptotics for the Fourier transform of the measure carried by Σ, and for $n \geqslant 3$, Σ is no longer a curve, and these asymptotics are not available. In fact, in both of our main theorems, namely 1 and 8, the key to using the square-function techniques is to obtain decay estimates beyond $1/2$. This in turn requires some sort of curvature in at least two different directions, and thus we must obtain precise asymptotics in at least one of these. However, precise asymptotics are only available through the Van der Corput Lemma for curves of the form (t, t^{β}). In Theorem 1, this occurs in the radial direction, since the homogeneity assumption leads to the curve (r, r^m) (see (18) below), while in Theorem 8, this occurs in the tangential direction, since the level set Σ is a finite type curve (see (56) below).

Finally, we note that in the case when Φ is homogeneous, $Z_{\Phi} \cap Z_{H\Phi} = \{(0,0)\}$ is equivalent to the property that $\nabla\Phi$ vanishes only at $(0,0)$ by the Euler homogeneity relations. However, this no longer holds in the mixed homogeneous case, for example when $\Phi(x_1, x_2) = \left(x_1 - x_2^2\right)\left(x_1 - 2x_2^2\right)$.

2. Oscillatory estimates. The purpose of this section is to prove Theorems 4 and 6. We have

$$F_{\alpha}(\xi, \lambda) = \int e^{i(x \cdot \xi - \lambda\Phi(x))} \psi(x) |\Phi(x)|^{\alpha} \, dx,$$

where ψ is a smooth compactly supported radial function. In fact, for the purpose of obtaining decay estimates, it suffices to assume that ψ is supported in the annulus $\mathscr{A} = \{x \in R^n : 1/2 \leqslant |x| \leqslant 2\}$. Indeed, if for some $0 \leqslant \gamma < \alpha + (n/m)$ we have the estimate

$$|F_{\alpha}(\xi, \lambda)| \leqslant C|\lambda|^{-\gamma}, \tag{15}$$

whenever ψ is supported in the annulus $\mathscr{A}$, then we automatically have the same decay estimate for ψ supported in the unit ball. To see this, let φ be supported in $\mathscr{A}$ so that

$$\sum_{k=0}^{\infty} \varphi(2^k x) = 1, \qquad |x| \leqslant 1.$$

Setting

$$F_{\alpha}^{k}(\xi, \lambda) = \int e^{i(x \cdot \xi - \lambda\Phi(x))} \varphi(2^k x) \psi(x) |\Phi(x)|^{\alpha} \, dx,$$

we obtain

$$F_\alpha(\xi,\lambda)=\sum_{k=0}^{\infty} F_\alpha^k(\xi,\lambda)=\sum_{k=0}^{\infty} 2^{-kn}2^{-km\alpha}\tilde{F}_\alpha^k(2^{-k}\xi,2^{-km}\lambda),$$

where the $\tilde{F}_\alpha^k(\xi,\lambda)=\int e^{i(x\cdot\xi-\lambda\Phi(x))}\varphi(x)\psi(2^{-k}x)|\Phi(x)|^\alpha\,dx$ uniformly satisfy the estimate (15). Thus

$$F_\alpha(\xi,\lambda)=\left(\sum_{2^{km}\geqslant\lambda}+\sum_{2^{km}<\lambda}\right)2^{-kn}2^{-km\alpha}\tilde{F}_\alpha^k(2^{-k}\xi,2^{-km}\lambda)=I+II,$$

where

$$|I|\leqslant\sum_{2^{km}\geqslant\lambda}2^{-kn}2^{-km\alpha}C\leqslant C\lambda^{-\alpha-(n/m)},$$

since the $\tilde{F}_\alpha^k$ are bounded, and

$$|II|\leqslant\sum_{2^{km}<\lambda}2^{-kn}2^{-km\alpha}C(2^{-km}\lambda)^{-\gamma}\leqslant C\lambda^{-\gamma},$$

since the $\tilde{F}_\alpha^k$ satisfy (15) and since $\gamma<\alpha+(n/m)$.

In polar coordinates,

$$F_\alpha(\xi,\lambda)=\int_{S^{n-1}}\int e^{i(r\omega\cdot\xi-\lambda r^m\Phi(\omega))}\psi(r)r^{m\alpha+n-1}|\Phi(\omega)|^\alpha\,dr\,d\omega,$$

$$=\int_{S^{n-1}\cap\{\omega:\Phi(\omega)\geqslant 0\}}+\int_{S^{n-1}\cap\{\omega:\Phi(\omega)<0\}}=F_\alpha^+(\xi,\lambda)+F_\alpha^-(\xi,\lambda).$$

We claim that both $F_\alpha^+(\xi,\lambda)$ and $F_\alpha^-(\xi,\lambda)$ satisfy the decay estimate (10). However, as the proofs are virtually identical, we consider only $F_\alpha^+(\xi,\lambda)$, and for convenience, we simply assume $\Phi\geqslant 0$ so that $F_\alpha(\xi,\lambda)=F_\alpha^+(\xi,\lambda)$. After making a change of variables sending $r\to r(\lambda\Phi(\omega))^{-1/m}$, we get that $F_\alpha(\xi,\lambda)$ is

$$\lambda^{-\alpha-(n/m)}\int_{S^{n-1}}\Phi(\omega)^{-(n/m)}\left\{\int\exp\left(i(r(\langle\omega,\xi\rangle)/(\lambda\Phi(\omega))^{1/m})-r^m)\right)\right.$$

$$\left.\times\, r^{m\alpha+n-1}\psi(r\lambda^{-1/m}\Phi(\omega)^{-1/m})\,dr\right\}d\omega$$

$$=\lambda^{-\alpha-(n/m)}\int_{S^{n-1}}\Phi(\omega)^{-n/m}G_\alpha(\lambda\Phi(\omega),\langle\omega,\xi\rangle/(\lambda\Phi(\omega))^{1/m})\,d\omega, \qquad (16)$$

where

$$G_\alpha(A,B) = \int e^{i(Br-r^m)} r^{m\alpha+n-1} \psi\left(\frac{r}{A^{1/m}}\right) dr, \tag{17}$$

and ψ is a smooth function supported in $(1/2, 2)$.

2.1. Radial curvature estimates. We have the following estimates for $G_\alpha(A,B)$.

LEMMA 9. *Suppose $1/m < \alpha + (n/m) < 1$, and let $G_\alpha(A,B)$ be as in (17) with* supp $\psi \subset (1/2,2)$. *Then for c sufficiently large,*

(i) $|G_\alpha(A,B)| \leqslant C_\alpha A^{\alpha+(n/m)}$,

(ii) $|G_\alpha(A,B)| \leqslant C_\alpha A^{\alpha+(n/m)-1}$, *for* $B < c^{-1}A^{(m-1)/m}$ *or* $B > cA^{(m-1)/m}$,

(iii) $G_\alpha(A,B) = e^{ic_m B^{m/(m-1)}} B^{(m/(m-1))(\alpha+(n/m)-(1/2))} \psi\left(\frac{(B/m)^{1/(m-1)}}{A^{1/m}}\right) + O(A^{\alpha+(n/m)-1})$,

for $1 \leqslant c^{-1}A^{(m-1)/m} \leqslant B \leqslant cA^{(m-1)/m}$,

(iv) $|G_\alpha(A,B)| \leqslant C_\alpha A^{\alpha+(n/m)}(1+A)^{-1/2}$. (18)

Remarks. These estimates are essentially in [RS, Lemma 2.2 on p. 369], [St1, p. 339–341], [Ho, p. 317] or [GS, Chapter VII]. We have $c_m = (m-1)m^{-m/(m-1)}$. Finally, only estimate (iv) is needed for the proof of (A) of Theorem 4. Of course (iv) follows immediately from the three previous inequalities, but can also be obtained directly by noting that

$$G_\alpha(A,B) = A^{\alpha+(n/m)} \int \exp(i(BA^{1/m}t - At^m)) t^{m\alpha+n-1} \psi(t)\, dt$$

has decay $C_\alpha A^{\alpha+(n/m)}(1+A)^{-1/2}$ since the curve (t, t^m) has nonvanishing curvature on $(1/2, 2)$.

Proof. Inequality (18)(i) is obvious upon taking absolute values inside the integral in (17). To prove (18)(ii), let $\phi(r) = Br - r^m$. The critical point is given by $r_0 = (B/m)^{1/(m-1)}$. When $B > cA^{(m-1)/m}$ for sufficiently large c, we have $\phi'(r) > B/2$ on the support of the integrand, and since $\phi''(r) = -m(m-1)r^{m-2}$, we thus have

$$\begin{aligned}
|G_\alpha(A,B)| &= \left|\int e^{i\phi(r)} r^{m\alpha+n-1} \psi(rA^{-1/m})\, dr\right| \\
&= \left|\int \frac{d}{dr}\left(e^{i\phi(r)}\right) \frac{1}{i\phi'(r)} r^{m\alpha+n-1} \psi(rA^{-1/m})\, dr\right| \\
&\leqslant \int |(d/dr)\{(1/\phi'(r)) r^{m\alpha+n-1} \psi(rA^{-1/m})\}|\, dr \\
&\leqslant C \int_0^{cA^{1/m}} \{B^{-2} r^{m-2} r^{m\alpha+n-1} + B^{-1} r^{m\alpha+n-2} + B^{-1} r^{m\alpha+n-1} A^{-1/m}\}\, dr \\
&\leqslant CA^{\alpha+(n/m)-1}, \quad \text{since } B^{-1} < cA^{(1/m)-1} \text{ and } m\alpha+n-1 > 0.
\end{aligned}$$

In the case $B < c^{-1}A^{(m-1)/m}$, $|\phi'(r)| \geqslant cA^{(m-1)/m}$ on the support of the integrand, and using the above integration by parts argument, we obtain

$$|G_\alpha(A,B)| \leqslant C\int_0^{cA^{1/m}} \{(A^{(m-1)/m})^{-2}r^{m-2}r^{m\alpha+n-1} + (A^{(m-1)/m})^{-1}r^{m\alpha+n-2}\}\,dr$$

$$+ C\int_0^{cA^{1/m}} (A^{(m-1)/m})^{-1}r^{m\alpha+n-1}A^{-1/m}\,dr$$

$$\leqslant CA^{\alpha+(n/m)-1}, \quad \text{since } m\alpha+n-1>0.$$

We now turn to (18)(iii). We have

$$G_\alpha(A,B) = \int e^{i\phi(r)}r^{m\alpha+n-1}\psi(r/A^{1/m})\,dr$$

$$= \left\{\int_0^{(1/2)r_0} + \int_{(1/2)r_0}^{(3/2)r_0} + \int_{(3/2)r_0}^{cA^{1/m}}\right\} e^{i\phi(r)}r^{m\alpha+n-1}\psi\left(\frac{r}{A^{1/m}}\right)dr$$

$$= I_1 + I_2 + I_3.$$

As in the previous calculation, using $\phi'(r) > B/2$, $\phi''(r) \approx r^{m-2}$ and $r_0 \approx B^{1/(m-1)}$, we obtain

$$|I_1| = \left|\int_0^{(1/2)r_0} \frac{d}{dr}(e^{i\phi(r)})(1/i\phi'(r))r^{m\alpha+n-1}\psi(r/A^{1/m})\,dr\right|$$

$$\leqslant \left|\int_0^{(1/2)r_0} e^{i\phi(r)}(d/dr)\{(1/i\phi'(r))r^{m\alpha+n-1}\psi(r/A^{1/m})\}\,dr\right|$$

$$+ \left|\{e^{i\phi(r)}(1/i\phi'(r))r^{m\alpha+n-1}\psi(r/A^{1/m})\}\Big|_0^{(1/2)r_0}\right|$$

$$\leqslant C\int_0^{(1/2)r_0} \{B^{-2}r^{m-2}r^{m\alpha+n-1} + B^{-1}r^{m\alpha+n-2} + B^{-1}r^{m\alpha+n-1}A^{-1/m}\}\,dr$$

$$+ CB^{-1}r_0^{m\alpha+n-1}$$

$$\leqslant C(B^{-2}r_0^{m-2+m\alpha+n} + B^{-1}r_0^{m\alpha+n-1} + B^{-1}r_0^{m\alpha+n}A^{-1/m}) \leqslant CA^{\alpha+(n/m)-1}, \qquad (19)$$

since $A \approx B^{m/(m-1)}$.

Using a normalization argument together with the Van der Corput Lemma as on p. 339–341 of [St3], we can obtain the asymptotic estimate

$$I_2 = e^{i\phi(r_0)}\left(\frac{B}{m}\right)^{(m/(m-1))(\alpha+(n/m)-(1/2))}\psi\left(\frac{r_0}{A^{1/m}}\right) + O(A^{\alpha+(n/m)-1}). \tag{20}$$

For the sake of completeness, we review the details. Using the change of variable $r = r_0(y+1)$ and the change of phase $\phi(r) - \phi(r_0) = r_0^m\Phi(y)$, we obtain

$$\begin{aligned} I_2 &= \int_{-1/2}^{1/2} \exp\Big(i[\phi(r_0) + r_0^m\Phi(y)]\Big)[r_0(y+1)]^{m\alpha+n-1}\psi\Big(\frac{r_0}{A^{1/m}}(y+1)\Big)r_0\,dy \\ &= e^{i\phi(r_0)}r_0^{m\alpha+n}\int_{-1/2}^{1/2} e^{ir_0^m\Phi(y)}(y+1)^{m\alpha+n-1}\psi((r_0/A^{1/m})(y+1))\,dy. \end{aligned}$$

By Proposition 3, p. 334, and the remarks in 1.3.4, p. 337, of [St3], we obtain using $\Phi(0) = \Phi'(0) = 0$, $\Phi''(0) = -m(m-1)$ and $|\Phi'''| \leqslant C$ (which uses the fact that $r_0 \approx A^{1/m} \geqslant c'$),

$$\begin{aligned} &\int_{-1/2}^{1/2} e^{ir_0^m\Phi(y)}(y+1)^{m\alpha+n-1}\psi\Big(\frac{r_0}{A^{1/m}}(y+1)\Big)\,dy \\ &\quad = e^{ir_0^m\Phi(0)}|\Phi''(0)|^{-1/2}r_0^{-m/2}\psi\Big(\frac{r_0}{A^{1/m}}\Big) + O(r_0^{-m}) \\ &\quad = \frac{-1}{\sqrt{m(m-1)}}\,r_0^{-m2}\psi\Big(\frac{r_0}{A^{1/m}}\Big) + O(r_0^{-m}). \end{aligned}$$

Combining the above two equalities yields (20) since $r_0 = (B/m)^{1/(m-1)} \geqslant cA^{1/m}$.

Finally we come to I_3. Since $|\phi'(r)| \approx r^{m-1} \approx A^{(m-1)/m}$ on the support of the integrand, we compute that

$$\begin{aligned} |I_3| &= \left|\int_{(3/2)r_0}^{cA^{1/m}} (d/dr)(e^{i\phi(r)})(1/i\phi'(r))r^{m\alpha+n-1}\,dr\right| \\ &= \left|\int_{(3/2)r_0}^{cA^{1/m}} e^{i\phi(r)}(d/dr)\{(1/i\phi'(r))r^{m\alpha+n-1}\}\,dr + \{e^{i\phi(r)}(1/i\phi'(r))r^{m\alpha+n-1}\}\Big|_{(3/2)r_0}^{cA^{1/m}}\right| \\ &\leqslant C\int_{(3/2)r_0}^{cA^{1/m}} \{(r^{m-2}/r^{2(m-1)})r^{m\alpha+n-1} + (1/r^{m-1})r^{m\alpha+n-2}\}\,dr \\ &\quad + \left|\{e^{i\phi(r)}(1/i\phi'(r))r^{m\alpha+n-1}\}\Big|_{(3/2)r_0}^{cA^{1/m}}\right| \\ &\leqslant CA^{\alpha+(n/m)-1}. \end{aligned} \tag{21}$$

Inequalities (19), (20), and (21) yield (18)(iii), and since (iv) is an immediate consequence of (i)–(iii), this completes the proof of Lemma 9.

Proof of (A) of Theorem 4. The case $\alpha + (n/m) < 1/2$ has already been proved in Remark 2 following the statement of Theorem 4. We next turn to the case where $\alpha + \rho < \alpha + (n/m)$, and by the argument at the beginning of this section, it suffices to prove the estimate when ψ is supported in the annulus $\mathscr{A}$. We decompose the expression in (16) into two pieces:

$$\lambda^{-\alpha-(n/m)} \int G_\alpha(A,B)\Phi(\omega)^{-n/m} d\omega = \int_{\{\Phi(\omega)<1/\lambda\}} + \int_{\{\Phi(\omega)\geqslant 1/\lambda\}}$$

$$= I + II, \tag{22}$$

where $A = \lambda\Phi(\omega)$ and $B = \omega \cdot \xi/(\lambda\Phi(\omega))^{1/m}$. Let $|\{\Phi^{-1} > \lambda\}|$ denote the Lebesgue measure of the set where $\Phi^{-1} > \lambda$. Using $\Phi^{-1} \in$ weak $L^\rho(S^{n-1})$ and estimate (18)(i) or (iv), we see that

$$|I| \leqslant \int \lambda^{-\alpha-(n/m)}(\lambda\Phi(\omega))^{\alpha+(n/m)}\Phi(\omega)^{-n/m}\, d\omega \leqslant \lambda^{-\alpha}|\{\Phi^{-1} > \lambda\}| \leqslant \lambda^{-(\rho+\alpha)}. \tag{23}$$

To estimate II in the case $\alpha + (n/m) \leqslant 1/2$, we have by (18)(iv)

$$|II| \leqslant \int \lambda^{-\alpha-(n/m)} C|\Phi(\omega)|^{-n/m}\, d\omega \leqslant C\lambda^{-((n/m)+\alpha)},$$

provided $\Phi^{-1} \in L^{n/m}(S^{n-1})$, which is the desired estimate in the case $\rho = n/m$. In the case $0 < \rho < n/m$, the hypothesis $\Phi^{-1} \in$ weak $L^\rho(S^{n-1})$ yields

$$|II| \leqslant \int \lambda^{-\alpha-(n/m)} C|\Phi(\omega)|^{-n/m}\, d\omega = C\lambda^{-\alpha-(n/m)} \int_0^\lambda (n/m)t^{(n/m)-1}|\{t < \Phi^{-1} < \lambda\}|\, dt$$

$$\leqslant C\lambda^{-\alpha-(n/m)} \int_0^\lambda (n/m)t^{(n/m)-1}C(1+t)^{-\rho} dt \leqslant C\lambda^{-(\alpha+\rho)}.$$

On the other hand, if $\alpha + (n/m) > 1/2$, we have by (18)(iv)

$$|II| \leqslant \int \lambda^{-\alpha-(n/m)} C(\lambda\Phi(\omega))^{\alpha+(n/m)-(1/2)}\Phi(\omega)^{-n/m}\, d\omega$$

$$= C\lambda^{-1/2} \int (\Phi(\omega)^{-1})^{(1/2)-\alpha}\, d\omega. \tag{24}$$

In the case $\alpha + \rho = 1/2$, this yields the desired estimate provided $\Phi^{-1} \in L^\rho(S^{n-1})$.

In the case $\alpha + \rho < 1/2$, the weaker hypothesis $\Phi^{-1} \in$ weak $L^{\rho}(S^{n-1})$ yields

$$|II| \leqslant C\lambda^{-1/2} \int_0^{\lambda} ((1/2) - \alpha) t^{(1/2)-\alpha-1} |\{\Phi^{-1} > t\}| \, dt$$

$$\leqslant C\lambda^{-1/2} \int_0^{\lambda} ((1/2) - \alpha) t^{(1/2)-\alpha-1} C(1+t)^{-\rho} dt \leqslant C\lambda^{-(\alpha+\rho)}.$$

These estimates prove (A) of Theorem 4, since we may assume that $|\xi| \leqslant C\lambda$. Otherwise, Proposition 1, p. 331 in [St3] shows that $F_\alpha(\xi, \lambda)$ has rapid decay in ξ since $|\nabla_x\{\lambda^{-1}x \cdot \xi - \Phi(x)\}| \geqslant c > 0$ if $|\xi| \geqslant C|\lambda|$ for C large enough.

So far we have handled the cases $\alpha + (n/m) < 1/2$ and $\alpha + \rho < \alpha + (n/m)$. Thus the only case of (A) of the theorem not handled is the case $\alpha + \rho = \alpha + (n/m) = 1/2$, to which we now turn. Neither the scaling argument in the second remark following Theorem 4, nor the scaling argument at the beginning of this section apply here, so we must consider

$$F_\alpha(\xi, \lambda) = \lambda^{-\alpha-(n/m)} \int_{S^{n-1}} \Phi(\omega)^{-n/m} G_\alpha\left(\lambda\Phi(\omega), \frac{\langle \omega, \xi \rangle}{(\lambda\Phi(\omega))^{1/m}}\right) d\omega,$$

where $G_\alpha(A, B) = \int e^{i(Br - r^m)} r^{m\alpha+n-1} \psi(r/A^{1/m}) \, dr$, and ψ is supported in $[0, 1]$. Now

$$|G_{(1/2)-(n/m)}(A, B)| = \left| A^{1/2} \int \exp\left(i(BA^{1/m}r - Ar^m)\right) r^{(m-2)/2} \psi(r) \, dr \right|$$

$$\leqslant C,$$

since $r^{(m-2)/2}$ is essentially the square root of the curvature of the curve (r, r^m). Here we have used the result that decay of order 1/2 occurs when the Fourier transform of the curve is weighted by the square root of the curvature (see, e.g., [BNW]). Thus we have

$$|F_{(1/2)-(n/m)}(\xi, \lambda)| \leqslant \lambda^{-1/2} \int_{\{\omega \in S^{n-1} : \lambda\Phi(\omega) \leqslant 1\}} \Phi(\omega)^{-n/m} C \, d\omega$$

$$\leqslant C\lambda^{-1/2},$$

since $\Phi^{-1} \in L^{n/m}(S^{n-1})$.

For the converse, we note that using the change of variable $t = \lambda r^m \Phi(\omega)$, we get

$$|F_\alpha(0,\lambda)| = \left| \int e^{i\lambda\Phi(x)} \psi(x) \Phi(x)^\alpha \, dx \right|$$

$$= \left| \int_{S^{n-1}} \int_0^\infty e^{i\lambda r^m \Phi(\omega)} \psi(r) r^{m\alpha+n-1} \Phi(\omega)^\alpha \, dr \, d\omega \right|$$

$$\geqslant \lambda^{-\alpha-(n/m)} \int_{S^{n-1}} \operatorname{Im}\left\{ \int_0^\infty e^{it} \psi((t/\lambda\Phi(\omega))^{1/m}) t^{\alpha+(n/m)-1} \, dt \right\} \Phi(\omega)^{-n/m} \, d\omega$$

$$\geqslant \lambda^{-\alpha-(n/m)} \int_{\{\omega \in S^{n-1}: \lambda\Phi(\omega) > C_\alpha\}} c_\alpha \Phi(\omega)^{-n/m} \, d\omega. \tag{25}$$

Indeed,

$$\int_0^a (\sin t) t^{\alpha+(n/m)-1} dt = (1 - \cos a) a^{\alpha+(n/m)-1}$$

$$+ \left(1 - \alpha - \frac{n}{m}\right) \int_0^a (1 - \cos t) t^{\alpha+(n/m)-2} \, dt \geqslant 0$$

for all $a \geqslant 0$ when $0 < \alpha + (n/m) < 1$. Since ψ is radially decreasing, an integration by parts shows that

$$\int_0^\infty (\sin t) t^{\alpha+(n/m)-1} \psi((t/\lambda\Phi(\omega))^{1/m}) \, dt \geqslant 0.$$

Finally, since $\int_0^\infty (\sin t) t^{\alpha+(n/m)-1} \, dt > 0$, we also have

$$\int_0^\infty (\sin t) t^{\alpha+(n/m)-1} \psi((t/\lambda\Phi(\omega))^{1/m}) \, dt \geqslant c_\alpha > 0$$

for $\lambda\Phi(\omega) \geqslant C_\alpha$. Thus

$$|F_\alpha(0,\lambda)| \geqslant C\lambda^{-\alpha} \int_{\{\omega \in S^{n-1}: (1/2)C_\alpha^{-1}\lambda < \Phi(\omega)^{-1} < C_\alpha^{-1}\lambda\}} c_\alpha \, d\omega$$

$$= C\lambda^{-\alpha} \left| \left\{ \frac{1}{2} C_\alpha^{-1} \lambda < \Phi^{-1} < C_\alpha^{-1} \lambda \right\} \right|,$$

and combined with (10) for $\xi = 0$, this yields $|\{(1/2)C_\alpha^{-1}\lambda < \Phi^{-1} < C_\alpha^{-1}\lambda\}| \leqslant$

$C\lambda^{-\rho}$, which implies $\Phi^{-1} \in$ weak $L^{\rho}(S^{n-1})$. Moreover, if $\rho = n/m$, then directly from (25) and (10) we have

$$\int_{\{\omega \in S^{n-1}: \lambda\Phi(\omega) > C_\alpha\}} \Phi(\omega)^{-n/m}\, d\omega \leqslant C_\alpha,$$

for all $\lambda > 0$, which yields $\Phi^{-1} \in L^{n/m}(S^{n-1})$.

Now we establish the last assertion in the remark following Theorem 4. Suppose that $\Phi \geqslant 0$ and $|\{\Phi = 0\}| > 0$. Then

$$F_0(0, \lambda) = \int_{S^{n-1}} \int_0^\infty e^{i\lambda r^m \Phi(\omega)} \psi(r) r^{n-1} dr\, d\omega = \int_{\{\Phi=0\}} + \int_{\{\lambda\Phi(\omega) > \gamma\}} + \int_{\{0 < \lambda\Phi(\omega) \leqslant \gamma\}}$$

$$= \left(\int_0^\infty \psi(r) r^{n-1}\, dr\right) |\{\Phi = 0\}| + \int_{\{\lambda\Phi(\omega) > \gamma\}} + \int_{\{0 < \lambda\Phi(\omega) \leqslant \gamma\}}.$$

But

$$\left|\int_{\{\lambda\Phi(\omega) > \gamma\}} \int_0^\infty e^{i\lambda r^m \Phi(\omega)} \psi(r) r^{n-1}\, dr\, d\omega\right|$$

$$= \left|\int_{\{\lambda\Phi(\omega) > \gamma\}} \left\{\int_0^\infty e^{it} \psi\left(\left(\frac{t}{\lambda\Phi(\omega)}\right)^{1/m}\right) t^{(n/m)-1} dt\right\} (\lambda\Phi(\omega))^{-n/m}\, d\omega\right|$$

$$\leqslant C\gamma^{-n/m} < (1/2)\left(\int_0^\infty \psi(r) r^{n-1}\, dr\right) |\{\Phi = 0\}|,$$

provided γ is large enough, and then

$$\left|\int_{\{0 < \lambda\Phi(\omega) \leqslant \gamma\}} \int_0^\infty e^{i\lambda r^m \Phi(\omega)} \psi(r) r^{n-1}\, dr\, d\omega\right| \leqslant C\, |\{0 < \lambda\Phi(\omega) \leqslant \gamma\}|$$

tends to 0 as $\lambda \to \infty$. This shows that $|F_0(0, \lambda)| > (1/2)(\int_0^\infty \psi(r) r^{n-1}\, dr)|\{\Phi = 0\}|$ for λ sufficiently large, contradicting (10). A similar argument works for $\tilde{F}_0$.

Proof of (B) of Theorem 4. This time we decompose the expression in (16) into three pieces:

$$\lambda^{-\alpha-(n/m)} \int G_\alpha(A, B) \Phi(\omega)^{-n/m}\, d\omega = \int_{\{\Phi(\omega) < \lambda^{-1}\}} + \int_{\{\lambda^{-1} < \Phi(\omega) < \lambda^{-m\varepsilon}\}} + \int_{\{\Phi(\omega) > \lambda^{-m\varepsilon}\}}$$

$$= I + II + III, \tag{26}$$

where $\varepsilon > 0$ will be fixed later. We will prove the following estimates for $\alpha + \rho > 1/2 > \alpha$:

$$|I| \leqslant \lambda^{-(\rho+\alpha)},$$

$$|II| \leqslant \lambda^{-(1/2)-m\varepsilon(\alpha+\rho-(1/2))},$$

$$|III| \leqslant \lambda^{-(1/2)-\varepsilon'(l,m,M)}, \quad \text{for some } \varepsilon'(l,m,M) > 0.$$

As before we may assume that $|\xi| \leqslant C\lambda$, since otherwise Proposition 1, p. 331 in [St3] shows that $F_\alpha(\xi, \lambda)$ has rapid decay in ξ. The estimate for term I is the same as for I in the proof of part (A) above. The estimate for term II is similar to that above, requiring only the integrability hypothesis (4) on Φ^{-1}, which controls the strength of the degeneracy of the curvature of Φ in the radial direction. On the other hand, the estimate for III requires the finite type hypothesis (5) on the level set Σ. We first consider the easier case of II.

To handle term II, we use estimate (18)(iv) to obtain

$$\begin{aligned}
|II| &\leqslant C \int_{\{\Phi^{-1} > \lambda^{m\varepsilon}\}} \lambda^{-\alpha-(n/m)} C(\lambda\Phi(\omega))^{\alpha+(n/m)-(1/2)} \Phi(\omega)^{-n/m}\, d\omega \\
&= C\lambda^{-1/2} \int_{\{\Phi^{-1} > \lambda^{m\varepsilon}\}} \left(\Phi(\omega)^{-1}\right)^{(1/2)-\alpha} d\omega \\
&= C\lambda^{-1/2} \int_{\lambda^{m\varepsilon}}^{\infty} ((1/2)-\alpha) t^{(1/2)-\alpha-1} |\{\Phi^{-1} > t\}|\, dt \\
&\quad + C\lambda^{-1/2} \int_{\{\Phi^{-1} > \lambda^{m\varepsilon}\}} (\lambda^{m\varepsilon})^{(1/2)-\alpha} d\omega \\
&\leqslant C\lambda^{-1/2} \int_{\lambda^{m\varepsilon}}^{\infty} t^{(1/2)-(\rho+\alpha)-1}\, dt + C\lambda^{-(1/2)+((1/2)-\alpha)m\varepsilon} \lambda^{-\rho m\varepsilon} \\
&= C\lambda^{-(1/2)-m\varepsilon(\alpha+\rho-(1/2))} \\
&= C\lambda^{-(1/2)-\varepsilon'},
\end{aligned} \tag{27}$$

since $\alpha + \rho > 1/2$ and $\alpha < 1/2$. It remains only to prove the estimate for III.

2.2. Angular curvature estimates. In order to estimate III using the aforementioned finite type condition, we must transfer the integration over the sphere to the level set Σ via the change of variables $\mu = \omega/\Phi(\omega)^{1/m}$. We need the following facts:

(a) $|\mu| = \Phi(\omega)^{-1/m}$;
(b) $C_1|\mu|^{-1} \leqslant |\nabla\Phi(\mu)| \leqslant C_2|\mu|^{m-1}$, and $|\nabla^2\Phi(\mu)| \leqslant C|\mu|^{m-2}$;
(c) $d\sigma(\mu)/|\nabla\Phi(\mu)| = \Phi(\omega)^{-n/m}\, d\omega$, where $d\sigma$ denotes the Lebesgue measure on Σ. (28)

The assertion (a) is obvious, (b) follows from the Euler homogeneity relations, and (c) follows from the observation that $m(d\sigma(\mu)/d\omega) = |\mu|^{n-1}\sec\theta$, where θ is the angle between $\nabla\Phi(\mu)$ and μ together with $\mu \cdot \nabla\Phi(\mu) = m\Phi(\mu) = m$. Changing variables, we see that

$$III = \lambda^{-\alpha-(n/m)} \int_{\{\Phi(\omega)^{-1/m}<\lambda^{\varepsilon}\}} G_\alpha\left(\lambda\Phi(\omega), \frac{\langle\omega, \xi\rangle}{\lambda^{1/m}\Phi(\omega)^{1/m}}\right)\Phi^{-n/m}\, d\omega$$

$$= \lambda^{-\alpha-(n/m)} \int_{\{\Phi(\mu)=1,\ |\mu|\leqslant\lambda^{\varepsilon}\}} G_\alpha\left(\lambda|\mu|^{-m}, \frac{\langle\mu, \xi\rangle}{\lambda^{1/m}}\right) d\sigma(\mu)/|\nabla\Phi(\mu)|. \tag{29}$$

Note that the condition $c^{-1}A^{(m-1)/m} \leqslant B \leqslant cA^{(m-1)/m}$ in (18)(iii) can be rewritten

$$c^{-1}(\lambda\Phi(\omega))^{m/(m-1)} \leqslant \frac{\omega\cdot\xi}{\lambda^{1/m}\Phi(\omega)^{1/m}} \leqslant c(\lambda\Phi(\omega))^{m/(m-1)},$$

which transforms under our change of variable into

$$c^{-1}(\lambda|\mu|^{-m})^{(m-1)/m} \leqslant \frac{\mu\cdot\xi}{\lambda^{1/m}} \leqslant c(\lambda|\mu|^{-m})^{(m-1)/m}.$$

Let

$$\Sigma_\varepsilon = \{\mu \in R^n : \Phi(\mu) = 1,\ |\mu| < \lambda^{\varepsilon}\}. \tag{30}$$

Set $R = c\lambda^{-\varepsilon(m-1)}$, where $c > 0$ is a small constant to be determined (so that (32) below holds). Let $\{\mu_j\} \subset \Sigma_\varepsilon$ be a maximal collection of points where pairwise distances apart are at least R. Then $\bigcup_j B(\mu_j, R)$ covers Σ_ε where $B(\mu_j, R)$ denotes the ball centered at μ_j of radius R. Let $\{\rho_j(\mu)\}_j$ denote a partition of unity subordinate to $\{B(\mu_j, R)\}_j$. Then

$$III = \sum_j \int_{\Sigma\cap B(\mu_j,R)} \lambda^{-\alpha-(n/m)} G_\alpha\left(\lambda|\mu|^{-m}, \frac{\mu\cdot\xi}{\lambda^{1/m}}\right)\rho_j(\mu)\,\frac{d\sigma(\mu)}{|\nabla\Phi(\mu)|} \tag{31}$$

minus the portion of the integral living outside Σ_ε. However, this latter portion is dominated by the estimates for term II above. We shall need the following facts:

(a) $|\nabla\Phi(\mu) - \nabla\Phi(\mu_j)| \leqslant \frac{1}{100}|\nabla\Phi(\mu_j)|, \quad \mu \in B(\mu_j, R)$,

(b) $\left|\frac{\mu\cdot\xi}{\lambda^{1/m}} - \frac{\mu_j\cdot\xi}{\lambda^{1/m}}\right| \leqslant C(\lambda|\mu|^{-m})^{(m-1)/m}, \quad \mu \in B(\mu_j, R)$, (32) provided that the aforementioned constant c is small enough, and $|\xi| \leqslant C|\lambda|$. As observed earlier, we can assume the latter since otherwise an integration by parts argument shows that $F_\alpha(\xi, \lambda)$ has rapid decay (Proposition 1, p. 331 in [St3]).

To prove (32)(a) above, note that

$$\left|\nabla\Phi(\mu) - \nabla\Phi(\mu_j)\right| \leqslant \sup_{B(\mu_j,R)} |\nabla^2\Phi(\mu)|R \leqslant C\lambda^{\varepsilon(m-2)}c\lambda^{-\varepsilon(m-1)}$$

$$= cC\lambda^{-\varepsilon} \leqslant \frac{1}{100}\left|\nabla\Phi(\mu_j)\right|$$

by estimate (28)(b). To prove (32)(b), note that

$$\left|\frac{\mu\cdot\xi}{\lambda^{1/m}} - \frac{\mu_j\cdot\xi}{\lambda^{1/m}}\right| \leqslant \frac{|\xi|}{\lambda^{1/m}}R \leqslant \lambda^{(m-1)/m}c\lambda^{-\varepsilon(m-1)} \leqslant c(\lambda|\mu|^{-m})^{(m-1)/m},$$

since $|\mu| < c\lambda^{\varepsilon}$ for $\mu \in B(\mu_j, R)$.

Let's fix a j for the moment. We wish to distinguish two separate cases. The first is where

$$c^{-1}(\lambda|\mu_j|^{-m})^{(m-1)/m} \leqslant \frac{\mu_j\cdot\xi}{\lambda^{1/m}} \leqslant c(\lambda|\mu_j|^{-m})^{(m-1)/m}, \tag{33}$$

and we will say that $j \in \mathscr{A}$ in this case. Note that because of (32)(b), we actually have

$$c'^{-1}(\lambda|\mu|^{-m})^{(m-1)/m} \leqslant \frac{\mu\cdot\xi}{\lambda^{1/m}} \leqslant c'(\lambda|\mu|^{-m})^{(m-1)/m}, \qquad \text{for } \mu \in B(\mu_j, R),\ j \in \mathscr{A},$$

which means that by (18)(iii) we have

$$\int_{\Sigma\cap B(\mu_j,R)} \lambda^{-\alpha-(n/m)} G_\alpha\left(\lambda|\mu|^{-m}, \frac{\mu\cdot\xi}{\lambda^{1/m}}\right)\rho_j(\mu)\,\frac{d\sigma(\mu)}{|\nabla\Phi(\mu)|}$$

$$= \lambda^{-\alpha-(n/m)} \int_{B(\mu_j,R)} \exp\left(ic_m(\mu\cdot\xi/\lambda^{1/m})^{m/(m-1)}\right)$$

$$\times\left(\frac{\mu\cdot\xi}{\lambda^{1/m}}\right)^{(m/(m-1))(\alpha+(n/m)-(1/2))} \rho_j(\mu)\,\frac{d\sigma(\mu)}{|\nabla\Phi(\mu)|}$$

$$+\,O\left[\lambda^{-\alpha-(n/m)}\int_{B(\mu_j,R)} C(\lambda|\mu|^{-m})^{(\alpha+(n/m)-1)}\rho_j(\mu)\,\frac{d\sigma(\mu)}{|\nabla\Phi(\mu)|}\right]$$

$$= IV_j + V_j, \qquad j \in \mathscr{A}.$$

For those j for which (33) fails, we say $j \in \mathscr{B}$, and by (32)(b) again we have for $j \in \mathscr{B}$,

$$\frac{\mu \cdot \xi}{\lambda^{1/m}} > c'(\lambda|\mu|^{-m})^{(m-1)/m} \quad \text{or} \quad \frac{\mu \cdot \xi}{\lambda^{1/m}} < c'^{-1}(\lambda|\mu|^{-m})^{(m-1)/m},$$

$$\mu \in B(\mu_j, R),\ j \in \mathscr{B},$$

which means that by (18)(ii) we have

$$\left| \int_{\Sigma \cap B(\mu_j,R)} \lambda^{-\alpha-(n/m)} G_\alpha\left(\lambda|\mu|^{-m}, \frac{\mu \cdot \xi}{\lambda^{1/m}}\right) \rho_j(\mu) \frac{d\sigma(\mu)}{|\nabla\Phi(\mu)|} \right|$$

$$\leqslant \lambda^{-\alpha-(n/m)} \int_{B(\mu_j,R)} C(\lambda|\mu|^{-m})^{(\alpha+(n/m)-1)} \rho_j(\mu) \frac{d\sigma(\mu)}{|\nabla\Phi(\mu)|}$$

$$= V_j, \quad j \in \mathscr{B}.$$

Thus from (31), we can dominate $|III|$ by

$$\sum_{j \in \mathscr{A}} |IV_j| + \sum_{j \in \mathscr{A} \cup \mathscr{B}} V_j.$$

We turn now to estimating IV_j for $j \in \mathscr{A}$. The estimate (32)(a) above and the implicit function theorem imply that we can find a normalized coordinate system $\{y_1, \ldots, y_n\}$ at μ_j such that $\nabla\Phi(\mu_j)$ is parallel to the vector $(0, 0, \ldots, 1)$ and a smooth function $\Psi(y_1, \ldots, y_{n-1})$ such that,

$$\Sigma \cap B(\mu_j, R) = \{\mu_j + (y_1, \ldots, y_n) \colon y_n = \Psi(y_1, \ldots, y_{n-1})\}, \tag{34}$$

and where

$$\sum_{2 \leqslant |\beta| \leqslant l} \left| \frac{\partial^{|\beta|}}{\partial y_1^{\beta_1} \cdots \partial y_{n-1}^{\beta_{n-1}}} \Psi(y) \right| \geqslant c\lambda^{-\varepsilon M}, \tag{35}$$

by the finite type assumption (see Remark 1.1).

We observe that $\Psi^{m/(m-1)}$ satisfies (35) with l replaced by $lm/(m-1)$. Hence, by a theorem of Stein (see [St1, p. 317]), we see that

$$\left| \int_{\Sigma \cap B(\mu_j,R)} \exp\left(ic_m \left(\frac{\mu \cdot \xi}{\lambda^{1/m}} \right)^{m/(m-1)} \right) \right.$$

$$\left. \times \left(\frac{\mu \cdot \xi}{\lambda^{1/m}} \right)^{(m/(m-1))(\alpha+(n/m)-(1/2))} \rho_j(\mu) \, \frac{d\sigma(\mu)}{|\nabla\Phi(\mu)|} \right| \tag{36}$$

$$= \left| \int_{\Sigma \cap B(\mu_j,R)} \exp\left(ic_m \left[\left(\frac{|\xi|}{\lambda^{1/m}} \lambda^{-\varepsilon M} \right)^{m/(m-1)} \right] \right. \right.$$

$$\left. \left(\lambda^{\varepsilon M} \text{ affine term} + \lambda^{\varepsilon M} \left(\frac{\xi_n}{|\xi|} \right) \Psi(y') \right)^{m/(m-1)} \right) \tag{37}$$

$$\left. \times \left(\frac{\mu \cdot \xi}{\lambda^{1/m}} \right)^{(m/(m-1))\,(\alpha+(n/m)-(1/2))} |\nabla\Phi(\mu)|^{-1} \rho_j(\mu) \, d\sigma(\mu) \right| \tag{38}$$

is dominated by

$$C \left[\left(\frac{|\xi|}{\lambda^{1/m}} \lambda^{-\varepsilon M} \right)^{m/(m-1)} \right]^{-(m-1)/lm} \left\{ \left(\frac{|\xi| \lambda^{\varepsilon}}{\lambda^{1/m}} \right)^{(m/(m-1))(\alpha+(n/m)-(1/2))} \lambda^{\varepsilon} + \text{smaller terms} \right\}$$

$$\leqslant C\lambda^{\varepsilon M'} \left(\frac{|\xi|}{\lambda^{1/m}} \right)^{(m/(m-1))(\alpha+(n/m)-(1/2))-(1/l)}, \tag{39}$$

for some $M' > 0$. We must now estimate the number of balls $B(\mu_j, R)$. However, this number is bounded above by

$$\frac{\operatorname{Vol}(\Sigma_\varepsilon)}{\operatorname{Vol}(\text{ball})} = C\lambda^{\varepsilon(m-1)n} \operatorname{Vol}(\Sigma_\varepsilon) = C\lambda^{\varepsilon(m-1)n} \int_{\Sigma_\varepsilon} d\sigma(\mu)$$

$$\leqslant C\lambda^{\varepsilon(m-1)n} \lambda^{\varepsilon(m-1)} \int_{\Sigma_\varepsilon} \frac{d\sigma(\mu)}{|\nabla\Phi(\mu)|}$$

$$\leqslant C\lambda^{\varepsilon(m-1)(n+1)} \int_{\{\Phi(\omega)^{-1} < \lambda^{m\varepsilon}\}} \Phi(\omega)^{-n/m} \, d\omega \leqslant C\lambda^{\varepsilon N}, \tag{40}$$

for some $N > 0$. Combining this estimate with (39), we see that

$$\sum_{j \in \mathscr{A}} |IV_j| \leqslant C\lambda^{-\alpha-(n/m)} \left(\frac{|\xi|}{\lambda^{1/m}}\right)^{(m/(m-1))(\alpha+(n/m)-(1/2))-(1/l)} \lambda^{\varepsilon N'}, \tag{41}$$

for some $N' > 0$. We now choose l large enough so that $(1/l) < \alpha + \rho - (1/2) \leqslant \alpha + (n/m) - (1/2)$. We then choose $\varepsilon > 0$ so that $\varepsilon N' < ((m-1)/m)(1/l)$. If we now use the fact that $|\xi| \leqslant C|\lambda|$, we see that

$$\sum_{j \in \mathscr{A}} |IV_j| \leqslant \lambda^{-(1/2)-\varepsilon'},$$

for some $\varepsilon' > 0$.

In order to complete the proof, we must estimate $\sum_{j \in \mathscr{A} \cup \mathscr{B}} V_j$. However, we have

$$\sum_{j \in \mathscr{A} \cup \mathscr{B}} V_j = \lambda^{-\alpha-(n/m)} \int_{\cup_j B(\mu_j, R)} C(\lambda|\mu|^{-m})^{(\alpha+(n/m)-1)} \frac{d\sigma(\mu)}{|\nabla\Phi(\mu)|}$$

$$\leqslant C\lambda^{-\alpha-(n/m)} \int_{\{\lambda^{\varepsilon m}\Phi(\omega) > 1\}} (\lambda\Phi(\omega))^{\alpha+(n/m)-1} \Phi(\omega)^{-(n/m)}\, d\omega \tag{42}$$

if ε is chosen to be small enough. Thus,

$$\sum_{j \in \mathscr{A} \cup \mathscr{B}} V_j \leqslant C\lambda^{-\alpha-(n/m)} \int_{\{\Phi^{-1} < \lambda^{\varepsilon m}\}} \Phi(\omega)^{-n/m}\, d\omega$$

$$\leqslant C\lambda^{-\alpha-(n/m)} \int_0^{\lambda^{\varepsilon m}} t^{(n/m)-1C}(1+t)^{-\rho}\, dt \leqslant C\lambda^{-\alpha-\rho}, \tag{43}$$

if $\varepsilon \leqslant 1/m$. Hence $\sum_{j \in \mathscr{A} \cup \mathscr{B}} V_j \leqslant C\lambda^{-(1/2)-\varepsilon'}$ for some $\varepsilon' > 0$, and the proof is complete.

Proof of Theorem 6. Again we assume that $\Phi \geqslant 0$ for convenience. This time we have

$$F_\alpha(\xi, \lambda) = \int e^{i(x \cdot \xi - \lambda\Phi(x))} \psi(x) \Phi(x)^\alpha\, dx,$$

where ψ is a smooth radial function with small compact support. In polar coordinates,

$$F_\alpha(\xi,\lambda) = \int_{S^{n-1}} \int e^{ir(\omega\cdot\xi - \lambda\Gamma(r)\Phi(\omega))}\psi(r)\Gamma(r)^\alpha r^{n-1}\Phi(\omega)^\alpha \, dr \, d\omega.$$

By the finite type assumption (7) on Γ, we can write $\Gamma(r) = r^m g(r)$, where g is smooth and $g(0) \neq 0$. After making the change of variables $r \to r(\lambda\Phi(\omega))^{-1/m}$, we get

$$\begin{aligned} F_\alpha(\xi,\lambda) &= \lambda^{-\alpha-(n/m)} \int_{S^{n-1}} \Phi(\omega)^{-n/m} \frac{1}{m} \\ &\quad \times \left\{ \int \exp\left(i\left(r\frac{\omega\cdot\xi}{(\lambda\Phi(\omega))^{1/m}} - r^m g\left(\frac{r}{(\lambda\Phi(\omega))^{1/m}}\right)\right)\right) \right. \\ &\quad \left. \times\, r^{m\alpha+n-1} g\left(\frac{r}{(\lambda\Phi(\omega))^{1/m}}\right)^\alpha \psi(r\lambda^{-1/m}\Phi(\omega)^{-1/m})\, dr \right\} d\omega \\ &= \lambda^{-\alpha-(n/m)} \int_{S^{n-1}} \Phi(\omega)^{-n/m} G_{\alpha,g}\left(\lambda\Phi(\omega), \frac{\langle\omega,\xi\rangle}{(\lambda\Phi(\omega))^{1/m}}\right) d\omega, \end{aligned}$$

where $G_{\alpha,g}(A,B) = \int \exp(i(Br - r^m g(r/A^{1/m})))r^{m\alpha+n-1}g(r/A^{1/m})^\alpha\psi(r/A^{1/m})\,dr$. This time the phase function $\phi(r) = Br - r^m g(r/A^{1/m})$ satisfies

$$\phi'(r) = B - r^{m-1}\left\{ mg\left(\frac{r}{A^{1/m}}\right) + \left(\frac{r}{A^{1/m}}\right) g'\left(\frac{r}{A^{1/m}}\right)\right\},$$

and so using $g(0) \neq 0$, together with a sufficiently small support for ψ, we have that the critical point $r_0 \approx B^{1/(m-1)}$ as before, and then we can obtain the estimates (18) for $G_{\alpha,g}$ in place of G_α. The proof is similar to that for G_α, but more technical, and we omit the details. The proof of Theorem 6 now proceeds as in the proof of (B) of Theorem 4.

The following example shows that the finite type hypothesis (5) on Σ is needed in part (B) of Theorem 4.

Example 10. Let $n = 2$, $\xi = (\lambda, 0)$, $\alpha = (1/2) - (2/m) + \varepsilon$ *(where* $\varepsilon \geqslant 0$*) and set* $\Phi(\omega) = \rho_\delta(\cos^m\theta)$, $\omega = (\cos\theta, \sin\theta)$, *where* ρ_δ *is a smooth function satisfying* $\rho_\delta(t) = 1/2$ *for* $0 \leqslant t \leqslant 1/2$, *and* $\rho_\delta(t) = t$ *for* $(1/2) + \delta \leqslant t \leqslant 1$. *We claim that*

$$\left| \int e^{i(\lambda\Phi(x) - \xi\cdot x)}\Phi(x)^\alpha \, dx \right| \approx \lambda^{-1/2}. \tag{44}$$

To see this, we write

$$\int e^{i(\lambda\Phi(x)-\xi\cdot x)}\Phi(x)^{\alpha}dx$$

$$= \int_{S^1}\int_0^1 e^{i(\lambda r^m\Phi(\omega)-r\lambda\omega_1)}r^{m\alpha+1}\Phi(\omega)^{\alpha}\,dr\,d\omega$$

$$= \frac{1}{m}\int_{S^1}\int_0^{\Phi(\omega)} \exp\left(i\left(\lambda t-\left(\frac{\lambda\cos\theta)}{\Phi(\omega)^{1/m}}\right)t^{1/m}\right)\right)t^{\alpha+(2/m)-1}\,dt\,\Phi(\omega)^{-2/m}\,dt\,d\omega,$$

and consider separately integration over the three regions

$$R=\{\theta\colon \cos^m\theta \geqslant \tfrac{1}{2}+\delta\},$$

$$S=\{\theta\colon \tfrac{1}{2}\leqslant \cos^m\theta \leqslant \tfrac{1}{2}+\delta\},$$

$$T=\{\theta\colon \cos^m\theta \leqslant \tfrac{1}{2}\}.$$

Now

$$\int_R \cdots = \int_R\int_0^{\cos^m\theta} e^{i\lambda(t-t^{1/m})}t^{\alpha+(2/m)-1}(\cos\theta)^{-2}\,dt\,d\omega$$

$$= \int_0^1 e^{i\lambda(t-t^{1/m})}t^{\alpha+(2/m)-1}\int_{\{\omega\colon \cos^m\theta\geqslant((1/2)+\delta)\vee t\}}(\cos\theta)^{-2}\,dt\,d\omega$$

$$\approx e^{i\lambda(t_0-t_0^{1/m})}t_0^{\alpha+(2/m)-1}\left(\lambda\frac{1}{m}\left(1-\frac{1}{m}\right)t_0^{(1/m)-2}\right)^{-1/2} = C\lambda^{-1/2}e^{ic\lambda},$$

where t_0 is the critical point of $t\to t-t^{1/m}$, i.e., $t_0=m^{-m/(m-1)}$. Here we have used standard stationary phase as in [So1]. Also,

$$\left|\int_S\cdots\right| \leqslant \int_{\{\omega\colon (1/2)\leqslant\cos^m\theta\leqslant(1/2)+\delta\}}(\cos\theta)^{-2}\left|\vphantom{\int}\right.$$

$$\times\int_0^{\rho_\delta(\cos^m\theta)}\exp\left(i\lambda\left(t-\frac{\cos\theta}{(\rho_\delta(\cos^m\theta))^{1/m}}\,t^{1/m}\right)\right)t^{\alpha+(2/m)-1}\,dt\left.\vphantom{\int}\right|\,d\omega$$

$$\leqslant C\int_{\{\omega\colon (1/2)\leqslant\cos^m\theta\leqslant(1/2)+\delta\}}(\cos\theta)^{-2}\lambda^{-1/2}\,d\omega \leqslant C\delta\lambda^{-1/2}.$$

Finally,

$$\left|\int_T \cdots\right| = \left|\int_{\{\omega:\,\cos^m\theta\leqslant 1/2\}} (1/2)^{-2/m}\int_0^{1/2} e^{i\lambda(t-(2t)^{1/m}\cos\theta)}t^{\alpha+(2/m)-1}\,dt\,d\omega\right|$$

$$= \left|4^{1/m}\int_0^{1/2} e^{i\lambda t}t^{\alpha+(2/m)-1}\left\{\int_{\{\omega:\,\cos^m\theta\leqslant 1/2\}} e^{-i\lambda(2t)^{1/m}\cos\theta}d\omega\right\}dt\right|$$

$$\approx \left|\int_0^{1/2} e^{i\lambda t}t^{\alpha+(2/m)-1}\left\{e^{i\lambda(2t)^{1/m}}(\lambda(2t)^{1/m})^{-1/2}\right\}dt\right|$$

$$= \lambda^{-1/2}\left|\int_0^{1/2} e^{i\lambda(t+(2t)^{1/m})}t^{\alpha+(2/m)-(1/2m)-1}\,dt\right|$$

$$\leqslant C\lambda^{-1/2}\max\left\{C\int_0^{\lambda^{-m}} t^{\alpha+(2/m)-(1/2m)-1}\,dt,\, C\lambda^{-1}\right\} = C\lambda^{-3/2}.$$

Choosing δ sufficiently small now yields (44).

3. Three dimensions. The purpose of this section is to prove (A) of Theorem 8. Let $Z_\Phi = \{x\colon \Phi(x) = 0\}$. Let $H\Phi(x)$ denote the determinant of the Hessian matrix of Φ, and let $Z_{H\Phi} = \{x\colon H\Phi(x) = 0\}$. We begin by obtaining some technical information about the structure of Z_Φ and $Z_{H\Phi}$.

LEMMA 11. *Let Φ be mixed homogeneous of degree (a_1, a_2). Then, either $Z_\Phi = \{(0,0)\}$, or $Z_\Phi = \bigcup Z_\Phi^j$, where $Z_\Phi^j = \{x\colon x_1^{a_1} = C_j x_2^{a_2}\}$.*

LEMMA 12. *Let Φ be mixed homogeneous of degree (a_1, a_2). Then, $H\Phi$ is mixed homogeneous of degree (b_1, b_2), where*

$$b_1 = \frac{a_2(a_1-2)+a_1(a_2-2)}{a_2}, \qquad b_2 = \frac{a_2(a_1-2)+a_1(a_2-2)}{a_1}. \tag{45}$$

The proofs of Lemmas 11 and 12 follow from elementary calculations. If $Z_{H\Phi} = \{(0,0)\}$, the proof of Theorem 8 follows from the scaling estimates given in the following subsection. On the other hand, if $Z_{H\Phi} \neq \{(0,0)\}$, Lemmas 11 and 12 imply that under the assumptions of Theorem 8, $Z_\Phi = \bigcup_{j=1}^N V_j$ and $Z_{H\Phi} = \bigcup_{j=1}^{N'} V_j'$, where $V_j = \{x\colon x_1^{a_1} = A_j x_2{}^{a_2}\}$, and $V_j' = \{x\colon x_1^{a_1} = B_j x_2{}^{a_2}\}$ (note that $b_1/b_2 = a_1/a_2$), where $A_j \neq B_{j'}$, $\forall j, j'$. Note that N and N' are finite since Φ is a polynomial. In other words, the zero sets of Φ and $H\Phi$ both consist of "parabolas" of the same degree, which only intersect at the origin. In this case we construct thin "parabolic sectors" S_j, such that each sector contains exactly one $Z^j_{H\Phi}$, and $S_j \cap Z_\Phi = \{(0,0)\}$ for each j. More precisely, each sector is bounded by two "parabolas" given by the equations $x_1^{a_1} = D_k x_2{}^{a_2}$, $k = 1, 2$, where the D_k's are chosen to satisfy the conditions stated above. This case is treated in the following subsection.

3.1. Scaling and nondegenerate curvature estimates. We begin by treating the case where $H\Phi$ vanishes only at the origin, and in this case, our methods extend to arbitrary dimension. We say that $\Phi(x_1, \dots, x_n)$ is mixed homogeneous of degree $(a_1, \dots, a_n)$ if

$$\Phi(t^{1/a_1}x_1, \dots, t^{1/a_n}x_n) = t\Phi(x_1, \dots, x_n).$$

LEMMA 13. *Let $S = \{(x, x_{n+1}) \in R^{n+1} : x_{n+1} = \Phi(x)\}$, where Φ is mixed homogeneous of degree $(a_1, \dots, a_n)$. Suppose that $|H\Phi|$ is elliptic in the sense that $H\Phi$ does not vanish away from the origin. Let $-(1/a_1) - \dots - (1/a_n) < \alpha < (n/2) - (1/a_1) - \dots - (1/a_n)$. Then with $F_\alpha(\xi, \lambda) = \int e^{i(x\cdot\xi+\lambda\Phi(x))}|\Phi(x)|^\alpha \psi(x)\,dx$, we have*

$$|F_\alpha(\xi, \lambda)| \leqslant C(1+|\xi|+|\lambda|)^{-(1/a_1)-\dots-(1/a_n)-\alpha}. \tag{46}$$

In order to prove Lemma 13, we shall need the following well-known stationary phase result (see, e.g., [So1]).

LEMMA 14. *Let S be a smooth hypersurface in R^n with nonvanishing Gaussian curvature and $d\mu$ a C_0^∞ measure on S. Then*

$$|\widehat{d\mu}(\xi)| \leqslant \text{const.}(1+|\xi|)^{-(n-1)/2}. \tag{47}$$

Moreover, suppose that $\Gamma \subset R^n\backslash 0$ is the cone consisting of all ξ which are normal to some point $x \in S$ belonging to a fixed relatively compact neighborhood $\mathcal{N}$ of supp $d\mu$. *Then*

$$\left(\frac{\partial}{\partial\xi}\right)^\alpha \widehat{d\mu}(\xi) = O(1+|\xi|)^{-N} \qquad \forall N,\ \textit{if}\ \xi \notin \Gamma,$$

$$\widehat{d\mu}(\xi) = \sum e^{-ix_j\cdot\xi}a_j(\xi), \qquad \textit{if}\ \xi \in \Gamma,$$

where the finite sum is taken over all $x_j \in \mathcal{N}$ having ξ as the normal and

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha a_j(\xi)\right| \leqslant C_\alpha(1+|\xi|)^{-((n-1)/2)-|\alpha|}. \tag{48}$$

Proof of Lemma 13. Let

$$F_{\alpha,0}(\xi, \lambda) = \int e^{i(x\cdot\xi+\lambda\Phi(x))}|\Phi(x)|^\alpha \psi_0(x)\,dx, \tag{49}$$

where ψ_0 is a smooth cutoff function supported in the annulus $\{x: (1/2) \leqslant |x| \leqslant 4\}$ and satisfying

$$\sum_{k=0}^{\infty} \psi_0(2^{k/a_1}x_1, \ldots, 2^{k/a_n}x_n) \equiv 1.$$

We can express $F_\alpha(\xi, \lambda)$ in the form

$$\sum_j 2^{-(j/a_1)-\cdots-(j/a_n)}2^{-\alpha j}F^j_{\alpha,0}(2^{-j/a_1}\xi_1, \ldots, 2^{-j/a_n}\xi_n, 2^{-j}\lambda), \tag{50}$$

where

$$F^j_{\alpha,0}(\xi, \lambda) = \int e^{i(x\cdot\xi+\lambda\Phi(x))}|\Phi(x)|^\alpha \psi_0(x)\psi(2^{-j/a_1}x_1, \ldots, 2^{-j/an}x_n)\, dx$$

satisfies the same estimates (see below) as $F_{\alpha,0}$ uniformly in j. For convenience we work only with $F_{\alpha,0}$ from here on. We should note that we can assume that $|\xi| \leqslant C|\lambda|$, since if $|\lambda| \leqslant c|\xi|$ for a sufficiently small c, $F_\alpha(\xi, \lambda)$ has rapid decay in $|\xi|$.

Lemma 14 tells us that $|F_{\alpha,0}(\xi, \lambda)| \leqslant C(1+|\lambda|)^{-n/2}$. Hence, the sum in (50) is dominated by

$$C|\lambda|^{-n/2} \sum_{2^{-j}|\lambda|\geqslant 1} (2^{-j})^{-n/2} 2^{-(j/a_1)-\cdots-(j/a_n)}2^{-\alpha j} + C\sum_{2^{-j}|\lambda|\leqslant 1} 2^{-(j/a_1)-\cdots-(j/a_n)}2^{-\alpha j} \tag{51}$$

$$\leqslant C'|\lambda|^{-(1/a_1)-\cdots-(1/a_n)-\alpha}, \tag{52}$$

since the series converge if $-(1/a_1) - \cdots - (1/a_n) < \alpha < (n/2) - (1/a_1) - \cdots - (1/a_n)$. This completes the proof.

We can now dispose of the contribution to $F_\alpha(\xi, \lambda)$ arising away from the zero set of $H\Phi$. Recall that $Z_\Phi = \bigcup_{j=1}^N V_j$ and $Z_{H\Phi} = \bigcup_{j=1}^{N'} V'_j$, where $V_j = \{x: x_1^{a_2} = A_j x_2{}^{a_1}\}$, and $V'_j = \{x: x_1^{a_2} = B_j x_2{}^{a_1}\}$, where $A_j \neq B_{j'}$, $\forall j, j'$, and that we constructed thin "parabolic sectors" S_j, such that each sector contains exactly one $Z^j_{H\Phi}$, and $S_j \cap Z_\Phi = \{(0,0)\}$ for each j. Let $E = R^2 \backslash (\cup_j S_j)$. Clearly, $Z_{H\Phi} \cap E = \{(0,0)\}$. Lemma 13 with $n = 2$ can be used to get the desired estimates for F_α localized to E. In the proof of Lemma 13 the cutoff function ψ_0 was chosen to have support in the annulus where $1 \leqslant |x| \leqslant 2$. In order to adapt the proof of Lemma 13 to the current situation, we must chose ψ_0 supported in $E \cap \{x: (1/2) \leqslant |x| \leqslant 4\}$, and $\psi_0 = 1$ in $E \cap \{x: (5/4) \leqslant |x| \leqslant (7/4)\}$. A crucial observation is that E is invariant under a change of variables sending $(x_1,x_2) \to (2^j x_1, 2^{j(a_1/a_2)}x_2)$. Hence for the proof of Theorem 8, it suffices to estimate F_α localized to the S_j's, to which we now turn.

3.2. Curvature vanishing beyond the origin. Fix an index j for the remainder of the proof. Let ψ be a smooth cutoff function such that the support of ψ is contained in the unit ball, and let χ be the characteristic function of the sector S_j. We consider

$$F_\alpha(\xi, \lambda) = \int e^{-i(x \cdot \xi + \lambda \Phi(x))} |\Phi(x)|^\alpha \chi(x) \psi(x)\, dx. \tag{53}$$

We note that it suffices to consider a region where $|\xi| \leqslant C|\lambda|$, since if $|\xi| > C|\lambda|$ for a sufficiently large C, a familiar integration by parts argument shows that we get a rapid decay in ξ.

We shall use a weighted polar coordinate system; see, e.g, [FR]. We need the following observation.

LEMMA 15. *Consider a change of coordinates given by $x_1 = r\omega_1$, $x_2 = r^{a_1/a_2}\omega_2, \ldots, x_n = r^{a_1/a_n}\omega_n$, where $\omega = (\omega_1, \ldots, \omega_n)$ is the usual set of coordinates on the unit sphere. Suppose that $a_1 \geqslant a_j \forall j$. Then, away from the origin, $\{(r, \omega)\}$ is a C^1 coordinate system. The Jacobian of this change of coordinates is given by $r^{(a_1/a_2)+\cdots+(a_1/a_n)} g(\omega)$, where $1 \leqslant g(\omega) \leqslant C(a_1, \ldots, a_n)$.*

Using Lemma 15 with $n = 2$, we rewrite F_α in the form

$$\int \int_{\theta_j - \varepsilon}^{\theta_j + \varepsilon} e^{-i(\langle \omega, (r\xi_1, r^{a_1/a_2}\xi_2)\rangle + r^{a_1}\Phi(\omega)\lambda)} r^{(a_1/a_2)+a_1\alpha} |\Phi(\omega)|^\alpha \psi(r)\, d\theta\, dr, \tag{54}$$

where $\omega = (\cos(\theta), \sin(\theta))$, and $\omega^j = (\cos(\theta^j), \sin(\theta^j))$ is the unique isolated point on the circle in S_j where $H\Phi$ vanishes. If we note that Φ does not vanish in the range of integration above, we can make a change of variables sending $r \to r\Phi^{-1/a_1}(\omega)$. We then set $\mu = (\cos(\theta)/\Phi^{1/a_1}(\omega), \sin(\theta)/\Phi^{1/a_2}(\omega))$. If we note that $\Phi(\mu) = 1$, we see that

$$F_\alpha(\xi, \lambda) = \int \int_{\Sigma_j} e^{-i(r\mu_1\xi_1 + r^{a_1/a_2}\mu_2\xi_2 + \lambda r^{a_1})} r^{(a_1/a_2)+a_1\alpha} \tilde{\psi}(r)\, d\sigma(\mu)\, dr, \tag{55}$$

where

$$\Sigma_j = \{\mu \colon \Phi(\mu) = 1\} \cap S_j,$$

$$\tilde{\psi}(r) = \psi(\Phi(\omega)^{-1/a_1} r),$$

$$d\sigma = |\nabla\Phi|^{-1}\, d\eta,$$

and $d\eta = |\nabla\Phi(\mu)| \Phi(\omega)^{-(1/a_1)-(1/a_2)}$ is arc length measure on Σ_j. Since $\nabla\Phi \neq 0$ and $\Phi \neq 0$ on S_j, $d\sigma$ is a smooth measure on Σ_j.

We first analyze the integral over Σ_j. Since S_j contains only one "parabola" where $H\Phi$ vanishes, the curvature on Σ_j vanishes only at the unique point $\mu^j = (\cos\theta^j/\Phi^{1/a_1}(\omega^j), \sin\theta^j/\Phi^{1/a_2}(\omega^j))$. Moreover, since Φ is a polynomial, repeated application of the implicit function theorem shows that the curvature cannot vanish of order exceeding $a_1 - 2$. Hence, Σ_j is a smooth curve of finite type M where $3 \leqslant M \leqslant a_1$. Note that if $M = 2$, then $H\Phi$ would only vanish at the origin, a contradiction.

In local coordinates, after perhaps applying a rotation and a translation, we can write the integral over Σ_j in (55) as

$$I(r, \xi_1, \xi_2) = \int e^{-i(tr\xi_1 + \phi(t)r^{a_1/a_2}\xi_2)}\rho(t)\,dt, \tag{56}$$

where ρ is a smooth cutoff function supported in $[-\delta, \delta]$, $\delta > 0$ small, $\phi(t) = g(t)t^M$, and g is a smooth function such that $g(0) \neq 0$. Let ρ_0 denote a smooth cutoff function supported in $((\delta/2), \delta) \cup (-(\delta/2), -\delta)$, such that $\sum_k \rho_0(2^k t) = 1$. Let $I_k(\xi_1, \xi_2) = \int e^{-i(tr\xi_1 + \phi(t)r^{a_1/a_2}\xi_2)}\rho_0(2^k t)\,dt$. Each I_k is defined over a piece of Σ_j where the curvature does not vanish. In order to take advantage of this fact, we make a change of variables sending $t \to 2^{-k}t$. We get

$$I_k(r, \xi_1, \xi_2) = 2^{-k}\int e^{-i(tr2^{-k}\xi_1 + g(t2^{-k})2^{-kM}t^M r^{a_1/a_2}\xi_2)}\rho_0(t)\,dt.$$

Note that $I_k(r, \xi_1, \xi_2) \approx 2^{-k}I_0(r, 2^{-k}\xi_1, 2^{-Mk}\xi_2)$, since $g(t2^{-k}) \to g(0) \neq 0$, as $k \to \infty$. Using Lemma 14 we can write

$$I_k(r, \xi_1, \xi_2) = 2^{-k}e^{iq_k(r2^{-k}\xi_1, r^{a_1/a_2}2^{-kM}\xi_2)}b_k(r2^{-k}\xi_1, r^{a_1/a_2}2^{-kM}\xi_2),$$

where b is a symbol of order $-1/2$, q_k is homogeneous of degree 1, and the Hessian matrix of q_k has rank 1 everywhere. Note that

$$q_k(r2^{-k}\xi_1, r^{a_1/a_2}2^{-kM}\xi_2) = r^{(a_1(a_2-2))/(a_2(a_1-2))}q_k(2^{-k}\xi_1, 2^{-kM}\xi_2).$$

Also note that since the Gauss map is smooth, for k large we have

$$q_k(2^{-k}\xi_1, 2^{-kM}\xi_2) \approx q(2^{-k}\xi_1, 2^{-kM}\xi_2),$$

where $q(\xi_1, \xi_2)$ is the "limiting" phase function given by Lemma 14 corresponding to the curve $(t, g(0)t^M)$. It was shown in [I2] that $q(2^{-k}\xi_1, 2^{-kM}\xi_2) = q(\xi_1, \xi_2)$, and in fact that

$$q(\xi_1, \xi_2) = c_M\frac{\xi_1^{M/(M-1)}}{\xi_2^{1/(M-1)}}. \tag{57}$$

More precisely, there exists a uniform constant $C > 0$ such that

$$C^{-1}|q(\xi_1, \xi_2)| \leqslant |q_k(2^{-k}\xi_1, 2^{-kM}\xi_2)| \leqslant C|q(\xi_1, \xi_2)|. \tag{58}$$

Similarly, $\{b_k(\xi_1, \xi_2)\}_k$ is contained in a bounded subset of symbols of order $-1/2$, i.e.,

$$|D_\xi^\beta b_k(\xi_1, \xi_2)| \leqslant C_\beta |\xi|^{-(1/2)-|\beta|},$$

where the C_β are uniform constants.

After making a change of variables in (55) sending $r \to r\lambda^{-1/a_1}$, followed by setting $s = r^{(a_1(a_2-1))/(a_2(a_1-1))}$, what results is

$$F_\alpha(\xi, \lambda) = \lambda^{-(1/a_1)-(1/a_2)-\alpha} \sum_{k=0}^{\infty} 2^{-k}$$

$$\times \int \exp\left(i\left(sq_k(2^{-k}\xi_1, 2^{-Mk}\xi_2)\lambda^{-(a_2-1)/(a_2(a_1-1))} - s^{(a_2(a_1-1))/(a_2-1)}\right)\right)$$

$$\times s^{(a_2(a_1-1))/(a_2-1)\alpha} s^{(a_1-1)/(a_2-1)} s^{(a_1-a_2)/(a_1(a_2-1))}$$

$$\times b_k(s^{(a_2(a_1-1))/(a_1(a_2-1))}\, 2^{-k}\xi_1\lambda^{-1/a_1}, s^{(a_1-1)/(a_2-1)} 2^{-kM}\xi_2\lambda^{-1/a_2})\, ds.$$

Note that for k large, $|q_k(\xi_1, \xi_2)| \approx |\xi| \approx |\xi_2|$ since by Lemma 14, we restrict attention to those ξ in the cone of normals to the piece of Σ_j lying in the support of $\rho(2^k t)$. Let $\Psi(s) = sA_k - s^{(a_2(a_1-1))/(a_2-1)}$, where $A_k = q_k(2^{-k}\xi_1, 2^{-Mk}\xi_2) \cdot \lambda^{-(a_2-1)/(a_2(a_1-1))}$. Note that

$$A_k \approx A = q(\xi_1, \xi_2)\lambda^{-(a_2-1)/(a_2(a_1-1))}$$

by (57) and (58). Then $\Psi'(s_0) = 0$ if $s_0 = CA_k^{(a_2-1)/(a_1a_2-2a_2+1)}$, and $\Psi''(s) = Cs^{(a_1a_2-3a_2+2)/(a_2-1)}$. By the Van der Corput Lemma, we see that the expression above is bounded by

$$C\lambda^{-(1/a_1)-(1/a_2)-\alpha} \sum_{k=0}^{\infty} 2^{-k} \left| b_k\left(s_0^{(a_2(a_1-1))/(a_1(a_2-1))}\, 2^{-k}\xi_1\lambda^{-1/a_1},\right.\right.$$

$$\left.\left. \times\, s_0^{(a_1-1)/(a_2-1)}\, 2^{-kM}\xi_2\lambda^{-1/a_2}\right)\right| |A|^{(1/2)(-(a_1a_2-3a_2+2))/(a_1a_2-2a_2+1)}$$

$$\times |A|^{(a_2(a_1-1)\alpha)/(a_1a_2-2a_2+1)} |A|^{(a_1-1)/(a_1a_2-2a_2+1)} |A|^{(a_1-a_2)/(a_1(a_1a_2-2a_2+1))}.$$

We shall now estimate

$$\sum_{k=0}^{\infty} 2^{-k} b_k\left(s_0^{(a_2(a_1-1))/(a_1(a_2-1))}\, 2^{-k}\xi_1\lambda^{-1/a_1},\, s_0^{(a_1-1)/(a_2-1)} 2^{-kM}\xi_2\lambda^{-1/a_2}\right).$$

Let $B=\xi_2\lambda^{-1/a_2}$, and note that $|B|\leqslant|A|$. Then

$$\sum_{k=0}^{\infty} 2^{-k} b_k\left(s_0^{(a_2(a_1-1))/(a_1(a_2-1))}\, 2^{-k}\xi_1\lambda^{-1/a_1},\, s_0^{(a_1-1)/(a_2-1)}\, 2^{-kM}\xi_2\lambda^{-1/a_2}\right)$$

$$= \sum_{|A|^{(a_1-1)/(a_1a_2-2a_2+1)}|B|\geqslant 2^{a_1k}} 2^{-k} b_k\left(s_0^{(a_2(a_1-1))/(a_1(a_2-1))}\, 2^{-k}\xi_1\lambda^{-1/a_1},\right.$$

$$\left. s_0^{(a_1-1)/(a_2-1)} 2^{-kM}\xi_2\lambda^{-1/a_2}\right)$$

$$+ \sum_{|A|^{(a_1-1)/(a_1a_2-2a_2+1)}|B|\leqslant 2^{a_1k}} 2^{-k} b_k\left(s_0^{(a_2(a_1-1))/(a_1(a_2-1))}\, 2^{-k}\xi_1\lambda^{-1/a_1},\right.$$

$$\left. s_0^{(a_1-1)/(a_2-1)} 2^{-kM}\xi_2\lambda^{-1/a_2}\right)$$

$$= I + II.$$

If we use the fact that b_k is a symbol of order $-1/2$ and that $M\geqslant 3$, we see that term I satisfies

$$|I| \leqslant \sum_{|A|^{(a_1-1)/(a_1a_2-2a_2+1)}|B|\geqslant 2^{a_1k}} 2^{-k} C(|A|^{(a_1-1)/(a_1a_2-2a_2+1)}|B|)^{-1/2} 2^{kM/2}$$

$$\leqslant C(|A|^{(a_1-1)/(a_1a_2-2a_2+1)}|B|)^{(1/a_1)((M/2)-1)-(1/2)}$$

$$\leqslant C|A|^{-(a_1-1)/(a_1(a_1a_2-2a_2+1))}|B|^{-1/a_1}$$

since $M\leqslant a_1$. Using the fact that b_k is bounded we see that term II satisfies

$$|II| \leqslant \sum_{|A|^{(a_1-1)/(a_1a_2-2a_2+1)}|B|\leqslant 2^{a_1k}} 2^{-k} C \leqslant C|A|^{-(a_1-1)/(a_1(a_1a_2-2a_2+1))}|B|^{-1/a_1}.$$

Putting our estimates together, we get

$$|F_\alpha(\xi,\lambda)| \leqslant C\lambda^{-(1/a_1)-(1/a_2)-\alpha}|A|^{(1/2)(-(a_1a_2-3a_2+2))/(a_1a_2-2a_2+1)}$$

$$\times |A|^{(a_2(a_1-1)\alpha)/(a_1a_2-2a_2+1)}|A|^{(a_1-1)/(a_1a_2-2a_2+1)}$$

$$\times |A|^{(a_1-a_2)/(a_1(a_1a_2-2a_2+1))}\,|A|^{-(a_1-1)/(a_1(a_1a_2-2a_2+1))}|B|^{-1/a_1}.$$

Recall that for $|\xi| \geqslant C|\lambda|$, with C large enough, integration by parts yields rapid decay in ξ for $F_\alpha(\xi,\lambda)$. We now restrict attention to the case $c|\lambda| \leqslant |\xi| \leqslant C|\lambda|$, with a fixed small constant c, and later we will use a scaling argument to obtain the general case. Let $\gamma = a_1a_2 - 2a_2 + 1$. Using the fact that $|B| \approx |A|^{((a_1-1)(a_2-1))/\gamma}$ when $|\xi| \approx |\lambda|$, we see that the exponent of $|A|$ above is

$$-\frac{1}{2}\frac{a_1a_2-3a_2+2}{\gamma}+\frac{a_2(a_1-1)}{\gamma}\alpha+\frac{a_1-1}{\gamma}+\frac{a_1-a_2}{a_1\gamma}-\frac{a_1-1}{a_1\gamma}-\frac{(a_1-1)(a_2-1)}{a_1\gamma}$$

$$=\frac{1}{\gamma}\left\{\left(\alpha-\frac{1}{2}\right)a_2(a_1-1)+a_1-1\right\},$$

which is nonpositive when

$$\alpha \leqslant \frac{1}{2}-\frac{1}{a_2}.$$

Thus $|F_\alpha(\xi,\lambda)| \leqslant C\lambda^{-(1/a_1)-(1/a_2)-\alpha}$ when $c|\lambda| \leqslant |\xi| \leqslant C|\lambda|$ and $\alpha \leqslant (1/2)-(1/a_2)$.

In order to complete the proof of the general case, we must extend the argument above to all $|\xi| \leqslant c|\lambda|$. We need the following scaling argument. Let

$$F_\alpha(\xi,\lambda) = \int e^{-i(x\cdot\xi+\lambda\Phi(x))}|\Phi(x)|^\alpha\psi(x)\,dx$$

as before. Let

$$F_\alpha^k(\xi,\lambda) = \int e^{-i(x\cdot\xi+\lambda\Phi(x))}|\Phi(x)|^\alpha\rho(2^k|x|_{(a_1,a_2)})\psi(x)\,dx,$$

where ρ is defined as before, and $|x|_{(a_1,a_2)}= (x_1^{2a_1}+x_2^{2a_2})^{1/2a_1}$. After making a change of variables, we see that

$$F_\alpha^k(\xi,\lambda) \approx 2^{-k}2^{-k(a_1/a_2)}2^{-ka_1\alpha}F_\alpha^0(2^{-k}\xi_1, 2^{-k(a_1/a_2)}\xi_2, 2^{-a_1k}\lambda)$$

and

$$F_\alpha(\xi, \lambda) = \sum_{k=0}^{\infty} F_\alpha^k(\xi, \lambda).$$

Note that F_α^0 is defined over the piece of the hypersurface where the normals are contained in a fixed cone away from the coordinate axes. This follows from the assumption that $\nabla\Phi \neq (0,0)$ away from $(0,0)$, and the fact that the normal at the point $(x_1, x_2, \Phi(x_1, x_2))$ is given by $(\nabla\Phi, -1)$. A standard integration by parts argument shows that for (ξ, λ) away from a slightly larger cone, i.e., for $|\xi| < c_1|\lambda|$ or $|\xi| > c_2|\lambda|$, F_α^0 decays rapidly in (ξ, λ). Hence, by what we just proved above, we have

$$\begin{aligned} |F_\alpha^0(\xi, \lambda)| &\leqslant C|\lambda|^{-(1/a_1)-(1/a_2)-\alpha}, && \text{for } c_1|\lambda| \leqslant |\xi| \leqslant c_2|\lambda|, \\ &\leqslant C_N|\lambda|^{-N} && \text{for any } N, \text{ otherwise,} \end{aligned} \tag{59}$$

where c_1 and c_2 are uniform constants. We break up the sum as follows:

$$\begin{aligned} &\sum_{k=0}^{\infty} 2^{-k}2^{-k(a_1/a_2)}2^{-ka_1/\alpha}F_\alpha^0(2^{-k}\xi_1, 2^{-k(a_1/a_2)}\xi_{2,}2^{-a_1k}\lambda) \\ &\quad = \sum_{|\lambda| \geqslant 2^{a_1 k}} 2^{-k}2^{-k(a_1/a_2)}2^{-ka_1\alpha}F_\alpha^0(2^{-k}\xi_1, 2^{-k(a_1/a_2)}\xi_{2,}2^{-a_1k}\lambda) \\ &\quad + \sum_{|\lambda| \leqslant 2^{a_1 k}} 2^{-k}2^{-k(a_1/a_2)}2^{-ka_1\alpha}F_\alpha^0(2^{-k}\xi_1, 2^{-k(a_1/a_2)}\xi_2, 2^{-a_1k}\lambda) \\ &\quad = I + II. \end{aligned}$$

The second term II is bounded by $C|\lambda|^{-(1/a_1)-(1/a_2)-\alpha}$ using only $|F_\alpha^0| \leqslant C$. To bound the first term we use estimate (59) to obtain

$$|F_\alpha^0(2^{-k}\xi_1, 2^{-k(a_1/a_2)}\xi_{2,}2^{-a_1k}\lambda)| \leqslant C|2^{-a_1k}\lambda|^{-(1/a_1)-(1/a_2)-\alpha},$$

for $|(2^{-k}\xi_1, 2^{-k(a_1/a_2)}\xi_2)| \approx |2^{-a_1k}\lambda|$ and $\alpha \leqslant (1/2) - (1/a_2)$, and

$$|F_\alpha^0(2^{-k}\xi_1, 2^{-k(a_1/a_2)}\xi_{2,}2^{-a_1k}\lambda)| \leqslant C_N|2^{-a_1k}\lambda|^{-N},$$

for any N if $|(2^{-k}\xi_1, 2^{-k(a_1/a_2)}\xi_2)|$ is not comparable to $|2^{-a_1k}\lambda|$. Now for fixed ξ and λ, there are at most finitely many k, say $k \in \mathscr{F}$, such that

$$c_1|2^{-a_1k}\lambda| \leqslant |(2^{-k}\xi_1, 2^{-k(a_1/a_2)}\xi_2)| \leqslant c_2|2^{-a_1k}\lambda|,$$

i.e.,

$$c_1|\lambda| \leqslant \left|(2^{(a_1-1)k}\xi_1, 2^{a_1(1-(1/a_2))k}\xi_2)\right| \leqslant c_2|\lambda|.$$

Thus the first term satisfies

$$|I| \leqslant C_N \sum_{|\lambda| \geqslant 2^{a_1 k}} 2^{-k} 2^{-k(a_1/a_2)} 2^{-k a_1 \alpha} \left|2^{-a_1 k}\lambda\right|^{-N}$$

$$+ C \sum_{k \in \mathscr{F}:\, |\lambda| \geqslant 2^{a_1 k}} 2^{-k} 2^{-k(a_1/a_2)} 2^{-k a_1 \alpha} \left|2^{-a_1 k}\lambda\right|^{-(1/a_1)-(1/a_2)-\alpha}$$

$$\leqslant C|\lambda|^{-(1/a_1)-(1/a_2)-\alpha},$$

if we take $N > (1/a_1) + (1/a_2) + \alpha$. This completes the proof.

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