Packing in One, Two, and Three Dimensions

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CoronaVirus Lecture Series
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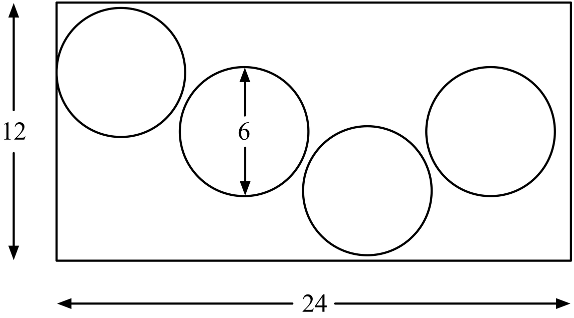
Slides at http://jamespropp.org/cvls20.pdf

Packing problem

"How many disks of diameter d can you fit into an a-by-b rectangle?"

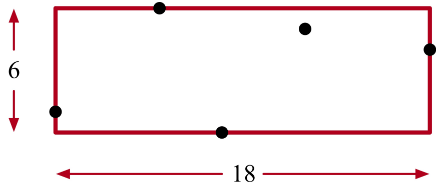
For instance, how many disks of diameter 6 feet can you fit into a 12 foot by 24 foot rectangle?



Can anyone see how a problem like that carries over to a timely application?

Distancing problem

"How many people can we fit into a 6-foot-by-18-foot rectangle if we require that nobody is within less than 6 feet of anyone else?"

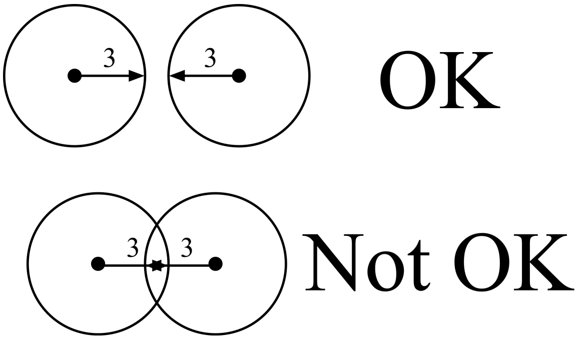


What's the connection?

Connecting the two problems

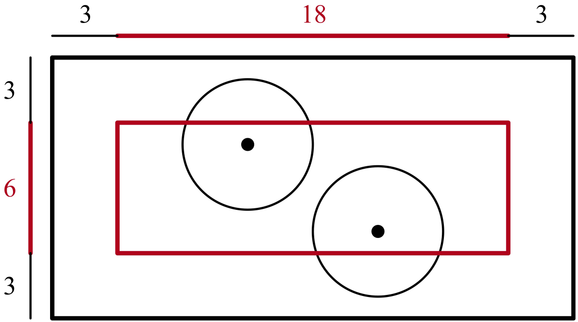
Imagine each person carries around a personal bubble whose diameter is 6 feet, that is, whose radius is 3 feet.

Requiring that two people be at least 6 feet apart is equivalent to requiring that their bubbles don't overlap.



Connecting the two problems

The bubbles extend beyond the 6-by-18 rectangle, 3 feet in each direction, so those bubbles lie in a 6+3+3 by 18+3+3 rectangle.



Connecting the two problems

If the people in the 6-by-18 hallway are all 6 feet apart, then their bubbles are disjoint disks in an imaginary 12-by-24 rectangle, **and vice versa**.

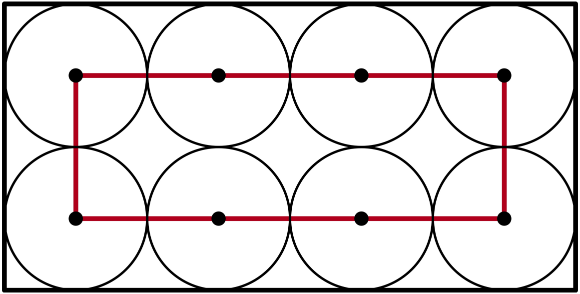
So figuring out good ways to pack disks can tell us good ways to physically distance.

Precept #1

Different representations suggest different tools.

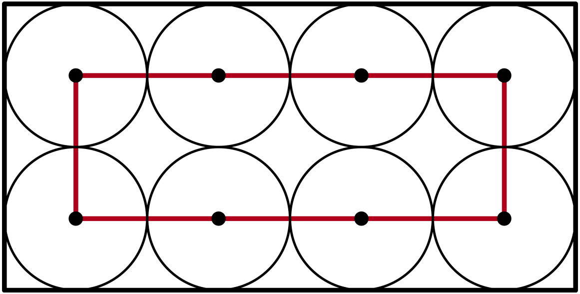
What's obvious?

We can fit 8 people into the hallway:



It's obvious that there's no way to add a 9th person without violating distancing.

What's obvious?



It's also true that there's no way to fit in 9 people even if you remove the other 8 and start from scratch, and maybe you think this is obvious, but one of my goals is to make you feel half an hour from now that it isn't really so obvious.

Down to 1 dimension

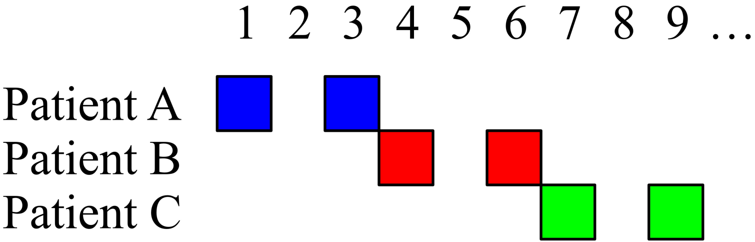
Scheduling problems can sometimes be thought of as 1-dimensional packing problems.

Say we've got some resource (it could be a person, it could be a place, it could be a thing) that needs to be shared on a schedule.

To make the problem specific, say it's a piece of medical equipment, and the treatment schedule is that each patient needs 1 hour of treatment, followed by an hour of rest, followed by 1 more hour of treatment.

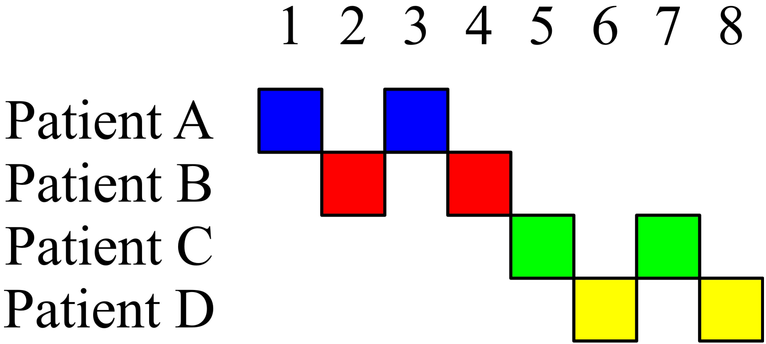
Scheduling problem

If we schedule the patients in three-hour blocks, then the machine is being used only 2/3 of the time.



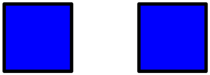
Scheduling problem

But if we schedule the patients in pairs, like this, then the machine is being used all the time, and we can treat more patients.



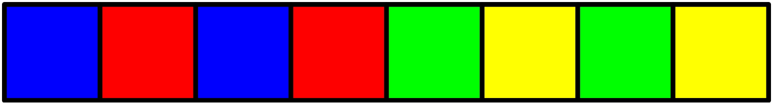
Scheduling as packing

You can think of our scheduling problem as a packing problem where the pieces being packed look like



(imagine a force field that keeps the two parts at a fixed distance)

but the two halves of each piece have to shift horizontally in tandem, maintaining the space between them.



2 on, 2 off, 1 on

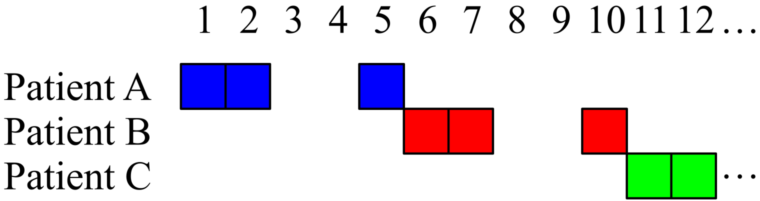
Here's a problem we'll do together.

Suppose each patient needs 2 hours on the machine, 2 hours of rest, then 1 hour of treatment.

How can we set up a schedule to make the most efficient use of the machine?

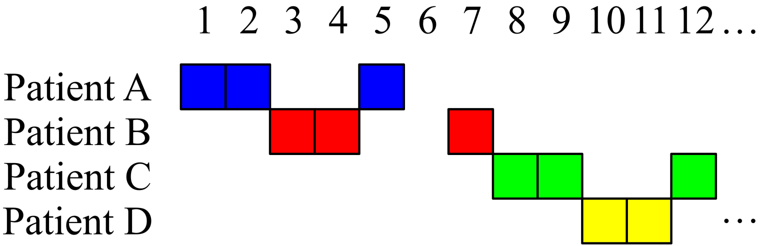
2 on, 2 off, 1 on

If we don't interleave patients at all, we get a schedule that is only $3/5 = 60\%$ efficient.



2 on, 2 off, 1 on

But there's a schedule where the machine gets used $6/7 \approx 86\%$ of the time.



2 on, 3 off, 1 on

Here's a problem you'll do in groups.

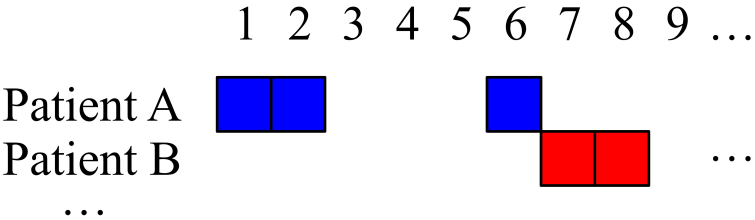
Suppose each patient needs 2 hours on the machine, 3 hours of rest, and then 1 hour on the machine.

How can we set up a schedule to make the most efficient use of the machine?

(Intermission)

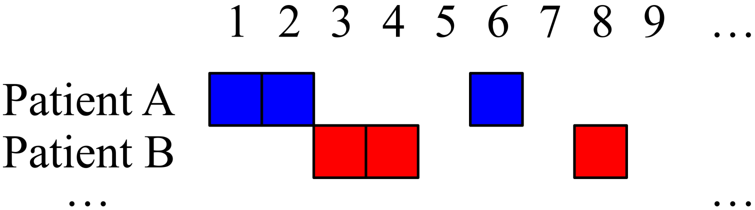
2 on, 3 off, 1 on

If we don't interleave patients at all, we get a solution with efficiency $3/6 = 50\%$:



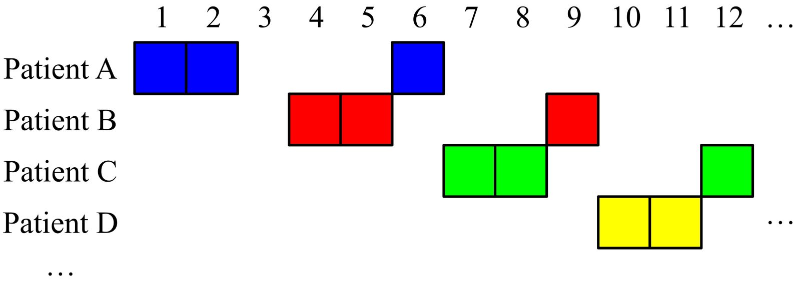
2 on, 3 off, 1 on

Here's a solution with efficiency $6/8 = 75\%$:



2 on, 3 off, 1 on

Here's a solution with efficiency 100% (aside from a hiccup at the start when the machine goes unused for a single hour):

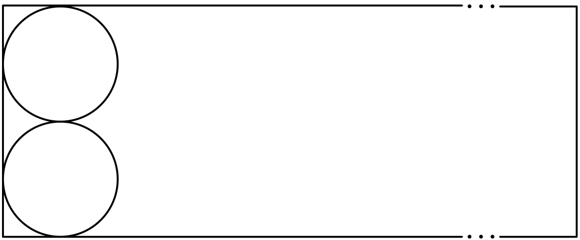


Precept #2

Long-term goals require long-term planning.

Back to two dimensions

Say I have a long skinny tray that's just wide enough to accommodate two quarters at the end, but it's a mile long.

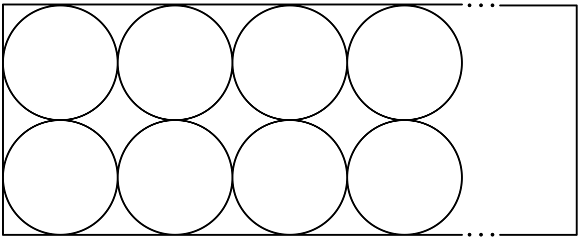


Say I offer you as many quarters as you can pack into the tray, laid flat with no stacking. No tricks allowed, just geometry.

What's the best way for you to get quarters?

Back to two dimensions

You could stick them into the corners in the obvious way:



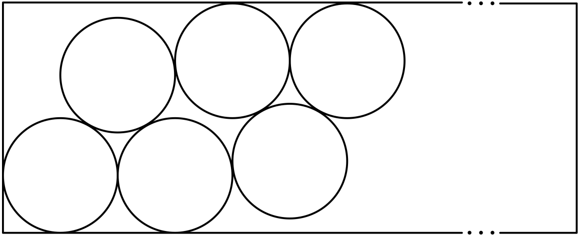
That's gotta be optimal, right? ...

Precept #2

Long-term goals require long-term planning.

Back to two dimensions

The optimum looks like this:



Precept #3

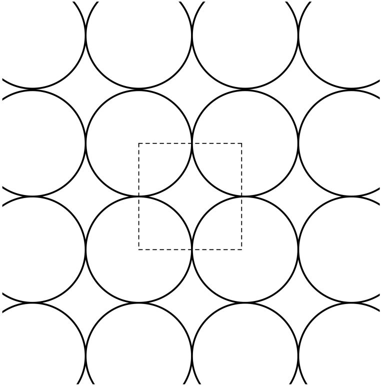
Be very, very, VERY careful about calling things “obvious”.

Packing without boundaries

This surprising phenomenon is related to the fact that in two dimensions, if we don't have boundaries to worry about, and we're just trying pack disks densely in the unbounded plane, the most efficient way to do this is what's called "hexagonal close packing".

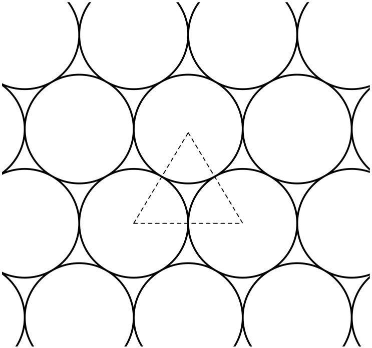
Square close packing

Square close packing has packing fraction $\pi/4 \approx 79\%$.



Hexagonal close packing

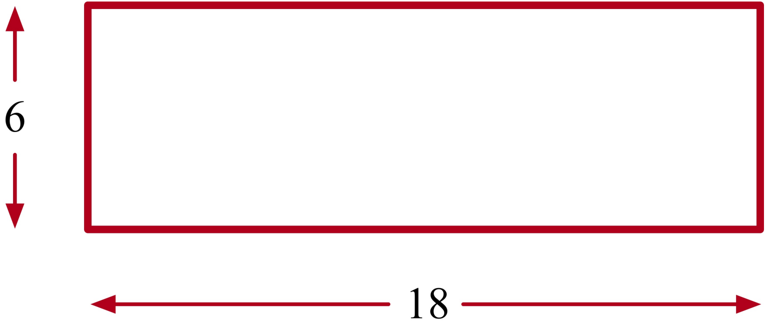
Hexagonal close packing has packing fraction $\pi\sqrt{3}/6 \approx 91\%$.



In 1940, Laszlo Fejes-Toth, building on earlier work by Axel Thue in 1890, showed that this is the optimal packing fraction.

Proving the not-so-obvious

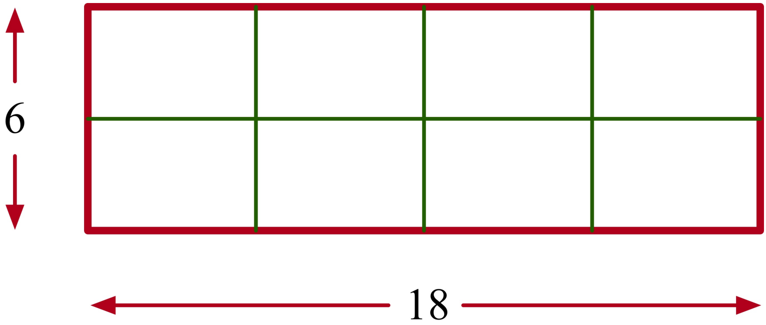
Theorem: If you place nine points in a 6-by-18 rectangle, two of them will be a distance less than 6.



(I hope you agree it's not obvious, or at least not as obvious as it was before I told you that it becomes false if we replace 18 by 18,000!)

Proving the not-so-obvious

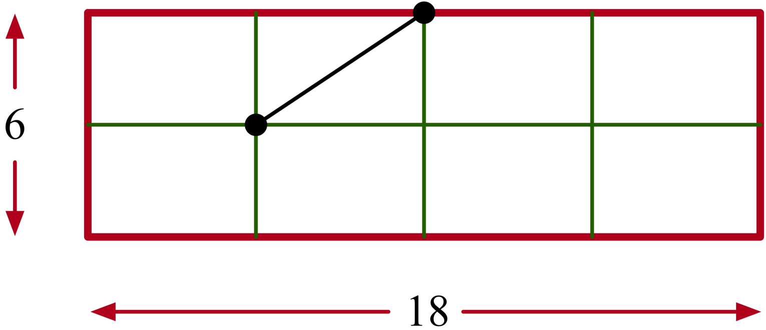
Proof: Divide the big 6-by-18 rectangle into eight small 6/2-by-18/4 rectangles.



If you place nine points in the big rectangle, two of them must belong to the same small rectangle (by the pigeonhole principle).

Proving the not-so-obvious

But that means the distance between those points is at most the length of the diagonal of the small rectangle, which is $\sqrt{(6/2)^2 + (18/4)^2} = \sqrt{29.25} < \sqrt{36} = 6$.)



Three dimensions

What is the most efficient way to pack spheres of diameter 1?

One obvious way to pack them into space so that no two of them overlap is to situate them in a cubical lattice, with centers at the points (x, y, z) where x, y, and z are all integers.

The packing efficiency is $\pi/6 \approx 52\%$.

Three dimensions

A better way is the "grocer's packing", with packing efficiency $\pi\sqrt{2}/6 \approx 74\%$. You lay down a 2-dimensional close packing, and then another layer, and then another, etc.

The optimality of this packing was proved by Tom Hales in 1998.

Thanks!

Slides for this talk are at http://jamespropp.org/cvls20.pdf .

See my Mathematical Enchantments essays Sphere-packing and Believe it, then don't: toward a pedagogy of discomfort.

I recommend Patrick Honner, The Math of Social Distancing Is a Lesson in Geometry.

In June I gave a virtual talk called Packings in one, two, and three dimensions: a macro-meso-microscopic view (slides, abstract, and video available).

The preprint Germ order for one-dimensional packings (by Abrams, Landau, Landau, Pommersheim, Propp, and Russell) applies calculus (specifically power series) to the study of packings.