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Maximal Averages over Surfaces

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Let

$$\mathcal{M}f(x) = \sup_{t>0} |f * \delta_t(\psi\, d\sigma)(x)|$$

denote the maximal operator associated with surface measure $d\sigma$ on a smooth surface S. We prove that if S is convex and has finite order contact with its tangent lines, then $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$, $p>2$, if and only if $d(x,\mathcal{H})^{-1} \in L^{1/p}_{\mathrm{loc}}(S)$ for all tangent planes $\mathcal{H}$ not passing through the origin. Let

$$\mathcal{M}'f(x) = \sup_{t>0} |f * \delta'_t(\psi\, d\sigma)(x)|$$

be the maximal operator associated with a nonisotropic dilation δ'_t of surface measure $d\sigma$. We prove that $\mathcal{M}'$ often behaves far better than $\mathcal{M}$ due to a rotational curvature in the time parameter t. © 1997 Academic Press

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1. INTRODUCTION

Let S be a smooth hypersurface in $\mathbb{R}^n$, let $d\sigma$ denote Lebesgue measure on S, and let ψ denote a smooth cutoff function in $\mathbb{R}^n$. Let δ_t denote the usual dilation given by $\widehat{\delta_t h}(\xi) = \hat{h}(t\xi)$. We consider first the convolution operators

$$M_t f(x) = f * \delta_t(\psi\, d\sigma)(x),$$

and their associated maximal operators

$$\mathcal{M}f(x) = \sup_{t>0} |M_t f(x)|. \tag{1}$$

In [St2], Stein showed that when $S = \mathbb{S}^{n-1}$, the unit $(n-1)$-dimensional sphere, then

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{2}$$

holds for $p > n/(n-1)$, $n \geqslant 3$, where f is initially taken to be in the class of rapidly decreasing functions. The two-dimensional version of this result was proved by Bourgain [Bo]. The key feature of the spherical maximal operator is the non-vanishing Gaussian curvature of the sphere. Indeed, one obtains the same L^p bounds if the sphere is replaced by a piece of any hypersurface in $\mathbb{R}^n$ with everywhere non-vanishing Gaussian curvature (see [Gr]). More generally, one can treat the case where the surfaces vary in the presence of non-vanishing rotational curvature—see e.g. [St3, p. 494].

A fundamental unsolved problem is characterizing the L^p boundedness properties of the maximal operators associated to hypersurfaces where the Gaussian curvature is allowed to vanish, i.e., determining the best possible value of p_0 such that (2) holds for all $p > p_0$ (such a $p_0 < \infty$ exists for finite type surfaces S by [SoSt]). In [IoSa1], we showed that a necessary condition for (2) to hold is that

$$d(x, \mathcal{H})^{-1} \in L^{1/p}_{\text{loc}}(S), \tag{3}$$

where $\mathcal{H}$ is any hyperplane not passing through the origin, and $d(x, \mathcal{H})$ denotes the distance from x on S to $\mathcal{H}$. For $p > 2$, we know of no counterexample to the converse.

Conjecture 1. For S smooth and $p > 2$, condition (3) is necessary and sufficient for the maximal inequality (2).

We remark that for $1 < p < 2$, condition (3) is not sufficient, even for convex surfaces. Indeed, if S in $\mathbb{R}^3$ is given as the graph of $z = x^{2m} + y^2$, the maximal operator is bounded on L^p if and only if $p > 4m/(2m+1)$, while condition (3) holds if and only if $p > 4m/(2m+2)$. See Proposition 5.1 in [NaSeWa] and the discussion there. We will not pursue the case $p < 2$ in this paper. We also mention here an old open conjecture, which in the case $\gamma = \frac{1}{2}$ is due to E. M. Stein.

Conjecture 2. Suppose S is smooth and $|\hat{\sigma}(\xi)| \leqslant C(1+|\xi|)^{-\gamma}$, $0 < \gamma \leqslant \frac{1}{2}$. Then the maximal inequality (2) holds for $p > 1/\gamma$.

1.1. *Convex Finite-Type Surface*

The first main result of this paper (see Theorem 6 below) is that Conjecture 1 holds if S is a smooth convex hypersurface in $\mathbb{R}^n$, $n \geqslant 3$, of finite-type in the sense of Bruna *et al.* [BrNaWa], i.e., every tangent line makes finite order contact with S. As we will see below, this also establishes Conjecture 2 for such surfaces.

Earlier results in this direction include [Io1, Io2], where the conjecture was proved for plane curves, and [IoSa1], where the conjecture was proved when the surface S is the graph of a homogeneous function Φ with finite-type level set $\Sigma = \{\Phi = 1\}$ (in the weaker sense that every tangent hyperplane makes finite order contact with Σ). In [NaSeWa], Nagel *et al.* proved that the maximal inequality (2) holds for smooth convex finite-type hypersurfaces satisfying conditions involving the integrability and scaling properties of the non-isotropic balls related to the distance from the hypersurface to its tangent hyperplanes, namely $d(x, \mathscr{H})$ in (3). More precisely, they define non-isotropic balls $\mathscr{B}(x, \delta)$ on S as the set of $y \in S$ such that $d(y, \mathscr{H}_x) < \delta$ where $\mathscr{H}_x$ is the tangent plane to S at x. Their sufficient condition for the maximal inequality (2) is then as follows: Suppose there exist $\epsilon > 0$ and $\gamma > \frac{1}{2}$ such that for all $\delta > 0$,

$$\left(\int_S |\mathscr{B}(x, \delta)|^{\epsilon} \, d\sigma(x)\right)^{1/\epsilon} \leqslant C\delta^{\gamma}.$$

Then the maximal inequality (2) holds for $p > 2(1 + 1/\epsilon)$. We also mention the closely related result in [BrNaWa] that decay of the Fourier transform of surface-carried measure is controlled by the volume of the nonisotropic balls,

$$|\widehat{\psi \, d\sigma}(\xi)| \leqslant C \sup_{x \in \operatorname{supp} \psi} |\mathscr{B}(x, |\xi|^{-1})|. \tag{4}$$

One of the two main tools we use in dealing with convex hypersurfaces of finite type is a result due to Schulz [Sc], which says that after perhaps rotating the coordinates, any smooth convex finite-type function Φ can be written in the form $\Phi(x) = Q(x) + R(x)$, where Q is a convex mixed homogeneous polynomial (i.e., there exist even integers $(a_1, ..., a_{n-1})$ such that

$Q(s^{1/a_1}x_1, ..., s^{1/a_{n-1}}x_{n-1}) = sQ(x), s > 0)$ that vanishes only at the origin, and $R(x)$ is a remainder term in the sense that it tends to zero under the non-isotropic dilation of Q. The $(n-1)$-tuple $(a_1, ..., a_{n-1})$ is referred to as the *multi-type* of Φ at 0. In practice, this means that the discussion of the maximal operator associated to convex hypersurfaces reduces to the local analysis of the principal term $Q(x)$, and in the range $p > 2$ the maximal operator turns out to be bounded on L^p for $p > 1/((1/a_1) + \cdots + (1/a_{n-1}))$. A calculation, using the homogeneity and positivity of Q, shows that this corresponds precisely to the integrability condition (3).

The second main tool used in proving maximal theorems when $p \geqslant 2$ is the square function technique of Sogge and Stein [SoSt] (see also Cowling and Mauceri [CoMa1]). Essentially, this says that if the Fourier transform of a compactly supported distribution τ has decay of order $-\frac{1}{2} - \epsilon$, i.e.,

$$|\hat{\tau}(\xi)| \leqslant C(1 + |\xi|)^{-1/2 - \epsilon}, \qquad \epsilon > 0, \tag{5}$$

then the maximal convolution operator $\mathscr{M}_\tau$ corresponding to the dilates of τ is bounded on L^2. A modified proof (see the proof of Theorem 15 below) shows that the epsilon in (5) can be replaced by a log factor, or more generally by

$$|\hat{\tau}(\xi)| \leqslant C(1 + |\xi|)^{-1/2} \gamma(|\xi|), \qquad \gamma \searrow \quad \text{and} \quad \sum_{n=0}^{\infty} \gamma(2^n) < \infty. \tag{6}$$

(One consequence of this is the existence of $\mathscr{M}$ whose interval of boundedness is closed. See the subsection on examples below, where the condition on γ is also shown to be sharp.)

Alternatively, we can use Sogge's theorem on one non-vanishing principal curvature [So2]. This says that if a surface S has everywhere at least one non-vanishing principal curvature, then the corresponding maximal operator $\mathscr{M}$ is bounded on $L^p(\mathbb{R}^n)$ for $p > 2, n \geqslant 2$ (thus establishing an instance of Conjecture 2). For our purposes, it is essential to know that the operator norm on $L^p(\mathbb{R}^n)$ is bounded by $C_{p,n} d(0, S)^{1/p}$, where $d(0, S)$ denotes the distance from S to the origin.

We now comment briefly on how these ideas tie together in proving the equivalence of (2) and (3) for smooth convex finite-type surfaces. First, a partition of unity argument reduces matters to the case where we can write the surface as the graph of a convex function Φ to which the Schulz decomposition applies: $\Phi = Q + R$. We then decompose the surface S in dyadic shells away from the origin, using the non-isotropic dilation associated with the multi-type $(a_1, ..., a_{n-1})$ of Φ. We then rescale and blow up the kth dyadic shell to the unit annulus. This has the effect of multiplying the integral by $2^{-k(n-1)}$, and more crucially, of translating this piece of the surface by $c_0 2^{km}$ in the vertical direction, where $1/m = (1/(n-1))(1/a_1 + \cdots + 1/a_{n-1})$. Now the Euler homogeneity relations, together with the finite type hypothesis, imply that there is at least one non-vanishing principal curvature everywhere on this blown up surface. At this point we can either apply directly Sogge's theorem with the operator bound $C_{p,n}(c_0 2^{km})^{1/p}$, and sum the estimates $C2^{(km/p)-k(n-1)}$ when $p > m/(n-1) = 1/((1/a_1) + \cdots + (1/a_{n-1}))$, or we can use the Schulz lemma once more, along with (4), to show that the Fourier transform of surface-carried measure decays like $|\xi|^{-(1/2)-\epsilon}$ for some $\epsilon > 0$, and apply the square function theorem of Sogge and Stein [SoSt].

1.2. *Maximal Operators with More General Dilation Groups*

We now recall a generalization of the maximal operator $\mathcal{M}$ given by Greenleaf in [Gr]. Given an n-tuple $(\beta_1, ..., \beta_n)$ of nonnegative real numbers, consider $\mathcal{M}'$ defined by

$$\mathcal{M}' f(x) = \sup_{t>0} |M'_t f(x)|, \tag{7}$$

where the convolution operator M'_t is given by

$$M'_t f(x) = f * \delta'_t(\psi \, d\sigma)(x),$$

but where δ'_t now denotes the nonisotropic dilation given by $\widehat{\delta'_t h}(\xi) = \hat{h}(t^{\beta_1}\xi_1, ..., t^{\beta_n}\xi_n)$. In the case $\beta_1 = \beta_2 = \cdots = \beta_n$, $\mathcal{M}'$ reduces to the familiar operator $\mathcal{M}$. While all of the maximal inequalities for $\mathcal{M}$ extend with little change in the proof to $\mathcal{M}'$, we will see below that $\mathcal{M}'$ often behaves much better due to a "rotational curvature" in the time parameter t.

We note that the proof of Theorem 2 in [IoSa1] shows that a necessary condition for

$$\|\mathcal{M}' f\|_{L^p(\mathbb{R}^n)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{8}$$

to hold is that (3) holds for any hyperplane $\mathcal{H}$ not passing through the origin, and its normal does not change direction under the action of the dilations δ'_t. For example, in the case $\beta_1 = \beta_2 = \cdots = \beta_{n-1} \neq \beta_n$, the hyperplanes under question would be the horizontal planes (perpendicular to $\vec{e}_n$) and the vertical planes (perpendicular to vectors with vanishing nth component).

The next result of this paper (see Theorem 7 below) is that in the case $\beta_1 = \beta_2 = \cdots = \beta_{n-1} \neq \beta_n$, condition (3) over all horizontal hyperplanes $\mathcal{H}$ is sufficient for the nonisotropic maximal function inequality (8) when the surface S is given as the graph of a mixed homogeneous function (without any positivity assumption) with finite-type level set Σ (in the weaker sense of Sogge and Stein, i.e., every tangent plane (of dimension $n-2$) has finite order contact with Σ).

More generally and significantly, our main theorem here (see Theorem 9 below) shows that this result persists for certain parametric surfaces of codimension 1 and 2 with homogeneity relative to a more general dilation group. See Example 2 below for an illustration where $\mathcal{M}'$ behaves much better than $\mathcal{M}$, and also the example at the end of the paper which exhibits a surface of codimension 1 to which the theorem applies, yet the decay of the Fourier transform of surface-carried measure is much worse than might be expected.

These results rely on the phenomenon that the uniform L^∞ decay in (6) can be weakened to an average L^2 decay,

$$\left\{\int_1^2 |\hat{\tau}(t^{\beta_1}\xi_1,..., t^{\beta_n}\xi_n)|^2\, dt\right\}^{1/2} \leqslant C(1+|\xi|)^{-1/2}\,\gamma(|\xi|), \tag{9}$$

in order to obtain L^2 boundedness of $\mathcal{M}'_\tau$ on L^2, where $\mathcal{M}'_\tau$ is the obvious analogue of $\mathcal{M}_\tau$ relative to the dilations δ'_t (see Theorem 8 below). Note that when the β_j are not all equal, and ξ does not point in a coordinate direction, then the projection of the vector $(t^{\beta 1}\xi_1,..., t^{\beta n}\xi_n)$ onto the unit sphere traces out a non-constant path, thereby avoiding the worst normals to the surface S most of the time.

An immediate consequence of this phenomenon is that for the surface S_α given as the graph of $z=1+x+e^{-|y|^{-\alpha}}$, the corresponding maximal operator $\mathcal{M}'_{S_\alpha}$ is bounded on L^2 for $\alpha<1$ (when $\beta_1 \neq \beta_3$, of course), while $\mathcal{M}$ is not bounded on L^p for any $\alpha>0$ and $p<\infty$. See Example 1 below. The point here is that this surface essentially rotates while being dilated by δ'_t, giving rise to a type of degenerate rotational curvature *in time t* that results in an average decay of $-\frac{1}{2}$, and accounts for the vastly improved mapping properties of $\mathcal{M}'$ over $\mathcal{M}$. Note that this is quite distinct from the notion of rotational curvature (in space) as in [St3]. Indeed, in our translation invariant case, non-vanishing rotational curvature (in space) is equivalent to non-vanishing Gaussian curvature of the surface S.

We remark that there is no local smoothing phenomenon for $p>2$ in the setting of average L^2 decay. Indeed, the Fourier transform of the surface-carried measure on S_α has an average L^2 decay of order $\frac{1}{2}$, in fact, (9) holds with $\gamma(|\xi|)=(\log(1/|\xi|))^{-1/\alpha}$, yet an argument involving the Besicovitch set shows that $\mathcal{M}'_{S_\alpha}$ fails to be bounded on L^p for $\alpha>p$. Perhaps more striking is that the plane $S_\infty (z=1+x)$ has average L^2 decay $\frac{1}{2}$, yet $\mathcal{M}'_{S_\alpha}$ fails to be bounded on $L^p(\mathbb{R}^3)$ for all $p<\infty$. This follows from the two-dimensional version: if S is a non-vertical line segment in the plane (that does not pass through the origin if it is horizontal), then $\mathcal{M}'_S$ fails to be bounded on $L^p(\mathbb{R}^2)$ for all $p<\infty$. Since the average L^2 decay of S is $\frac{1}{2}$, this contrasts with Bourgain's circular maximal theorem. See Example 4 below for these results.

We now comment briefly on the ideas used, beginning with a review of the methods in [IoSa1]. In [IoSa1], we considered measures $d\beta_\alpha$ given by weighting S with powers of $|\Phi|$;

$$d\beta_\alpha(y) = |\Phi(y')|^\alpha (\psi\, d\sigma)(y).$$

Given an additional finite type assumption on the level set $\Sigma = \{x: \Phi(x) = 1\}$, we proved that $|\widehat{\beta_\alpha}(\xi)| \leqslant C|\xi|^{-(1/2)-\epsilon}$ for some $\epsilon > 0$ if $\alpha > \frac{1}{2} - \rho$. The decay $|\xi|^{-1/2}$ was obtained merely from the curvature of Φ in the radial direction together with (3), which in this case reduces to $\Phi^{-1} \in L^\rho(\mathbb{S}^{n-2})$, and $\alpha > \frac{1}{2} - \rho$. To obtain the crucial stronger decay of $|\xi|^{-(1/2)-\epsilon}$, we used the finite-type hypothesis on the level set Σ. With this established, a theorem of Sogge and Stein ([SoSt]; see also Cowling and Mauceri [CoMa2]), involving square function techniques, was used to establish the L^2 boundedness of

$$\mathcal{M}_\alpha f(x) = \sup_{t>0} \left| \int_S f(x' - ty', x_n - ty_n)\, d\beta_\alpha(y) \right|, \tag{10}$$

when $\frac{1}{2} > \alpha > \frac{1}{2} - \rho$. A simple application of Hölder's inequality, together with the local integrability of $|\Phi(y')|^{-\rho}$, then showed that $\mathcal{M}_0 = \mathcal{M}$ is bounded on L^p for $p > 1/\rho$.

In the case of surfaces that are graphs of functions of mixed homogeneity, and more generally of parametric surfaces given by (mixed) homogeneous functions, the crucial decay estimate $|\widehat{\beta_\alpha}(\xi)| \leqslant C|\xi|^{-(1/2)-\epsilon}$ is problematic. Indeed, if S is a k-dimensional surface ($k = n-1$ or $n-2$) given parametrically by

$$S = \{(\Phi_1(x), ..., \Phi_{n-1}(x), \Phi(x) + c_0) \in \mathbb{R}^n : x \in \mathbb{R}^k\},$$

where the Φ_j (respectively, Φ) are homogeneous of degree m_j (respectively, m), then

$$\widehat{\beta_\alpha}(\xi) = \int_{\mathbb{S}^{k-1}} \left[\int_0^\infty e^{i\{\sum_{j=1}^{n-1} \xi_j \Phi_j(\omega) r^{m_j} + \xi_n \Phi(\omega) r^m\}} r^{n-1+m\alpha} \psi(r)\, dr \right] \Phi^\alpha(\omega)\, d\omega.$$

The integral in square brackets is essentially the Fourier transform of the curve $\vec{\sigma} = (r^{m_1}, ..., r^{m_{n-1}}, r^m)$ in $\mathbb{R}^n$. If the exponents $m_1, ..., m_{n-1}, m$ are distinct, then the curve is nondegenerate, and the best possible bound for such a curve is essentially $C|\xi_n|^{-1/n}$, far short of that required when $n \geqslant 3$.

In fact, the decay of the entire integral in $\widehat{\beta_\alpha}(\xi)$ can be essentially no better than this for certain parametric surfaces. See the example at the end of the paper for a hypersurface given parametrically as the graph of homogeneous functions, for which $\mathcal{M}'$ is bounded on L^2, yet the decay of the Fourier transform of surface-carried measure is worse than $-\frac{1}{2}$. See also Example 2 below for a simpler codimension 2 surface, for which $\mathcal{M}'$ is bounded on L^2, yet $\mathcal{M}$ fails to be bounded on L^m (m large). We nevertheless conjecture that the decay of $\widehat{\beta_\alpha}$ is at least $-\frac{1}{2}$ in the case of a surface of codimension 1 given as the graph of a mixed homogeneous function. Note that for homogeneous Φ, the phase is $(\sum_{j=1}^{n-1} \omega_j \xi_j)\, r + \xi_n \Phi(\omega)\, r^m$, which is the Fourier transform of a nondegenerate curve in $\mathbb{R}^2$, hence with decay $C\,|\xi_n|^{-1/2}$.

On the other hand, it is the case that an average L^2 decay of order $-\frac{1}{2}-\epsilon$ holds for $\widehat{\beta_\alpha}$, namely (see Theorem 23 and also (61) below)

$$\left\{\int_1^2 |\widehat{\beta_\alpha}(t^{\beta_1}\xi', t^{\beta_n}\xi_{n+1})|^2\, dt\right\}^{1/2} \leqslant C(1+|\xi|)^{-(1/2)-\epsilon}, \tag{11}$$

provided $\beta_n \neq \beta_1$. Essentially, this holds because the dilations in t prevent the dual variable from becoming stationary in bad directions. This allows us to invoke a universal Van der Corput estimate (Theorem 17 below) of the type

$$\left|\int e^{i\phi(s)}\psi(s)\, ds - \sum_{k=1}^{m} e^{i\phi(r_k)} \left(\frac{2\pi i}{\phi''(r_k)}\right)^{1/2} \psi(r_k)\right|$$
$$\leqslant C_\epsilon \left(\sum_{k=1}^{m} |\phi''(r_k)|^{-(1/2)-\epsilon} + \sum_{\ell=1}^{n} |\phi'(s_\ell)|^{-1} + \sum_{i=1}^{2} |\phi'(\tau_i)|^{-1}\right),$$

where $\{r_k\}_{k=1}^{m}$ and $\{s_\ell\}_{\ell=1}^{n}$ are the real zeroes of ϕ' and ϕ'' respectively in $[\tau_1, \tau_2]$ which contains the support of ψ, and ϕ satisfies the crucial condition

$$|\phi'''(s)|^{1/3} \leqslant C\,|\phi''(r_k)|^{(1/2)-\epsilon} \qquad \text{for} \quad |s-r_k| \leqslant |\phi''(r_k)|^{\epsilon-(1/2)},$$
$$k = 1, \ldots, m.$$

An additonal tool needed is an extension to finite-type functions of the reverse Hölder inequality of Ricci and Stein for small negative powers of polynomials (see Proposition 22 below). Once we have (11), it remains to prove the analogue of the theorem of Sogge and Stein in [SoSt] with the hypothesis of uniform decay replaced by an average L^2 decay. This is accomplished with the aid of a Littlewood–Paley decomposition in Theorem 15 below.

A different average L^2 decay estimate (of order $-(n-1)/2$) is introduced independently by Marletta and Ricci in [MaRi] to obtain certain 2-parameter maximal theorems for surfaces passing through the origin.

1.3. *Statement of Main Theorems*

We begin with a characterization of the L^p bounds for maximal averaging operators associated to convex finite-type hypersurfaces for $p>2$, $n \geqslant 3$.

THEOREM 3. *Let Φ be a smooth convex function, let S be the graph of Φ, suppose S is of finite type in the sense that every tangent line has finite order contact with S, and let $\mathcal{M}f$ be defined as in (1) above. If $p>2$, then estimate (2) holds if (3) holds for all hyperplanes $\mathcal{H}$.*

Conversely, if the estimate (2) holds, then (3) holds for all hyperplanes $\mathcal{H}$ that do not pass through the origin.

In order to illuminate the role played by homogeneity in this theorem, we recall the following result due to Schulz [Sc].

DEFINITION 4. We say that a smooth function $Q\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is mixed homogeneous of degree $(a_1, a_2, ..., a_{n-1})$, $a_j > 0$, if $Q(s^{1/a_1}x_1, ..., s^{1/a_{n-1}}x_{n-1}) = sQ(x)$, $s>0$.

LEMMA 5. *Let Φ be a smooth convex function such that $\Phi(0, ..., 0) = 0$ and $\nabla\Phi(0, ..., 0) = (0, ..., 0)$. Suppose that Φ has no tangents of infinite order at the origin. Then, after perhaps applying a rotation, we can write*

$$\Phi(x) = Q(x) + R(x),$$

where $Q(x)$ is a convex mixed homogeneous polynomial of degree $(a_1, ..., a_{n-1})$, $Q(x) \neq 0$ for $x \neq 0$, and $R(x)$ is a remainder term in the sense that

$$\lim_{s \to 0} \frac{R(s^{1/a_1}x_1, ..., s^{1/a_{n-1}}x_{n-1})}{s} = 0.$$

The $(n-1)$-tuple $(a_1, ..., a_{n-1})$ is referred to as the *multi-type* of Φ at the origin. More generally, given a convex surface S with no tangents of infinite order, we define the *multi-type* of S at the point $x \in S$ to be the $(n-1)$-tuple obtained from the lemma after translating and rotating coordinates so that $\Phi(x) = 0$ and $\nabla\Phi(x) = 0$.

The next theorem is a local result that establishes a bridge between condition (3) and the maximal inequality (2).

THEOREM 6. *Let Φ be a smooth convex function of finite type, let S be the graph of Φ, and let $\mathcal{M}f$ be defined as in (1) above. Let $(a_1, ..., a_{n-1})$ be* *the multi-type of S at a point $x \in S$ where the tangent plane $\mathscr{H}$ does not pass through the origin, and suppose that ψ has sufficiently small support near x. If $p > 2$, then estimate* (2) *holds if and only if*

$$p > \frac{1}{(1/a_1) + (1/a_2) + \cdots + (1/a_{n-1})},$$

which in turn holds if and only if (3) *holds for the hyperplane $\mathscr{H}$ tangent to S at x.*

Remark 1. In order to obtain Theorem 3 from Theorem 6, simply cover the support of the given cutoff function ψ with small neighbourhoods in which Theorem 6 applies, and use a partition of unity along with the "worst case" multi-types $(a_1, ..., a_{n-1})$ that arise.

Remark 2. We will also see below that the decay of the Fourier transform of surface-carried measures $\psi\, d\sigma$ for such surfaces satisfies

$$|\widehat{\psi\, d\sigma}(\xi)| \leqslant C(1 + |\xi|)^{-\gamma},$$

where $\gamma = \inf_{x \in S}((1/a_1) + (1/a_2) + \cdots + (1/a_{n-1}))$, and $(a_1, ..., a_{n-1})$ is the multi-type of S at x. Thus Theorem 3 verifies, for smooth convex surfaces of finite type, Conjecture 2 which states that decay of order $-\gamma$ implies boundedness of the maximal operator for $p > 1/\gamma$.

Here is our theorem for the non-isotropic maximal operator $\mathscr{M}'$ on mixed homogeneous surfaces. Our main theorem, an extension to parametric surfaces of codimension 1 or 2, will be given below (Theorem 9).

THEOREM 7. *Suppose $\Phi(x)$ is mixed homogeneous of degree $(a_1, ..., a_{n-1})$, with $a_j > 1$, namely*

$$\Phi(\lambda^{1/a_1} x_1, ..., \lambda^{1/a_{n-1}} x_{n-1}) = \lambda \Phi(x), \qquad \lambda > 0, \quad x \in \mathbb{R}^{n-1}.$$

Set $1/m = (1/(n-1))((1/a_1) + \cdots + (1/a_{n-1}))$. Suppose further that

$$\Phi(\omega)^{-1} \in L^{\rho}(\mathbb{S}^{n-2}), \qquad 0 < \rho \leqslant \min\left\{\frac{n-1}{m}, \frac{1}{2}\right\},$$

and $\sum = \{x : \Phi(x) = 1\}$ is of finite type with polynomial bounds, namely,

$$\sum_{2 \leqslant |\alpha| \leqslant l} \left| \frac{\partial^{|\alpha|}}{\partial y^{\alpha}} \Phi(x) \right| \geqslant c|x|^{-M}, \tag{12}$$

for some $M \geqslant 0$, $\ell \geqslant 2$ and where $\alpha = (\alpha_1, ..., \alpha_{n-2})$ is a multi-index, and $(y_1, ..., y_{n-2})$ is a coordinate system orthogonal to $\nabla\Phi(x)$ at x. Let $\mathscr{M}'$ be *defined as in* (7) *above with* S *given as the graph of* $\Phi + c_0$, *and with* $0 \leqslant \beta_1 = \beta_2 = \cdots = \beta_{n-1} < \beta_n$. *Then* $\mathscr{M}'$ *is bounded on* $L^p(\mathbb{R}^n)$, *i.e.,* (8) *holds, for* $p > 1/p$. *Moreover, the constant* C_p *in* (8) *is at most* $C'_p(1+|c_0|)^{1/p}$ (*in the case* $\beta_1 = 0$, *our proof yields an additional factor* $\log(1+|c_0|)$).

Conversely, if (8) *holds for a given* p *and* $c_0 \neq 0$, *then* $p > m/(n-1)$ *and* $\Phi(\omega)^{-1} \in L^{1/p}(\mathbb{S}^{n-2})$.

Note that the notion of finite type used here for Σ is considerably weaker than that used for S in Theorem 3 (the current notion involves finite order contact for tangent planes as opposed to tangent lines).

Our point of departure in dealing with $\mathscr{M}'$ is the following extension of the square function theorem of Sogge and Stein [SoSt] (see also [CoMa2]). Let δ'_t be as above and set $\hat{\delta}'_t \xi = (t^{\beta_1}\xi_1, ..., t^{\beta_n}\xi_n)$ so that $\widehat{\delta'_t h}(\xi) = \hat{h}(\hat{\delta}'_t \xi)$.

THEOREM 8. *Suppose* τ *is a distribution supported in a ball* B *of radius* C_1 *with* $|\hat{\tau}(\xi)| \leqslant C_1$ *and* $\max\{|x| : x \in \operatorname{supp} \tau\} \leqslant C_2$. *Suppose moreover that*

$$\left\{\int_1^2 |\hat{\tau}(\hat{\delta}'_t\xi)|^2 \, dt\right\}^{1/2} \leqslant C_1(1+|\xi|)^{-1/2}\gamma(|\xi|),$$

$$\left\{\int_1^2 |\nabla\hat{\tau}(\hat{\delta}'_t\xi)|^2 \, dt\right\}^{1/2} \leqslant C_2(1+|\xi|)^{-1/2}\gamma(|\xi|),$$

where γ *is bounded and nonincreasing on* $[0, \infty)$, *and* $\sum_{n=0}^{\infty} \gamma(2^n) < \infty$. *For* $t > 0$, *define* $\hat{\tau}_t(\xi) = \hat{\tau}(\hat{\delta}'_t \xi)$ *as above with* $\beta_i \geqslant 0$ *for* $i = 1, 2, ..., n$, *and set*

$$\mathscr{M}'_\tau f(x) = \sup_{t>0} |f * \tau_t(x)|.$$

Then

$$\|\mathscr{M}'_\tau f\|_{L^2} \leqslant C\sqrt{C_1 C_2}\, \|f\|_{L^2} \qquad \textit{for all} \quad f \in \mathscr{S}$$

(*in the case some* $\beta_i = 0$, *our proof yields an additional factor* $\log(C_2/C_1)$).

The summability condition on γ is sharp (see Example 3 below). When we apply this theorem later in the paper, the constant C_2 will capture the distance from the origin of the support of τ.

Theorem 7 on mixed homogeneous surfaces can be strengthened in two ways. First, the mixed homogeneities can be replaced by a more general group of dilations. Second, and more importantly, the surface S can have codimension 1 or 2 and be given parametrically with certain restrictions. In order to state the more general theorem concerning $\mathscr{M}'$, we first introduce some notation.

Let P be a real $k \times k$ matrix with trace k, and define the group of associated dilations $\{T_r\}_{r>0}$ by $T_r x = e^{P \ln r} x$ for $x \in \mathbb{R}^k$. See [dG] and [Ri]. We say that a function Φ is P-homogeneous of degree m if $\Phi(T_r x) = r^m \Phi(x)$. In the special case P is the diagonal matrix with entries $m/a_1, \ldots, m/a_k$, where $(1/m) = (1/k) \sum_{i=1}^{k} (1/a_i)$, so that $T_\lambda x = (\lambda^{m/a_1} x_1, \ldots, \lambda^{m/a_k} x_k)$ and $\Phi(T_\lambda x) = \lambda^m \Phi(x)$, we say that Φ is mixed homogeneous of degree $(a_1, \ldots, a_k)$. Let S be a smooth k-dimensional surface in $\mathbb{R}^n$ given parametrically by

$$S = \{(\Phi_1(x), \ldots, \Phi_{n-1}(x), \Phi(x) + c_0) \in \mathbb{R}^n : x \in \mathbb{R}^k\}. \tag{13}$$

Set

$$\mathscr{R}(x) = (\Phi_1(x), \ldots, \Phi_{n-1}(x))$$

for $x \in \mathbb{R}^k$. Let σ denote a smooth compactly supported measure on S, and define $\mathscr{M}'$ as above.

THEOREM 9. *Suppose a k-dimensional surface S, $k = n-1$ or $n-2$ (but $k \geqslant 2$), is given parametrically as in* (13) *where $\Phi(x)$ is P-homogeneous of degree m, and $\Phi_j(x)$ is P-homogeneous of degree $m_j \neq m$. Suppose further that*

(i) *There is $0 < p \leqslant \min\{k/m, \frac{1}{2}\}$ such that*

$$\Phi(\omega)^{-1} \in L^p(\mathbb{S}^{k-1}).$$

(ii) *The image of*

$$\sum = \{x : \Phi(x) = 1\}$$

under the map $\mathscr{R}$ is of finite type with polynomial bounds.

(iii) *For each $v \in \mathbb{S}^{n-2}$,*

$$\operatorname{rank} \left[\frac{\partial^2}{\partial x_i \partial x_j} \left\{ \sum_{k=1}^{n-1} v_k \Phi_k(x) \right\} \right]_{1 \leqslant i, j \leqslant k} \geqslant 2$$

whenever

$$\nabla_x \left\{ \sum_{k=1}^{n-1} v_k \Phi_k(x) \right\} = 0.$$

Let $\mathscr{M}'$ be defined as in (7) *above with $0 \leqslant \beta_1 = \beta_2 = \cdots = \beta_{n-1} < \beta_n$. Then $\mathscr{M}'$ is bounded on $L^p(\mathbb{R}^n)$, i.e.,* (8) *holds, for $p > 1/p$. Moreover, the constant C_p in* (8) *is at most $C'_p(1 + |c_0|)^{1/p}$ (in the case $\beta_1 = 0$, our proof yields an additional factor* $\log(1 + |c_0|)$).

Note that the sufficiency half of Theorem 7 is included in Theorem 9 as the special case when $k=n-1$ and $\mathscr{R}(x)=x$ is the identity map.

Remark 3. Condition (iii) is used only to obtain a uniform decay $|\hat{\sigma}(\xi', \xi_n)| \leqslant C|\xi|^{-1}$ for ξ in the cone $\mathscr{C}=\{(\xi', \xi_n) \in \mathbb{R}^n : |\xi_n| \leqslant c|\xi'|\}$ for c small. This can be replaced with the hypothesis that there is $\epsilon > O$ such that $|\hat{\sigma}(\xi', \xi_n)| \leqslant C|\xi|^{-(1/2)-\epsilon}$ for $\xi \in \mathscr{C}$. Note that the only meaningful choices of dimension k for the surface S in Theorem 9 are $k=n-1$ or $k=n-2$, since otherwise decay of order $-\frac{1}{2}-\epsilon$ cannot be guaranteed in the cone $\mathscr{C}$. Thus the philosophy of the hypotheses in Theorem 9 can be summarized as follows. We assume via (iii) that the decay of the Fourier transform of the surface-carried measure in the near horizontal directions is good enough for L^2 boundedness of $\mathscr{M}'$ (even $\mathscr{M}$). Hypotheses (i) and (ii) guarantee that, on average, the decay in the near vertical directions will be good enough for L^2 boundedness of $\mathscr{M}'_\alpha$ (cf. (10)) when $\alpha+\rho > \frac{1}{2}$. Indeed, (i) gives decay $-\frac{1}{2}$ while (ii) yields the extra $-\epsilon$. Thus (i) is an essentially sharp hypothesis, while (iii) is not.

We give a codimension 2 example to illustrate Theorem 9 in the following subsection (see Example 2), and a codimension 1 example at the end of the paper. Both examples exhibit better behaviour for $\mathscr{M}'$ than is shown by the usual maximal operator $\mathscr{M}$.

1.4. Examples. (1) Consider the smooth surface S_α given as the graph of $z=1+x+e^{-|y|^{-\alpha}}$. We claimed above that the maximal operator $\mathscr{M}'$ is bounded on L^2 for $\alpha<1$, while $\mathscr{M}$ is not bounded on any L^p space for $\alpha>0$ and $p<\infty$. Indeed, taking $\mathscr{H}$ to be the hyperplane $z=1+x$ in (3) shows that $\mathscr{M}$ is not bounded on any L^p space, $p<\infty$, since

$$\int_0^1 \int_0^1 (e^{-|y|^{-\alpha}})^{-1/p}\, dx\, dy = \int_0^1 e^{(1/p)|y|^{-\alpha}}\, dy = \infty, \qquad p<\infty, \quad \alpha>0.$$

On the other hand, one directly computes that the Fourier transform of a smooth compactly supported product measure σ on S_α is rapidly decreasing for $|\xi| \geqslant C|\lambda|$, so (9) automatically holds, and otherwise satisfies

$$\begin{aligned} |\hat{\sigma}(\xi, \lambda)| &= \left|\int e^{i\{x\xi_1 + y\xi_2 + \lambda(1-x+e^{-|y|^{-\alpha}})\}}\, d\sigma\right| \\ &= \left|\int e^{i\{x\xi_1+\lambda x\}}\, d\sigma_1(x)\right| \left|\int e^{i\{y\xi_2+\lambda e^{-|y|^{-\alpha}}\}}\, d\sigma_2(y)\right| \\ &\leqslant C(1+|\xi_1+\lambda|)^{-1} (\ln(2+|\lambda|))^{-1/\alpha}. \end{aligned}$$

In this case also the average L^2 decay condition (9) holds:

$$\int_1^2 |\hat{\sigma}(t^{\beta_1}\xi_1, t^{\beta_2}\xi_2, t^{\beta_3}\lambda)|^2 \, dt \leqslant (\ln(2+\lambda))^{-2/\alpha} \int_1^2 (1+|t^{\beta_1}\xi_1 + t^{\beta_3}\lambda|)^{-2} \, dt$$
$$\leqslant C(1+|\lambda|)^{-1} (\ln(2+|\lambda|))^{-2/\alpha},$$

provided $\beta_1 \neq \beta_3$. Hence $\mathscr{M}'$ is bounded on L^2 for $\alpha < 1$ by Theorem 8 above. In Example 4 below, we will turn to the negative L^p mapping results for the operators $\mathscr{M}'_{S_\alpha}$.

(2) To illustrate Theorem 9 in the codimension-2 case, consider the two-dimensional surface S in $\mathbb{R}^4$ given parametrically by

$$(\Phi_1(x), \Phi_2(x), \Phi_3(x), \Phi(x) + c_0) = (x_1, x_2, x_1^2 - x_2^2, x_1^{2m} + x_2^{2m} + c_0)$$

for $x = (x_1, x_2) \in \mathbb{R}^2$, and m large. Note that the component functions are homogeneous in the usual sense of degrees 1, 1, 2, and $2m$ respectively. We first verify condition (iii) of Theorem 9. For $\vec{v} = (v_1, v_2, v_3) \in \mathbb{S}^2$, the Hessian of $\vec{v} \cdot \mathscr{R}(x) = v_1 x_1 + v_2 x_2 + v_3(x_1^2 - x_2^2)$ is

$$D^2 \vec{v} \cdot \mathscr{R}(x) = \begin{bmatrix} 2v_3 & 0 \\ 0 & -2v_3 \end{bmatrix},$$

while the gradient is

$$\nabla \vec{v} \cdot \mathscr{R}(x) = \begin{pmatrix} v_1 + 2v_3 x_1 \\ v_2 - 2v_3 x_2 \end{pmatrix}.$$

Thus

$$\begin{aligned} 1 = \|\vec{v}\| &\leqslant |v_1| + |v_2| + |v_3| \\ &\leqslant C(|\nabla \vec{v} \cdot \mathscr{R}(x)| + |v_3|) \\ &\leqslant C(|\nabla \vec{v} \cdot \mathscr{R}(x)| + \sqrt{\det D^2 \vec{v} \cdot \mathscr{R}(x)}), \end{aligned}$$

which easily yields (iii). We remark that this argument actually yields the stronger inequality

$$1 \leqslant C(|\nabla\{\vec{v} \cdot \mathscr{R}(x) + \epsilon\Phi(x)\}| + \sqrt{\det D^2\{\vec{v} \cdot \mathscr{R}(x) + \epsilon\Phi(x)\}} + \epsilon),$$

which shows that the oscillatory integral $\int e^{i\lambda\{\vec{v} \cdot \mathscr{R}(x) + \epsilon\Phi(x)\}} \psi(x) \, dx$ decays like λ^{-1} for ϵ sufficiently small. (This illustrates the way in which (iii) enters into the proof of Theorem 9.) It remains only to verify (i) and (ii).

Now Φ is positive on $\mathbb{S}^1$ so that (i) is trivial. The image of $\Sigma = \{\Phi(x) = 1\}$ under $\mathcal{R}$ is the parametric curve

$$(\cos^{1/m}\theta, \sin^{1/m}\theta, \cos^{2/m}\theta - \sin^{2/m}\theta),$$

which is of finite type. Thus Theorem 9 applies to yield that the maximal operator $\mathcal{M}'$ for this surface S is bounded on $L^2(\mathbb{R}^4)$, while for $c_0 \neq 0$, the usual maximal operator $\mathcal{M}$ is not bounded on $L^m(\mathbb{R}^4)$ by (3) with $\mathcal{H}$ the hyperplane $x_4 = c_0$. Note that in this case, the Fourier transform of surface-carried measure decays no better than $|\xi|^{-1/m}$ in the direction e_4.

(3) We now give an example to illustrate the sharpness of the summability condition on γ in Theorem 8, which will also yield an example of a maximal operator whose interval of boundedness is closed. For the smooth surface S given as the graph of $z = 1 + x^2 + e^{-|y|^{-\alpha}}$, $0 < \alpha < 1$, we claim that $\mathcal{M}$ is bounded on L^p if and only if $p \geqslant 2$, and moreover that $\mathcal{M}$ is not bounded on L^2 if $\alpha \geqslant 1$. Indeed, one computes that the Fourier transform of a smooth compactly supported product measure σ on S is rapidly decreasing for $|\xi| \geqslant C|\lambda|$, and otherwise satisfies

$$\begin{aligned}
|\hat{\sigma}(\xi, \lambda)| &= \left| \int e^{i\{x\xi_1 + y\xi_2 + \lambda(1 + x^2 + e^{-|y|^{-\alpha}})\}}\, d\sigma \right| \\
&= \left| \int e^{i\{x\xi_1 + \lambda x^2\}}\, d\sigma_1(x) \right| \left| \int e^{i\{y\xi_2 + \lambda e^{-|y|^{-\alpha}}\}}\, d\sigma_2(y) \right| \\
&\leqslant C(1+\lambda)^{-1/2} (\ln(2+\lambda))^{-1/\alpha}.
\end{aligned}$$

Thus (6) holds if $\alpha < 1$, and so $\mathcal{M}$ is bounded on L^2. On the other hand, the necessary condition (3), with $\mathcal{H}$ the horizontal hyperplane $z = 1$, shows that $\mathcal{M}$ is not bounded on L^p for $p < 2$ and any $\alpha > 0$, since

$$\begin{aligned}
&\int_0^1 \int_0^1 (x^2 + e^{-|y|^{-\alpha}})^{-1/p}\, dx\, dy \\
&\quad = \int_0^1 \left\{ \int_0^{e^{(1/2)|y|^{-\alpha}}} (t^2+1)^{-1/p}\, dt \right\} e^{(1/p - 1/2)|y|^{-\alpha}}\, dy \\
&\quad \approx \int_0^1 e^{(1/p - 1/2)|y|^{-\alpha}}\, dy = \infty, \qquad \text{for} \quad p < 2, \alpha > 0,
\end{aligned}$$

and that $\mathcal{M}$ is not bounded on L^2 for $\alpha \geqslant 1$, since for $p = 2$ the above integral is essentially

$$\int_0^1 \left\{ \int_0^{e^{(1/2)|y|^{-\alpha}}} (t^2+1)^{-1/2}\, dt \right\} dy \approx \int_0^1 |y|^{-\alpha}\, dy = \infty, \qquad \text{for} \quad \alpha \geqslant 1.$$

(4) We now return to the surfaces S_α in Example 1 and use the Besicovitch set to demonstrate that $\mathscr{M}'_{S_\alpha}$ is not bounded on $L^p(\mathbb{R}^3)$ for $\alpha > p$ (here we consider the case $\beta_1 = \beta_2 = 0$, $\beta_3 = 1$). In order to first show this for the plane $S_\infty(z = 1 + x)$, we consider the maximal operator $\mathscr{M}'_S$ associated to a line segment S in the plane,

$$\mathscr{M}'_S f(x_1, x_2) = \sup_{1<t<2} \int_0^3 f(x_1 + s, x_2 + t(c + s))\, ds, \qquad c \in \mathbb{R}.$$

Note that for $c = 0$, $\mathscr{M}'_S$ is the familiar Kakeya maximal function which is well known to be unbounded on all $L^p(\mathbb{R}^2)$, $p < \infty$, by use of the Besicovitch set (see e.g. 3.3 on p. 455 of [St3]). When $c \neq 0$, $\mathscr{M}'_S$ is a translate of the Kakeya maximal function, but for future purposes we modify the argument as follows. Following the presentation in Chapter X of [St3], there is, for each $\epsilon > 0$, a set E_ϵ that is a union of 2^N rectangles $R_1, \ldots, R_{2^N}$ each having side lengths 1 and 2^{-N}. We also need their "reaches" $\tilde{R}_j$ obtained by translating R_j two units in the negative direction, along the longer side of R_j. (We choose to work with reaches below E_ϵ, rather than above as in [St3], for convenience in using $\mathscr{M}'_S$.) Fixing $c > 0$ for the moment, we consider the modified reaches $\hat{R}_j$ obtained by translating the reach $\tilde{R}_j$ directly down a distance $t_j c$ where t_j is the slope of the rectangle $\tilde{R}_j$. We have the following three properties:

(i) $|E_\epsilon| \leqslant \epsilon$.

(ii) *The $\hat{R}_j$ are pairwise disjoint, so that* $|\bigcup_{j=1}^{2^N} \hat{R}_j| = 1$.

(iii) $\mathscr{M}'_S \chi_{E_\epsilon}(x) \geqslant 1$ *for* $x \in \bigcup_{j=1}^{2^N} \hat{R}_j$.

Property (i) is Theorem 1(i) on p. 435 of [St3], and (ii) follows immediately from the corresponding assertion for the $\tilde{R}_j$, which is Theorem 1(ii) on p. 435 of [St3]. Property (iii) follows from the fact that $x \in \hat{R}_j$ implies $(x_1, x_2 + t_j c) \in \tilde{R}_j$ which in turn implies

$$(x_1 + s, x_2 + t_j(c + s)) \in R_j \subset E_\epsilon$$

for a set of $s \in [0, 3]$ having measure at least 1. If now $\mathscr{M}'_S$ is weak type bounded for some $p < \infty$, then

$$1 = \left| \bigcup_{j=1}^{2^N} \hat{R}_j \right| \leqslant |\{\mathscr{M}'_S \chi_{E_\epsilon}(x) \geqslant 1\}| \leqslant C\, |E_\epsilon| = C\epsilon$$

for all $\epsilon > 0$, a contradiction.

Turning now to the surfaces S_α with $\alpha < \infty$, we extend the sets constructed above, with $c = 1$, to three-dimensional sets by considering them in the x_1, x_3 plane, and crossing them with the interval $[-\frac{1}{2}, \frac{1}{2}]$ in the x_2 direction. More precisely, let

$$E_\epsilon^* = \{(x_1, x_2, x_3): (x_1, x_3) \in E_\epsilon, -\tfrac{1}{2} \leqslant x_2 \leqslant \tfrac{1}{2}\},$$
$$\hat{R}_j^* = \{(x_1, x_2, x_3): (x_1, x_3) \in \hat{R}_j, -\tfrac{1}{2} \leqslant x_2 \leqslant \tfrac{1}{2}\},$$

etc. We now have the following four properties:

(i) $|E_\epsilon^*| \leqslant \epsilon$.

(ii) *The* $\hat{R}_j^*$ *are pairwise disjoint, so that* $|\bigcup_{j=1}^{2^N} \hat{R}_j^*| = 1$.

(iii) $N \approx (1/\epsilon) \log(1/\epsilon)$.

(iv) $\mathscr{M}'_{S_\alpha} \chi_{E_\epsilon^*}(x) \geqslant c_1(\epsilon/\log(1/\epsilon))^{1/\alpha}$ *for* $x \in \bigcup_{j=1}^{2^N} \hat{R}_j^*$.

Properties (i) and (ii) are obvious from the above. Property (iii) follows from the formula displayed on p. 440 of [St3], and (iv) uses (iii) in the following way. If $x = (x_1, x_2, x_3) \in \hat{R}_j^*$, then

$$(x_1 + s, x_2 + u, x_3 + t_j(1 + s + e^{-|u|^{-\alpha}})) \in R_j^* \subset E_\epsilon^*$$

for a set of $s \in [0, 3]$ having measure 1, and essentially the set of u satisfying $e^{-|u|^{-\alpha}} \leqslant 2^{-N}$, since the R_j^* are $1 \times 1 \times 2^{-N}$ rectangles. Thus for $x \in \hat{R}_j^*$,

$$\begin{aligned}
\mathscr{M}'_{S_\alpha} \chi_{E_\epsilon^*}(x) &\geqslant |\{(s, u): (x_1 + s, x_2 + u, x_3 + t_j(1 + s + e^{-|u|^{-\alpha}})) \in R_j^*\}| \\
&\geqslant |\{u: e^{-|u|^{-\alpha}} \leqslant 2^{-N}\}| \\
&= |\{u: |u| \leqslant N^{-1/\alpha}\}| \\
&\approx c_1 \left(\frac{\epsilon}{\log(1/\epsilon)}\right)^{1/\alpha}.
\end{aligned}$$

If now $\mathscr{M}'_{S_\alpha}$ is weak type p, we have

$$\begin{aligned}
1 = \left|\bigcup_{j=1}^{2^N} \hat{R}_j^*\right| &\leqslant \left|\left\{\mathscr{M}'_{S_\alpha} \chi_{E_\epsilon^*} \geqslant c_1 \left(\frac{\epsilon}{\log(1/\epsilon)}\right)^{1/\alpha}\right\}\right| \\
&\leqslant C\left[c_1 \left(\frac{\epsilon}{\log(1/\epsilon)}\right)^{1/\alpha}\right]^{-p} |E_\epsilon| \\
&= C\epsilon^{1-(p/\alpha)}(\log(1/\epsilon))^{p/\alpha}
\end{aligned}$$

for all $\epsilon > 0$, which implies $\alpha \leqslant p$.

In the next three sections, we prove Theorem 6 concerning $\mathscr{M}$ for convex surfaces, and in the following four sections we prove Theorem 7 concerning $\mathscr{M}'$ for surfaces of mixed homogeneity, and give the modifications needed for the proof of Theorem 9 on parametric surfaces of codimension 1 and 2. As all of our proofs in the next three sections involve representing a surface as a graph, it will be convenient to work in $\mathbb{R}^{n+1}$ rather than in $\mathbb{R}^n$ for these sections.

2. CONVEX SURFACES

Our main theorem on convex surfaces, Theorem 3, follows from Theorem 6 as indicated in Remark 1. Theorem 6 in turn will follow from the Schulz lemma and the next result, which provides a local link between conditions (2) and (3).

THEOREM 10. *Let Φ be a smooth function with the following properties. Suppose that after perhaps applying a rotation, $\Phi(x) = Q(x) + R(x)$, where Q is mixed homogeneous of degree $(a_1,..., a_n)$ with $(1/a_1) + \cdots + (1/a_n) \leqslant 1$, $Q(x) \neq 0$ for $x \neq 0$, and*

$$\lim_{s \to 0} \frac{R(s^{1/a_1}x_1,..., s^{1/a_n}x_n)}{s} = 0. \tag{14}$$

Let $M_t f(x, x_{n+1})$ and $\mathcal{M}f(x, x_{n+1})$ be as in (1) above where S is the surface given as the graph of $\Phi(x) + c_0$, with ψ of sufficiently small support. Suppose that $(1/a_1) + (1/a_2) + \cdots + (1/a_n) \leqslant \frac{1}{2}$. Then

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^{n+1})} \leqslant C_p \|f\|_{L^p(\mathbb{R}^{n+1})}, \qquad \text{for} \quad p > \frac{1}{(1/a_1) + \cdots + (1/a_n)}, \tag{15}$$

where $f \in S(\mathbb{R}^{n+1})$, the class of rapidly decreasing functions.

Remark 4. The proof will show that the hypothesis $Q(x) \neq 0$ for $x \neq 0$ can be replaced with the weaker hypothesis $\nabla Q(x) \neq 0$ for $x \neq 0$. Of course, in our application, Lemma 5 supplies the former.

By Theorem 2 of [IoSa1] this result is sharp since it is easily computed that

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} = \inf \left\{ p : \int |Q(x)|^{-p} \psi(x)\, dx < \infty \right\},$$

which shows that (3) holds if and only if $p > 1/((1/a_1) + \cdots + (1/a_n))$, since if ψ has sufficiently small support, then

$$\int |\Phi(x)|^{-\gamma} \psi(x)\, dx \approx C \int |Q(x)|^{-\gamma} \psi(x)\, dx, \qquad \gamma > 0.$$

Indeed, for $x \in \mathbb{S}^{n-1}$, and $T_s x = (s^{1/a_1}x_1, ..., s^{1/a_n}x_n)$, we have

$$\Phi(T_s x) = Q(T_s x) + R(T_s x) = s\left[Q(x) + \frac{R(T_s x)}{s}\right] \approx sQ(x),$$

for small s by (14) and the fact that $\min_{x \in \mathbb{S}^{n-1}} Q(x) > 0$.

Our proof of Theorem 10 will use the following result due to Sogge [So2].

THEOREM 11. *Let S be a smooth hypersurface in $\mathbb{R}^n, n \geq 2$, having the property that, at each $x \in S$, at least one principal curvature is non-zero. With $\mathcal{M}$ as in (1) above, then $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$ for $p > 2$.*

We will actually need to know the more precise conclusion,

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \leq C_p d(0, S)^{1/p} \|f\|_{L^p(\mathbb{R}^n)}, \tag{16}$$

where $d(0, S) = \max\{|x| : x \in S\}$. This is easily seen by tracing through the initial steps in Sogge's proof as follows. Let Φ be a defining function for the surface S, i.e., $S = \{x : \Phi(x) = 0\}$ and $\nabla\Phi \neq 0$ on S. Then the averaging operator $M_t f(x)$, as in (1) above, can be written, with a small abuse of notation, as

$$\begin{aligned} Af(x, t) &= \int_{\mathbb{R}^n} t^n \delta_0(\Phi(t(x + y)))\, \psi(t(x + y))\, f(y)\, dy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} t^n e^{i\tau\Phi(t(x+y))} \psi(t(x + y))\, f(y)\, d\tau\, dy \\ &= \sum_{j=-\infty}^{\infty} \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} t^n e^{i\tau\Phi(t(x+y))} \beta(2^{-j}\tau)\, \psi(t(x + y))\, f(y)\, d\tau\, dy \\ &= \sum_{j=-\infty}^{\infty} A_j f(x, t), \end{aligned}$$

where $\sum_{j=-\infty}^{\infty} \beta(2^{-j}\tau) = 1$ for $\tau \neq 0$. (Here δ_0 denotes the Dirac delta function.) If we apply d/dt to $A_j f(x, t)$, then we bring down the factor $i\tau(x + y) \cdot \nabla\Phi(t(x + y))$ into the integrand. Since $\tau \approx 2^j$ and $|x + y| \leq Cd(0, S)$ (in the range $t \in [1, 2]$), we see that $(d/dt)\, A_j f(x, t)$ looks like $2^j d(0, S)$ times $A_j f(x, t)$. If one uses this observation in inequality (3) of [So2], one obtains that

$$\left\|\sup_{t>0}\left|\sum_{j=0}^{\infty} A_j f(x, t)\right|\right\|_{L^p(\mathbb{R}^n)} \leq C_p d(0, S)^{1/p} \|f\|_{L^p(\mathbb{R}^n)}.$$

Now $\sup_{t>0} |\sum_{j=-\infty}^{0} A_j f(x, t)|$ is controlled by a skewed maximal function of the type in (31) below, and thus is weak type 1,1 with constant $Cd(0, S)$, hence bounded on $L^p(\mathbb{R}^n)$ with norm $C_{p,n} d(0, S)^{1/p}$ by interpolation. Altogether this establishes (16).

Another proof of Theorems 3 and 6 uses the following result due to Sogge and Stein (see [SoSt] and also [CoMal]). A more general result is proved in Theorem 15 below.

THEOREM 12. *Let ϕ be a compactly supported distribution on $\mathbb{R}^n$. Suppose that for some $\epsilon > 0$ we have*

$$|\hat{\phi}(r\omega)|, |\nabla\hat{\phi}(r\omega)| \leqslant C(1+r)^{-(1/2)-\epsilon}, \qquad \omega \in \mathbb{S}^{n-1}, \quad r > 0.$$

Let $\widehat{\delta_t f}(\xi) = \hat{f}(t\xi)$ and $\phi^\# f = \sup_{t>0} |\delta_t \phi * f|$. Then

$$\|\phi^\# f\|_2 \leqslant C\, \|f\|_2, \qquad f \in S(\mathbb{R}^n).$$

Using Theorem 12, Theorem 6 follows from Hölder's inequality (see [IoSa1] and the proof below), and the following oscillatory estimate.

THEOREM 13. *Let Φ be a smooth convex function whose graph is of finite type in the sense that it has finite-order contact with every tangent line. Suppose further that Φ has the decomposition given in Theorem 10. Let*

$$F_\alpha(\xi, \lambda) = \int e^{i(x\cdot\xi + \lambda\Phi(x))}\, |\Phi(x)|^\alpha\, \psi(x)\, dx,$$

where ψ is a smooth cutoff function supported in a neighborhood of the origin, with sufficiently small support. Then there exists an $\epsilon > 0$ such that

$$|F_\alpha(\xi, \lambda)|, |\nabla F_\alpha(\xi, \lambda)| \leqslant C(1+|\xi|+|\lambda|)^{-(1/2)-\epsilon},$$

if $\alpha > (1/2) - (1/a_1) - \cdots - (1/a_n)$.

The next theorem summarizes what decay can be obtained with our methods without explicitly assuming convexity of the surface.

THEOREM 14. *Let Φ be a smooth function with the following properties. Suppose that after perhaps applying a rotation, $\Phi(x) = Q(x) + R(x)$, where Q is mixed homogeneous of degree $(a_1, \ldots, a_n)$ with $(1/a_1) + \cdots + (1/a_n) \leqslant 1$, $Q(x) \neq 0$ for $x \neq 0$, and R satisfies (14), i.e.,*

$$\lim_{s\to 0} \frac{R(s^{1/a_1}x_1, \ldots, s^{1/a_n}x_n)}{s} = 0.$$

Let μ denote the number of distinct a_j's. Suppose that

$$\int |\Phi(x)|^{-\rho}\, \psi(x)\, dx < \infty,$$

where $0 < \rho < 1/(\mu+1)$ *and* $\rho \leqslant (1/a_1) + \cdots + (1/a_n)$. *Let*

$$F(\xi, \lambda) = \int e^{i(x \cdot \xi + \lambda \Phi(x))}\, \psi(x)\, dx.$$

Then if ψ has sufficiently small support,

$$F(\xi, \lambda) \leqslant C(1 + |\xi| + |\lambda|)^{-\rho}.$$

Remark 5. The hypothesis $Q(x) \neq 0$ for $x \neq 0$ in Theorem 14 can be dropped if we assume $R(x) \equiv 0$.

Remark 6. The motivation for the integrability assumption of Theorem 14 is the following. Consider $\int_B e^{i\lambda Q(x)} dx$, where Q is homogeneous of degree $m \geqslant 2$, where without loss of generality $Q \geqslant 0$, and B denotes the unit ball. In polar coordinates we get

$$\int_0^1 \int_{\mathbb{S}^{n-1}} e^{i\lambda r^m Q(\omega)} r^{n-1}\, d\omega\, dr = \int_0^1 \int_{\{\lambda Q(\omega) \leqslant 1\}} + \int_0^1 \int_{\{\lambda Q(\omega) > 1\}} = \mathrm{I} + \mathrm{II}.$$

Now

$$|\mathrm{I}| \leqslant C \left| \left\{ \omega \colon Q(\omega) \leqslant \frac{1}{\lambda} \right\} \right| \leqslant \lambda^{-\rho} \int Q^{-\rho}(\omega)\, d\omega \leqslant C\lambda^{-\rho},$$

if $\int_{\mathbb{S}^{n-1}} Q(\omega)^{-\rho}\, d\omega < \infty$. After a change of variables sending $r \to r(\lambda Q(\omega))^{-1/m}$ we get

$$|\mathrm{II}| = \left| \lambda^{-n/m} \int_0^{(\lambda Q(\omega))^{1/m}} e^{ir^m} r^{n-1}\, dr \int_{\{\lambda Q(\omega) > 1\}} Q^{-n/m}(\omega)\, d\omega \right|$$

$$\leqslant C\lambda^{-n/m} \int_{\{\lambda Q(\omega) > 1\}} Q^{-n/m}(\omega)\, d\omega,$$

since an easy integration by parts argument shows that

$$\int_0^N e^{ir^m} r^{n-1}\, dr < \infty,$$

independent of N. But this last expression can be rewritten as

$$\lambda^{-\rho} \int_{\{\lambda Q(\omega) > 1\}} (\lambda Q(\omega))^{\rho - (n/m)} Q(\omega)^{-\rho} \, d\omega$$

$$\leqslant \lambda^{-\rho} \int_{\mathbb{S}^{n-1}} Q(\omega)^{-\rho} \, d\omega \leqslant C\lambda^{-\rho},$$

if $\int_{\mathbb{S}^{n-1}} Q(\omega)^{-\rho} \, d\omega < \infty$. Thus we have shown that $|\int_B e^{i\lambda Q(x)} \, dx| \leqslant C\lambda^{-\rho}$, provided that $\int_{\mathbb{S}^{n-1}} Q(\omega)^{-\rho} \, d\omega < \infty$. The same calculation works if Q is mixed homogeneous.

The next two sections of this paper consist of various refinements of the idea in the remark above, and its application to the more complicated phase function $\langle x, \xi \rangle + \lambda(Q(x) + R(x))$, where R is the remainder described in the statement of Theorem 14. Theorems 10 and 13 are proved in the next section, and Theorem 14 is proved in the following section.

3. MAXIMAL THEOREMS ON CONVEX SURFACES OF FINITE TYPE

The purpose of this section is to prove Theorem 10, and we begin with a proof using Sogge's Theorem 11. We then prove Theorem 13 and use it to give another proof of Theorem 10 via the square function techniques in Theorem 12.

Proof of Theorem 10. We begin by decomposing the surface S in dyadic shells according to the nonisotropic dilations associated with the multi-type $(a_1, ..., a_n)$. For this we write

$$\psi(x) = \sum_{k=0}^{\infty} \psi_k(x) = \sum_{k=0}^{\infty} \psi_0(2^{k/a_1}x_1, ..., 2^{k/a_n}x_n)$$

where $\psi_k(x) = \psi_0(2^{k/a_1}x_1, ..., 2^{k/a_n}x_n)$ and ψ_0 is a smooth cutoff function supported in the annulus $\{x\colon 1 \leqslant |x| \leqslant 2\}$. We define

$$\mathscr{M}_k f(x, x_{n+1}) = \sup_{t>0} |M_t^k f(x, x_{n+1})| = \sup_{t>0} |f * \delta_t(\psi_k \, d\sigma)(x, x_{n+1})|.$$

Let $\tau_k g(x) = g(2^{mk/a_1}x_1, ..., 2^{mk/a_n}x_n, 2^{mk}x_{n+1})$ and set $2^{mk/a} \circ y = (2^{mk/a_1}y_1, ..., 2^{mk/a_n}y_n)$. Then we have

$$\tau_{-k} M_t^k \tau_k f(x, x_{n+1})$$

$$= \int \tau_k f(2^{-mk/a} \circ x - ty, 2^{-mk} x_{n+1} - t(\Phi(y) + c_0)) \psi_k(y)\, dy$$

$$= \int f(x - t2^{mk/a} \circ y, x_{n+1} - t(2^{mk}\Phi(y) + 2^{mk} c_0)) \psi_0(2^{mk/a} \circ y)\, dy$$

$$= 2^{-kn} \int f(x - ty, x_{n+1} - t(2^{mk}\Phi(2^{-mk/a} \circ y) + 2^{mk} c_0)) \psi_0(y)\, dy,$$

and using

$$2^{mk}\Phi(2^{-mk/a} \circ y) = 2^{mk} Q(2^{-mk/a} \circ y) + 2^{mk} R(2^{-mk/a} \circ y)$$
$$= Q(y) + 2^{mk} R(2^{-mk/a} \circ y),$$

we can write $\tau_{-k} M_t^k \tau_k f(x, x_{n+1})$ as

$$2^{-kn} \int f(x - ty, x_{n+1} - t(Q(y) + 2^{mk} R(2^{-m/a_k} \circ y) + 2^{mk} c_0)) \psi_0(y)\, dy.$$

At this point we use the fact that our assumptions imply that the Hessian matrix of Q has rank $\geqslant 1$ on the support of ψ_0. To see this, note that by the Euler homogeneity relations, i.e., differentiating $Q(s^{1/a_1} x_1, ..., s^{1/a_n} x_n) = sQ(x)$ with respect to s, we obtain

$$Q(x) = \frac{x_1}{a_1} \frac{\partial Q}{\partial x_1} + \frac{x_2}{a_2} \frac{\partial Q}{\partial x_2} + \cdots + \frac{x_n}{a_n} \frac{\partial Q}{\partial x_n}.$$

Differentiating $Q(s^{1/a_1} x_1, ..., s^{1/a_n} x_n) = sQ(x)$ with respect to x_j we obtain

$$s^{1/a_j} \frac{\partial Q}{\partial x_j} (s^{1/a_1} x_1, ..., s^{1/a_n} x_n) = s \frac{\partial Q}{\partial x_j}.$$

Let Q_{ij} denote the mixed partial derivative with respect to x_i and x_j. Setting $u = s^{1-(1/a_j)}$ in the previous identity, differentiating with respect to u, and then setting $u = 1$, we get

$$\frac{a_j x_1}{a_1(a_j - 1)} Q_{j1}(x) + \frac{a_j x_2}{a_2(a_j - 1)} Q_{j2}(x) + \cdots + \frac{a_j x_n}{a_n(a_j - 1)} Q_{jn}(x) = \frac{\partial Q}{\partial x_j}(x).$$

Consequently, if the rank of the Hessian matrix of Q is 0 at any point away from the origin, the gradient must vanish at the same point, and so then also must Q, contradicting our assumption. Of course, these equations imply

$$|Q(x)| \leqslant C\,|x|\,|\nabla Q(x)| \leqslant C\,|x|^2\,|\nabla^2 Q(x)|,$$

and hence the quantitative estimate

$$\max_{i,j} |Q_{i,j}(x)| \geqslant c > 0, \qquad x \in \operatorname{supp} \psi_0. \tag{17}$$

Now by (14), $2^{mk}R(2^{-(m/a)k} \circ x)$ tends to 0 as $k \to \infty$, and we claim that this persists for second order derivatives also:

$$\lim_{k \to \infty} \frac{\partial^2}{\partial x_i\, \partial x_j} [2^{mk} R(2^{-(m/a)k} \circ x) = 0. \tag{18}$$

To see this, choose $N \geqslant 1 + \max\{a_1, ..., a_n\}$, and use Taylor's formula to write

$$\Phi(x) = P_N(x) + R_N(x) = Q(x) + P(x) + R_N(x),$$

where $P_N + R_N$ is the usual decomposition into a Taylor polynomial of degree N and a remainder term, and where P consists of the finitely many monomials in P_N that are not in Q. Thus $R = P + R_N$, and so if $G_\alpha(x) = x^\alpha$ is a monomial in P, then

$$0 = \lim_{s \to 0} s^{-1} G_\alpha(s^{1/a_1} x_1, ..., s^{1/a_n} x_n) = \lim_{s \to 0} s^{(\sum_{k=1}^n (\alpha_k/a_k)) - 1} x^\alpha,$$

which yields $\sum_{k=1}^n (\alpha_k/a_k) > 1$. It now follows immediately that

$$\begin{aligned}
&\lim_{s \to 0} \frac{\partial^2}{\partial x_i\, \partial x_j} \{s^{-1} G_\alpha(s^{1/a_1} x_1, ..., s^{1/a_n} x_n)\} \\
&\quad = \lim_{s \to 0} \{s^{-1 + (1/a_i) + (1/a_j)} \alpha_i \alpha_j [(s^{1/a_1} x_1)^{\alpha_1} \\
&\qquad \cdots (s^{1/a_i} x_i)^{\alpha_i - 1} \cdots (s^{1/a_j} x_j)^{\alpha_j - 1} \cdots (s^{1/a_n} x_n)^{\alpha_n}]\} \\
&\quad = \lim_{s \to 0} s^{(\sum_{k=1}^n (\alpha_k/a_k)) - 1} \alpha_i \alpha_j [x_1^{\alpha_1} \cdots x_i^{\alpha_i - 1} \cdots x_j^{\alpha_j - 1} \cdots x_n^{\alpha_n}] = 0,
\end{aligned}$$

since $\sum_{k=1}^n (\alpha_k/a_k) > 1$. Thus we have

$$\lim_{k \to \infty} \frac{\partial^2}{\partial x_i\, \partial x_j} [2^{mk} P(2^{-(m/a)k} \circ x)] = 0.$$

Finally, using the integral form of the remainder for $R_N(x)$, it is easy to see that $|(\partial^2/\partial x_i \partial x_j)\, R_N(x)| \leqslant C_N\, |x|^{N-1}$, and so

$$\lim_{s\to 0} \left| \frac{\partial^2}{\partial x_i\, \partial x_j} \{s^{-1} R_N(s^{1/a_1}x_1,..., s^{1/a_n}x_n)\} \right|$$
$$\leqslant \lim_{s\to 0} s^{-1+(1/a_i)+(1/a_j)} (s^{1/\max\{a_1,..., a_n\}})^{N-1} = 0,$$

by the definition of N. This finishes the proof of (18).

It follows that the surface given as the graph of $Q(y) + 2^{mk} R(2^{-m/a_k} \circ y) + 2^{mk} c_0$ has at least one nonvanishing principal curvature on the support of ψ_0, uniformly from below and above in k (by (17) and (18)), provided k is large enough. Alternatively, this can be achieved by taking the support of ψ, and hence also ψ_0, small enough. Thus we can apply the form of Sogge's theorem given in (16) to obtain that $\tau_{-k} \mathcal{M}_k \tau_k$, and hence also $\mathcal{M}_k$, is bounded on $L^p(\mathbb{R}^n)$ with norm at most $C_p 2^{-kn} 2^{km/p}$. Using Minkowski's inequality, we can sum these estimates and conclude that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$, for $p > m/n = 1/((1/a_1) + \cdots + (1/a_n))$ as required. This completes the proof of Theorem 10, and so also the proofs of Theorems 3 and 6 via Sogge's theorem.

We now turn to the alternate proof of Theorems 3 and 6 that doesn't involve Sogge's theorem. We begin with the proof of our decay estimate for the Fourier transform of a weighted surface-carried measure.

Proof of Theorem 13. Using $\Phi(x) = Q(x) + R(x)$, we have

$$F_\alpha(\xi, \lambda) = \int e^{i\{x\cdot\xi + \lambda(Q(x)+R(x))\}} (Q(x) + R(x))^\alpha\, \psi(x)\, dx.$$

Let

$$F_\alpha^k(\xi, \lambda) = \int e^{i\{x\cdot\xi + \lambda(Q(x)+R(x))\}} (Q(x) + R(x))^\alpha\, \psi_k(x)\, dx, \tag{19}$$

where $\psi_k(x) = \psi_0(2^{k/a_1}x_1,..., 2^{k/a_n}x_n)$ and ψ_0 is a smooth cutoff function supported in the annulus $\{x\colon 1 \leqslant |x| \leqslant 2\}$ and satisfying

$$\sum_{k=0}^{\infty} \psi_0(2^{k/a_1}x_1,..., 2^{k/a_n}x_n) \equiv 1.$$

Let $\tau_k x = (2^{k/a_1}x_1, \ldots, 2^{k/a_n}x_n)$. After making a change of variables sending $x \to \tau_{-k}x$, we get for (19)

$$2^{-(k/a_1) - \cdots - (k/a_n)} 2^{-k\alpha} \int e^{i\{\langle \tau_{-k}x, \xi\rangle + 2^{-k}\lambda Q(x)(1 + 2^k(R(\tau_{-k}x)/Q(x)))\}}$$

$$\times \left[Q(x) \left(1 + 2^k \frac{R(\tau_{-k}x)}{Q(x)}\right)\right]^\alpha \psi_0(x)\, dx. \tag{20}$$

We must estimate $\sum_{k=0}^{\infty} F_\alpha^k(\xi, \lambda)$. In the argument that follows, we shall be forced to take k sufficiently large, and so will only be able to estimate $\sum_{k=N_0}^{\infty} F_\alpha^k(\xi, \lambda)$, where N_0 is a large positive integer. Equivalently, we can estimate the full sum if ψ has sufficiently small support. We break up the sum as follows:

$$\sum_{k=N_0}^{\infty} = \sum_{\{\lambda \leqslant 2^k\}} + \sum_{\{\lambda > 2^k\}} = \mathrm{I} + \mathrm{II}.$$

To estimate I we use the fact that the integral in (20) is bounded since $Q(x) \geqslant c > 0$ for $x \in \operatorname{supp} \psi_0$, and $2^k R(\tau_{-k}x)$ is bounded by (14). It follows that I is dominated by

$$C \sum_{\lambda \leqslant 2^k} 2^{-(k/a_1) - \cdots - (k/a_n)} 2^{-k\alpha} \leqslant C\lambda^{-(1/a_1) - \cdots - (1/a_n) - \alpha}.$$

Suppose we could show that there is $\epsilon > 0$ such that

$$\left| \int e^{i\{\langle \tau_{-k}x, \xi\rangle + 2^{-k}\lambda Q(x)(1 + 2^k(R(\tau_{-k}x)/Q(x)))\}} \right.$$

$$\left. \times \left[Q(x) \left(1 + 2^k \frac{R(\tau_{-k}x)}{Q(x)}\right)\right]^\alpha \psi_0(x)\, dx \right| \leqslant C(2^{-k}\lambda)^{-(1/2) - \epsilon}, \tag{21}$$

for any $0 < \alpha < 1$. Then II is dominated by

$$C \sum_{\lambda > 2^k} 2^{-k((1/a_1) + \cdots + (1/a_n))} 2^{-k\alpha} (2^{-k}\lambda)^{-(1/2) - \epsilon} \leqslant C\lambda^{-(1/a_1) - \cdots - (1/a_n) - \alpha},$$

provided $\alpha + (1/a_1) + \cdots + (1/a_n) < \frac{1}{2} + \epsilon$. Altogether, this shows that

$$\left| \sum_{k=N_0}^{\infty} F_\alpha^k(\xi, \lambda) \right| \leqslant C\lambda^{-(1/2) - \epsilon'},$$

for $0 < \epsilon' = \alpha + (1/a_1) + \cdots + (1/a_n) - \frac{1}{2} < \epsilon$.

Thus the proof of Theorem 13 has been reduced to establishing (21). In order to prove the estimate (21), we shall need (17) together with a result in [BrNaWa]. Indeed, by Theorem B in [BrNaWa] (see (4) above), the decay of the Fourier transform of the surface-carried measure is controlled by

$$\sup_{x \in \operatorname{supp} \psi_0} |\mathscr{B}(x, |(\xi, \lambda)|^{-1})|,$$

where $\mathscr{B}(x, \delta)$ is the non-isotropic ball on S given as the set of $y \in S$ such that the distance to the tangent plane $\mathscr{H}_x$ at x is less than δ. Now Q has a non-vanishing principal curvature by (17), and using (14) and (18) we see that $Q(x)(1+2^k(R(\tau_{-k}x)/Q(x)))$ also does for k sufficiently large. But this function is a blowup of the graph of Φ, and hence satisfies the hypotheses required to apply the Schulz Lemma 5 at the point $x \in \operatorname{supp} \psi_0$. If $(b_1, ..., b_n)$ is the multi-type at the point x, then the nonvanishing principal curvature implies that one of the b_j is a 2. One now easily computes that

$$|\mathscr{B}(x, \delta)| \leqslant C\delta^{(1/2)-\epsilon}, \qquad x \in \operatorname{supp} \psi_0.$$

This completes the proof of (21), and hence also of the decay estimate for $F_\alpha(\xi, \lambda)$. The estimate for $\nabla F_\alpha(\xi, \lambda)$ is the same since differentiation only introduces additional factors of x and $Q(x)+R(x)$ into the integrand, thereby possibly increasing α. This completes the proof of Theorem 13.

Proof of Theorem 6. We give here an alternate proof of Theorem 6, using Theorems 12 and 13 in place of Theorem 11. In order to apply Hölder's inequality as in [IoSa1], we first need to know that Φ^{-p} is locally integrable whenever $p<(1/a_1)+ \cdots +(1/a_n)$. Now it is the case that Q^{-p} is locally integrable whenever $p<(1/a_1)+ \cdots +(1/a_n)$, since in weighted polar coordinates,

$$\int Q(x)^{-p}\, dx = \iint r^{-pa_1} r^{((a_1/a_2)+ \cdots (a_1/a_n))} Q(\omega)^{-p}\, dr\, d\omega.$$

The radial integral is bounded precisely when $p<(1/a_1)+ \cdots +(1/a_n)$, while the angular integral is bounded since Q is nonvanishing away from the origin. Finally, using (14), we obtain

$$\int_{|x| \leqslant \epsilon} |\Phi(x)|^{-p}\, dx \leqslant C, \qquad p<\frac{1}{a_1}+ \cdots +\frac{1}{a_n}, \tag{22}$$

for sufficiently small $\epsilon>0$.

Now define

$$\mathcal{M}_\alpha f(x, x_{n+1}) = \sup_{t>0} \int f(x - ty, x_{n+1} - t\Phi(y))\, |\Phi(y)|^\alpha\, \psi(y)\, dy.$$

Applying Hölder's inequality as in [IoSa1], we get

$$|\mathcal{M}f| \leqslant (\mathcal{M}_\alpha |f|^r)^{1/r} \left(\int |\Phi(y)|^{-\alpha(r'/r)}\, \psi(y)\, dy \right)^{1/r'} \leqslant C_{\alpha, r} (\mathcal{M}_\alpha |f|^r)^{1/r},$$

provided $\alpha(r'/r) < \rho$, i.e., $r > (\alpha + \rho)/\rho$, by (22). By Theorems 13 and 12, $\mathcal{M}_\alpha$ is bounded on L^2 if $\alpha + \rho > \frac{1}{2}$. Now fix $p > 1/\rho \geqslant 2$ and set $r = p/2$. Then $\frac{1}{2} - \rho < \rho(r-1)$ and thus we can choose α in $(\frac{1}{2} - \rho, \rho(r-1))$, which yields both $\alpha + \rho > \frac{1}{2}$ and $r > (\alpha + \rho)/\rho$. Then

$$\left(\int |\mathcal{M}f|^p \right)^{1/p} \leqslant C_{\alpha, r} \left(\int (\mathcal{M}_\alpha |f|^r)^2 \right)^{1/p} \leqslant C C_{\alpha, r} \left(\int |f|^p \right)^{1/p},$$

for $p > 1/\rho > 1/((1/a_1) + \cdots + (1/a_n))$. This completes our alternate proof of Theorem 6, and so also of Theorem 3.

4. DECAY OF THE FOURIER TRANSFORM ON SURFACES OF MIXED HOMOGENEITY

The purpose of this section is to prove Theorem 14. Assume without loss of generality that $Q(\omega) \geqslant 0$. We also assume that $\mu = n$. The general case requires only an obvious adjustment to the proof below.

Consider the weighted polar coordinate system given by $x_1 = r\omega_1$, $x_2 = r^{a_1/a_2}\omega_2, \ldots, x_n = r^{a_1/a_n}\omega_n$, where $\omega = (\omega_1, \ldots, \omega_n)$ denotes the standard coordinates on $\mathbb{S}^{n-1}$. It is not hard to check (see e.g. [FaRi]) that the Jacobian of this change of variables is $g(\omega)\, r^{(a_1/a_2) + \cdots + (a_1/a_n)}$, where $1 \leqslant g(\omega) \leqslant C$. We write

$$\begin{aligned} F(\xi, \lambda) &= \int e^{i\{x \cdot \xi + \lambda \Phi(x)\}} \psi(x)\, dx \\ &= \int_{\mathbb{S}^{n-1}} \int_0^\infty e^{i\{r\omega_1\xi_1 + \cdots + r^{a_1/a_n}\omega_n\xi_n + \lambda r^{a_1}\Phi(\omega)\}} \\ &\quad \times \psi(r)\, g(\omega) r^{(a_1/a_2) + \cdots (a_1/a_n)}\, dr\, d\omega \\ &= \iint_{\{\omega \in \mathbb{S}^{n-1}:\, \lambda Q(\omega) \leqslant 1\}} \cdots dr\, d\omega + \iint_{\{\omega \in \mathbb{S}^{n-1}:\, \lambda Q(\omega) > 1\}} \cdots dr\, d\omega \\ &= F_1(\xi, \lambda) + F_2(\xi, \lambda). \end{aligned}$$

Now

$$|F_1(\xi, \lambda)| \leqslant C\,|\{\omega: \lambda Q(\omega) \leqslant 1\}|$$

$$\leqslant \lambda^{-\rho} \int_{\mathbb{S}^{n-1}} Q^{-\rho}(\omega)\, d\omega \leqslant C\lambda^{-\rho}$$

by assumption.

Now we consider the second term F_2, where $\lambda Q(\omega) > 1$. We have

$$F(\xi, \lambda) = \iint_{\mathbb{S}^{n-1}} e^{i(r\omega_1 \xi_1 + \cdots + r^{(a_1/a_n)}\omega_n \xi_n + \lambda(r^{a_1} Q(\omega) + R(r\omega_1, \ldots, r^{(a_1/a_n)}\omega_n)))}$$

$$\times r^{(a_1/a_2) + \cdots + (a_1/a_n)} g(\omega)\, \psi(r)\, d\omega\, dr,$$

where without loss of generality we are taking ψ to be radial. Let $\psi_0 \in C_0^\infty(\mathbb{R})$ be a smooth cutoff function supported in the interval $[\frac{1}{2}, 4]$, such that $\sum_k \psi_0(2^k s) \equiv 1$. Let

$$F_k(\xi, \lambda) = \int_{\mathbb{S}^{n-1}} \int e^{i(r\omega_1 \xi_1 + \cdots + r^{a_1/a_n}\omega_n \xi_n + \lambda(r^{a_1} Q(\omega) + R(r\omega_1, \ldots, r^{a_1/a_n}\omega_n)))}$$

$$\times r^{(a_1/a_2) + \cdots + (a_1/a_n)} \psi_0(2^k r)\, g(\omega)\, d\omega\, dr$$

$$= \int_{\mathbb{S}^{n-1}} G_k(r, \omega, \xi, \lambda)\, g(\omega)\, d\omega.$$

We first analyze G_k. After making a change of variables sending $r \to 2^{-k} r$, we get $2^{-k(1 + (a_1/a_n) + \cdots + (a_1/a_n))}$ times

$$\int e^{i(2^{-k} r\omega_1 \xi_1 + \cdots + 2^{-k(a_1/a_2)} r^{a_1/a_n} \omega_n \xi_n + 2^{-a_1 k} \lambda Q(\omega)(r^{a_1} + (R(2^{-k} r\omega_1, \ldots, 2^{-k(a_1/a_n)} r^{a_1/a_n}\omega_n)/2^{-a_1 k} Q(\omega))))}$$

$$\times r^{(a_1/a_2) + \cdots + (a_1/a_n)} \psi_0(r)\, dr.$$

The expression above is the Fourier transform of a smooth measure supported on the curve

$$\Gamma(r) = \left(r, r^{(a_1/a_2)}, \ldots, r^{(a_1/a_n)}, r^{a_1} + \frac{R(2^{-k} r\omega_1, \ldots, 2^{-k(a_1/a_n)} r^{(a_1/a_n)}\omega_n)}{2^{-a_1 k} Q(\omega)}\right),$$

and evaluated at $(2^{-k}\omega_1 \xi_1, \ldots, 2^{-(a_1/a_n)}\omega_n \xi_n, 2^{-a_1 k}\lambda Q(\omega))$. It is not hard to check using (14) that, for k sufficiently large, Γ is nondegenerate (see Section 8.2 below) with constants independent of k and ω, away from $r = 0$. Therefore we have the estimate

$$|G_k(r, \omega, \xi, \lambda)| \leqslant C\,|2^{-a_1 k}\lambda Q(\omega)|^{-1/(n+1)}, \tag{23}$$

where C is independent of k, ξ, and ω. Also, we have the trivial estimate

$$|G_k(r, \omega, \xi, \lambda)| \leqslant C, \tag{24}$$

where C is a uniform constant.

Let N_0 denote a large positive integer. Since the estimate (23) is only valid for k sufficiently large, we shall estimate

$$\int_{\mathbb{S}^{n-1}} \sum_{k=N_0}^{\infty} G_k(r, \omega, \xi, \lambda)\, g(\omega)\, d\omega,$$

which will give us the estimates for $F_2(\xi, \lambda)$ in a sufficently small neighborhood of the origin. We break up the sum as follows:

$$\sum_{k=N_0}^{\infty} = \sum_{\{\lambda Q(\omega) \leqslant 2^{a_1 k}\}} + \sum_{\{\lambda Q(\omega) > 2^{a_1 k}\}} = \mathrm{I} + \mathrm{II}.$$

Using the estimate (24) we see that I is bounded by

$$C \sum_{\{\lambda Q(\omega) \leqslant 2^{a_1 k}\}} 2^{-k(1+(a_1/a_2)+\cdots+(a_1/a_n))} \leqslant C(\lambda Q(\omega))^{-(1/a_1)-\cdots-(1/a_n)}.$$

Integrating in ω, we get

$$C \int (\lambda Q(\omega))^{-(1/a_1)-\cdots-(1/a_n)}\, g(\omega)\, d\omega = C\lambda^{-\rho}$$

$$\times \int Q^{-\rho}(\omega)\, g(\omega) (\lambda Q(\omega))^{\rho-(1/a_1)-\cdots-(1/a_n)}\, d\omega$$

$$\leqslant C\lambda^{-\rho} \int Q^{-\rho}(\omega)\, d\omega \leqslant C\lambda^{-\rho}, \tag{25}$$

since $\int Q^{-\rho}(\omega)\, d\omega < \infty$, $\lambda Q(\omega) > 1$, and $g(\omega)$ is bounded.

Using the estimate (23) we see that II is bounded by

$$C \sum_{\{\lambda Q(\omega) > 2^{a_1 k}\}} 2^{-k(1+(a_1/a_2)+\cdots+(a_1/a_n))} (2^{-a_1 k} \lambda Q(\omega))^{-1/(n+1)}$$

$$\leqslant C \sum_{\{\lambda Q(\omega) > 2^{a_1 k}\}} 2^{a_1 k(1/(n+1)-((1/a_1)+\cdots+(1/a_n)))} Q^{-1/(n+1)}(\omega)$$

$$\leqslant C \sum_{\{\lambda Q(\omega) > 2^{a_1 k}\}} 2^{a_1 k(1/(n+1)-\rho)} Q^{-1/(n+1)}(\omega)$$

$$\leqslant C\lambda^{-\rho} Q^{-\rho}(\omega),$$

since $\rho \leqslant (1/a_1) + \cdots + (1/a_n)$ and $\rho < 1/(n+1)$. Integrating in ω we get

$$C\lambda^{-\rho} \int Q^{-\rho}(\omega)\, g(\omega)\, d\omega \leqslant C\lambda^{-\rho} \int Q^{-\rho}(\omega)\, d\omega \leqslant C\lambda^{-\rho}, \tag{26}$$

since $g(\omega)$ is bounded and $\int Q^{-\rho}(\omega)\, d\omega < \infty$. Combining the estimates (25) and (26) completes the proof.

5. AVERAGE SQUARE FUNCTION TECHNIQUES

In this section, we prove an analogue of the Sogge–Stein result in [SoSt] for an L^2 average decay. This improves on the result in [SoSt] in two ways: first, the L^2 average decay turns out to be a much weaker hypothesis than uniform decay when the dilations are truly nonisotropic, and second, the extra decay beyond $\frac{1}{2}$ in the hypothesis is minimized. In the final section we will use Theorem 23 below to help verify the hypotheses of Theorem 15 in the course of proving Theorem 7.

We recall the nonisotropic maximal operators $\mathcal{M}'$ introduced by Greenleaf in [Gr]. Given an n-tuple $(\beta_1, ..., \beta_n)$, we let $\mathcal{M}'$ denote the maximal operator

$$\mathcal{M}'f(x) = \sup_{t>0} M'_t f(x), \tag{27}$$

where the convolution operator M'_t is given by

$$M'_t f(x) = f * \delta'_t(\psi\, d\sigma)(x),$$

and where δ'_t denotes the nonisotropic dilation $\widehat{\delta'_t h}(\xi) = \hat{h}(t^{\beta_1}\xi_1, ..., t^{\beta_n}\xi_n)$. Set $\hat{\delta}'_t\xi = (t^{\beta_1}\xi_1, ..., t^{\beta_n}\xi_n)$ so that $\delta'_t h(\xi) = \hat{h}(\hat{\delta}'_t\xi)$. The following theorem is the appropriate nonisotropic square function estimate. Note that in the case $\beta_1 = \cdots = \beta_n > 0$, $\mathcal{M}'$ is simply the usual maximal operator $\mathcal{M}$, and in the case of surface-carried measures, the L^2 average decay reduces, for all practical purposes, to that of uniform decay (since the averages are taken along rays of fixed direction in the case of the usual dilations).

THEOREM 15. *Suppose τ is a distribution supported in a ball B of radius C_1 with $|\hat{\tau}(\xi)| \leqslant C_1$ and* $\max\{|x| : x \in \operatorname{supp} \tau\} \leqslant C_2$. *Suppose moreover that*

$$\left\{\int_1^2 |\hat{\tau}(\hat{\delta}'_t\xi)|^2\, dt\right\}^{1/2} \leqslant C_1(1+|\xi|)^{-1/2}\, \gamma(|\xi|),$$

$$\left\{\int_1^2 |\nabla\hat{\tau}(\hat{\delta}'_t\xi)|^2\, dt\right\}^{1/2} \leqslant C_2(1+|\xi|)^{-1/2}\, \gamma(|\xi|),$$

where γ is bounded and nonincreasing on $[0, \infty)$, and $\sum_{n=0}^{\infty} \gamma(2^n) < \infty$. For $t > 0$, define $\hat{\tau}_t(\xi) = \hat{\tau}(\hat{\delta}'_t \xi)$ as above with $\beta_i \geqslant 0$ for $i = 1, 2, \ldots, n$, and set

$$\mathcal{M}'_\tau f(x) = \sup_{t>0} |f * \tau_t(x)|.$$

Then

$$\|\mathcal{M}'_\tau f\|_{L^2} \leqslant C \sqrt{C_1 C_2} \|f\|_{L^2}, \qquad \textit{for all} \quad f \in \mathcal{S},$$

(*in the case some $\beta_i = 0$, our proof yields an additional factor* $\log(C_2/C_1)$).

Remark 7. The point of isolating the constants C_1 and C_2 above is that when we apply Theorem 15 in the final section, it will be to a piece of surface that has been translated a distance $c_0 2^{km}$, resulting in C_2, but not C_1, increasing by a factor of $1 + c_0 2^{km}$. The additional factor of $\log/(C_2/C_1)$ in the case that one of the β_i vanishes can probably be removed with a sharper argument, but in any event causes no problem in our application. Finally, the sharpness of the condition on γ follows from Example 3 in the Introduction.

Proof. We first consider the case $\beta_i > 0$ for all $i = 1, 2, \ldots, n$. Let $1 = \sum_{k=0}^{\infty} |\widehat{\phi_k}(\xi)|^2$ be the usual Littlewood–Paley decomposition, define τ^k by $\widehat{\tau^k}(\xi) = \widehat{\phi_k}(\xi)\, \hat{\tau}(\xi)$, and set

$$\mathcal{M}^{k'}_\tau f(x) = \sup_{t>0} |f * \tau^k_t(x)|.$$

By Minkowski's inequality and the hypothesis $\sum_{n=0}^{\infty} \gamma(2^n) < \infty$, it suffices to prove

$$\|\mathcal{M}^{k'}_\tau f\|_{L^2} \leqslant C \sqrt{C_1 C_2}\, \gamma(2^k)\, \|f\|_{L^2}, \qquad f \in \mathcal{S}, k \geqslant 0. \tag{28}$$

We note that for $k \geqslant 1$, we have

$$\begin{aligned} \left\{ \int_0^\infty |\widehat{\tau^k}(\hat{\delta}'_t \xi)|^2 \frac{dt}{t} \right\}^{1/2} &\leqslant C_1 2^{-k/2} \gamma(2^k), \\ \left\{ \int_0^\infty |\nabla \widehat{\tau^k}(\hat{\delta}'_t \xi)|^2 \frac{dt}{t} \right\}^{1/2} &\leqslant C_2 2^{-k/2} \gamma(2^k). \end{aligned} \tag{29}$$

Indeed, $\widehat{\tau^k}(\hat{\delta}'_t \xi) = \widehat{\phi_k}(\hat{\delta}'_t \xi)\, \hat{\tau}(\hat{\delta}'_t \xi) \neq 0$ only when $|\hat{\delta}'_t \xi| \approx 2^k$. Fixing k momentarily, let θ be such that $|\hat{\delta}'_\theta \xi| = 2^k$. Since the β_i are positive, there are constants c_1 and c_2 such that $\widehat{\tau^k}(\delta'_{s\theta}\xi)=\widehat{\tau^k}(\hat{\delta}'_s\hat{\delta}'_\theta\xi)\neq 0$ only when $c_1\leqslant s\leqslant c_2$. With the change of variable $t=s\theta$, we have, using our hypothesis on τ,

$$\begin{aligned}\int_0^\infty |\widehat{\tau^k}(\delta'_t\xi)|^2\frac{dt}{t}&=\int_0^\infty |\widehat{\tau^k}(\delta'_{s\theta}\xi)|^2\frac{ds}{s}=\int_{c_1}^{c_2}|\widehat{\tau^k}(\hat{\delta}'_s\hat{\delta}'_\theta\xi)|^2\frac{ds}{s}\\&\leqslant CC_1(1+|\hat{\delta}'_\theta\xi|)^{-1}\gamma(|\hat{\delta}'_\theta\xi|)^2\leqslant CC_1 2^{-k}\gamma(2^k)^2.\end{aligned}$$

The estimate for the term involving $\nabla\widehat{\tau^k}$ is similar, and this establishes (29).

Now we observe that

$$\lim_{t\to 0} f*\tau_t^k(x)=\lim_{t\to 0}\int e^{ix\cdot\xi}\hat{f}(\xi)\,\widehat{\tau^k}(\delta'_t\xi)\,d\xi=0,$$

for $f\in\mathscr{S}$ and $k\geqslant 1$, and so the fundamental theorem of calculus yields

$$\begin{aligned}|f*\tau_t^k(x)|^2&=\int_0^t\frac{d}{ds}\{f_k*\tau^k{}_s(x)\overline{f_k*\tau_s^k(x)}\}\,ds\\&\leqslant 2\int_0^\infty |f_k*\tau_s^k(x)|\left|f_k*\frac{d}{ds}\tau_s^k(x)\right|\,ds.\end{aligned}$$

This implies that

$$\begin{aligned}\|\mathscr{M}_\tau^{k'}f\|_{L^2}^2&=\int_{\mathbb{R}^n}\sup_{t>0}|f*\tau_t^k(x)|^2\,dx\\&\leqslant 2\left(\int_{\mathbb{R}^n}\int_0^\infty |f*\tau_s^k(x)|^2\frac{ds}{s}\,dx\right)^{1/2}\left(\int_{\mathbb{R}^n}\int_0^\infty\left|f*s\frac{d}{ds}\tau_s^k(x)\right|^2\frac{ds}{s}\,dx\right)^{1/2}\\&=2\left(\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2\int_0^\infty|\widehat{\tau^k}(\delta'_s\xi)|^2\frac{ds}{s}\,d\xi\right)^{1/2}\\&\quad\times\left(\int_{\mathbb{R}^n}|\hat{f}\xi)|^2\int_0^\infty|\hat{\delta}'_s\tilde{\xi}\cdot\nabla\widehat{\tau^k}(\hat{\delta}'_s\xi)|^2\frac{ds}{s}\,d\xi\right)^{1/2},\end{aligned}\tag{30}$$

where $\tilde{\xi}=(\beta_1\xi_1,...,\beta_n\xi_n)$, since $s(d/ds)(\hat{\delta}'_s\xi)=\hat{\delta}'_s\tilde{\xi}$. Using (29) together with the observation that $|\hat{\delta}'_s\tilde{\xi}|\approx 2^k$ when $\nabla\widehat{\tau^k}(\hat{\delta}'_s\xi)\neq 0$, we obtain from (30) that

$$\begin{aligned}\|\mathscr{M}_\tau^{k'}f\|_{L^2}^2&\leqslant CC_1 2^{-k/2}\gamma(2^k)\,\|f\|_{L^2}2^kC_2 2^{-k/2}\gamma(2^k)\,\|f\|_{L^2}\\&\leqslant CC_1C_2\gamma(2^k)^2\,\|f\|_{L^2}^2,\end{aligned}$$

which is (28) as required in the case $k\geqslant 1$.

The case $k=0$ of (28) is handled by the observation that

$$|\tau^0(x)| = |\tau * \phi_0(x)| \leqslant C_N C_1 \sum_{j=0}^{\infty} 2^{-j(n+N)} \chi_{2^j B}(x),$$

and so

$$\sup_{t>0} |f * \tau_t^0(x)| \leqslant C_N C_1 \sum_{j=0}^{\infty} 2^{-jN} M_{2^j B}(x),$$

where in the case $\beta_1 = \cdots = \beta_n = 1$ (the general case is handled similarly), M_B is the skewed maximal operator

$$M_B f(x) = \sup_{t>0} \left\{ \int_B |f(x-ty)|\, dy \right\}. \tag{31}$$

Now if we let $Q = \delta_{1/C_1} B$ so that Q is a ball of radius one and center, say, $x = (x', x_n) = (0, R)$ (where $R \approx C_2/C_1$), then

$$M_B f(x) = M_Q f(x) = \sup_{t>0} \left\{ \int_{|y| \leqslant 1} |f(x'-ty', x_n - t(R+y_n))|\, dy \right\}.$$

This operator M_B is weak type 1,1 with constant CR since

$$|\{x\colon M_B f(x) > \lambda\}| \leqslant CR\, |\{Mf(x) > \lambda\}|, \tag{32}$$

where M denotes the usual Hardy–Littlewood maximal function. In one dimension, this follows easily from the fact that if $\{I_j\}_j$ are the component intervals of the open set $\Omega_\lambda = \{Mf(x) > \lambda\}$ and if $\Omega_\lambda^R = \bigcup_j CRI_j$, then $\{x\colon M_B f(x) > \lambda\} \subset \Omega_\lambda^R$. In higher dimensions, if Q is a cube of side length d, we let $R \circ Q$ denote the rectangle with the same center as Q, but whose side length in the x_n direction is Rd, and in the other coordinate directions is d. If we set

$$\Omega_\lambda^R = \bigcup \{R \circ Q\colon Q \subset \Omega_\lambda\},$$

it again follows that $\{x\colon M_B f(x) > \lambda\} \subset \Omega_\lambda^R$, and hence that (32) holds. For the more general dilations δ_t', the set Ω_λ^R must be appropriately modified. If we now interpolate with the L^∞ estimate, we obtain that M_B is bounded on L^2 with norm $C\sqrt{R} \approx C\sqrt{C_2/C_1}$, and hence that $\sup_{t>0} |f * \tau_t^0(x)|$ is bounded on L^2 with norm $CC_1\sqrt{R} \approx C\sqrt{C_1 C_2}$, and this leads to (28) in the case $k=0$.

It remains only to consider the case when some of the β_i vanish. For simplicity of notation, we consider only the case $\beta_1 = \cdots = \beta_{n-1} = 0$ and $\beta_n = 1$, the general case being an obvious modification of this. Choose $\varphi \in \mathscr{S}$ with $\hat{\varphi}(\xi_n) = 1$ for $|\xi_n| \leqslant 1$ and let

$$\hat{\mu}(\xi) = \hat{\tau}(\xi) - \hat{\tau}(\xi', 0)\, \hat{\varphi}(\xi_n),$$

so that $\mu_t = \tau_t - \tau_0 \otimes \varphi_t$. Since

$$\sup_{t>0} |f * (\tau_0 \otimes \varphi_t)(x)| \leqslant C M_n(f * \tau_0)(x)$$

where M_n is a skewed maximal operator in the nth variable as in (31), and since

$$\begin{aligned} \|M_n(f * \tau_0)(x)\|_{L^2} &\leqslant C \sqrt{C_2/C_1}\, \|f * \tau_0(x)\|_{L^2} \\ &= C \sqrt{C_2/C_1}\, \|\hat{f}\hat{\tau}\|_{L^2} \leqslant C \sqrt{C_1 C_2}\, \|f\|_{L^2}, \end{aligned}$$

it suffices to prove

$$\|\mathscr{M}'_\mu f\|_{L^2} \leqslant C \sqrt{C_1 C_2}\, \|f\|_{L^2}, \qquad f \in \mathscr{S}, \tag{33}$$

where $\mu \in \mathscr{S}'$ satisfies

$$\begin{aligned} |\hat{\mu}(\xi)| &\leqslant C_1, \\ \left\{\int_1^2 |\hat{\mu}(\xi', t\xi_n)|^2\, dt\right\}^{1/2} &\leqslant C_1(1+|\xi|)^{-1/2}\, \gamma(\xi), \\ \left|\frac{\partial}{\partial \xi_n} \hat{\mu}(\xi)\right| &\leqslant C_2, \\ \left\{\int_1^2 \left|\frac{\partial}{\partial \xi_n} \hat{\mu}(\xi', t\xi_n)\right|^2 dt\right\}^{1/2} &\leqslant C_2(1+|\xi|)^{-1/2}\, \gamma(\xi), \\ |\hat{\mu}(\xi)| &\leqslant \min\{C_1, C_2 |\xi_n|\}. \end{aligned} \tag{34}$$

Indeed, the first four inequalities in (34) follow from the corresponding inequalities for τ and the rapid decay of $\hat{\varphi}$, while for the fifth we have that $|\xi_n| \leqslant 1$ implies $\hat{\varphi}(\xi_n) = 1$, and so

$$|\hat{\mu}(\xi', \xi_n)| = |\hat{\tau}(\xi', \xi_n) - \hat{\tau}(\xi', 0)| \leqslant |\xi_n| \left\|\frac{\partial}{\partial \xi_n} \hat{\tau}\right\|_{L^\infty} \leqslant C_2 |\xi_n|.$$

Now let $1 = \sum_{k=0}^{\infty} |\widehat{\phi_k}(\xi_n)|^2$ be the usual Littlewood–Paley decomposition in the nth variable, define μ^k by $\widehat{\mu^k}(\xi) = \widehat{\phi_k}(\xi_n)\,\hat{\mu}(\xi)$, and set

$$\mathscr{M}_\mu^{k'} f(x) = \sup_{t>0} |f * \mu_t^k(x)|.$$

By Minkowski's inequality and the hypothesis $\sum_{n=0}^{\infty} \gamma(2^n) < \infty$, it suffices to prove

$$\|\mathscr{M}_\mu^{k'} f\|_{L^2} \leqslant C\sqrt{C_1 C_2}\,\gamma(2^k)\,\|f\|_{L^2}, \qquad f \in \mathscr{S}, \quad k \geqslant 0. \tag{35}$$

The proof of (35) proceeds in the same way as the proof of (28), except that the case $k=0$ is handled differently. Arguing as in (30) with $k=0$ (and noting that $\lim_{t\to 0} \widehat{\mu^0}(\hat{\delta}'_t \xi) = 0$), we see that we must establish the boundedness of the expression $\int_0^\infty |\widehat{\mu^0}(\hat{\delta}'_s \xi)|^2 (ds/s)$. But if θ is such that $|\hat{\delta}'_\theta \xi| = 1$, then with $\eta = \hat{\delta}'_\theta \xi$, the substitution $s = t\theta$ yields

$$\begin{aligned}
\int_0^\infty |\widehat{\mu^0}(\hat{\delta}'_s \xi)|^2 \frac{ds}{s} &= \int_0^\infty |\widehat{\mu^0}(\hat{\delta}'_t \hat{\delta}'_\theta \xi)|^2 \frac{dt}{t} \\
&= \int_0^2 |\widehat{\mu^0}(\hat{\delta}'_t \eta)|^2 \frac{dt}{t} \\
&\leqslant \int_0^2 |\min\{C_1, C_2 t\}|^2 \frac{dt}{t} \\
&\leqslant C C_1^2 \log \frac{C_2}{C_1},
\end{aligned}$$

by the final inequality in (34). This completes the proof of Theorem 15.

6. VAN DER CORPUT ESTIMATES

In this section we present two nonasymptotic estimates of Van der Corput type. The first shows that if a sufficiently high derivative of the phase function ϕ is controlled, then decay of the oscillatory integral $\int e^{i\phi(s)}\psi(s)\,ds$ is governed by $|\phi''|^{-1/2}$ at the zeroes of ϕ', and by $|\phi'|^{-1}$ at the zeroes of ϕ''. The second shows that if the third derivative of ϕ is suitably controlled, then improved decay is obtained by subtracting off the appropriate asymptotic term. The latter result will play a pivotal role in the next section in establishing average L^2 decay of oscillatory integrals $\int e^{i\phi_t(s)}\psi(s)\,ds$ with a parameter t.

THEOREM 16. *Suppose the support of $\psi(s)$ is contained in $[\tau_1, \tau_2]$, and that $\phi(s)$ is smooth and real-valued on $[\tau_1, \tau_2]$. Let $\{r_k\}_{k=1}^m$ and $\{s_l\}_{l=1}^n$*

be the real zeroes of ϕ' *and* ϕ'' *respectively in* $[\tau_1, \tau_2]$. *Assume there is* $3 \leqslant N < \infty$ *satisfying*

$$|\phi^{(N)}(s)|^{1/N} \leqslant c_N |\phi''(r_k)|^{1/2} \qquad \text{for all} \quad s \in \operatorname{supp} \psi, \quad k = 1, 2, ..., m, \tag{36}$$

for a sufficiently small constant c_N. *Then there is a constant* C *such that*

$$\left| \int e^{i\phi(s)} \psi(s)\, ds \right| \leqslant C \left(\sum_{k=1}^{m} |\phi''(r_k)|^{-1/2} + \sum_{\ell=1}^{n} |\phi'(s_\ell)|^{-1} + \sum_{i=1}^{2} |\phi'(\tau_i)|^{-1} \right).$$

Proof. Let J_k denote the smallest interval containing r_k such that $|\phi'(t)| = |\phi''(r_k)|^{1/2}$ when t is an endpoint of J_k. Then by Lemma 5.13 in [GuSa] (see p. 29),

$$\begin{aligned} |\phi''(r_k)| &\leqslant \sup_{s \in J_k} |\phi''(s)| \\ &\leqslant C \left(\frac{\sup_{x, y \in J_k} |\phi'(x) - \phi'(y)|}{|J_k|} \right. \\ &\qquad \left. + \left(\sup_{x, y \in J_k} |\phi'(x) - \phi'(y)| \right)^{1 - 1/(N-1)} \|\phi^{(N)}\|_\infty^{1/(N-1)} \right) \\ &\leqslant \frac{C\, |\phi''(r_k)|^{1/2}}{|J_k|} + C\, |\phi''(r_k)|^{(1/2)(1 - 1/(N-1))} \|\phi^{(N)}\|_\infty^{1/(N-1)} \\ &\leqslant \frac{C\, |\phi''(r_k)|^{1/2}}{|J_k|} + C c_N\, |\phi''(r_k)| \end{aligned}$$

by (36), and since we may assume $Cc_N < \frac{1}{2}$, we conclude that

$$|J_k| \leqslant C\, |\phi''(r_k)|^{-1/2}.$$

Now write

$$\int e^{i\phi(s)} \psi(s)\, ds = \sum_{k=1}^{n} \int_{J_k} e^{i\phi(s)} \psi(s)\, ds + \int_{(\bigcup_{k=1}^{m} J_k)^c} e^{i\phi(s)} \psi(s)\, ds.$$

Then $|\int_{J_k} e^{i\phi(s)} \psi(s)\, ds| \leqslant C\, |J_k| \leqslant C\, |\phi''(r_k)|^{-1/2}$, and by construction (and calculus) we have

$$|\phi'| \geqslant \min_{\substack{1 \leqslant k \leqslant m \\ 1 \leqslant l \leqslant n \\ 1 \leqslant i \leqslant 2}} \{ |\phi''(r_k)|^{1/2}, |\phi'(s_\ell)|, |\phi'(\tau_i)| \}$$

on $(\bigcup_{k=1}^{m} J_k)^c$ which implies

$$\left|\int_{(\bigcup J_k)^c} e^{i\phi(s)}\psi(s)\,ds\right| \leqslant C\left(\sum_{k=1}^{m} |\phi''(r_k)|^{-1/2} + \sum_{\ell=1}^{n} |\phi''(s_\ell)|^{-1} + \sum_{i=1}^{2} |\phi'(\tau_i)|^{-1}\right)$$

as required.

THEOREM 17. *Suppose the support of $\psi(s)$ is contained in $[\tau_1, \tau_2]$, and that $\phi(s)$ is smooth and real-valued on $[\tau_1, \tau_2]$. Let $\{r_k\}_{k=1}^{m}$ and $\{s_\ell\}_{\ell=1}^{n}$ be the real zeroes of ϕ' and ϕ'' respectively in $[\tau_1, \tau_2]$. Assume there is $0 < \epsilon \leqslant \frac{1}{6}$ satisfying*

$$|\phi'''(s)|^{1/3} \leqslant C\,|\phi''(r_k)|^{(1/2)-\epsilon} \qquad \textit{for} \quad |s - r_k| \leqslant |\phi''(r_k)|^{\epsilon-(1/2)},$$
$$k = 1,\ldots, m. \tag{37}$$

Then there is C_ϵ such that

$$\left|\int e^{i\phi(s)}\psi(s)\,ds - \sum_{k=1}^{m} e^{i\phi(r_k)}\left(\frac{2\pi i}{\phi''(r_k)}\right)^{1/2}\psi(r_k)\right|$$
$$\leqslant C_\epsilon\left(\sum_{k=1}^{m} |\phi''(r_k)|^{-(1/2)-\epsilon} + \sum_{\ell=1}^{n} |\phi'(s_\ell)|^{-1} + \sum_{i=1}^{2} |\phi'(\tau_i)|^{-1}\right).$$

Proof. Since $\int e^{i\phi(s)}\psi(s)\,ds$ is bounded, it suffices to consider the case $|\phi''(r_k)| \geqslant C_1$ for $k = 1,\ldots, m$. We write

$$\int e^{i\phi(s)}\psi(s)\,ds = \sum_{k=1}^{m} \int_{I_{k,\epsilon}} e^{i\phi(s)}\psi(s)\,ds + \int_{(\bigcup_k I_{k,\epsilon})^c} e^{i\phi(s)}\psi(s)\,ds$$

where $I_{k,\epsilon} = (r_k - |\phi''(r_k)|^{-(1/2)+\epsilon},\ r_k + |\phi''(r_k)|^{-(1/2)+\epsilon})$. Note that (37) implies

$$\begin{aligned}
|\phi''(t)| &= |\phi''(r_k) + \phi''(t) - \phi''(r_k)| \\
&= |\phi''(r_k) + \phi'''(c)(t - r_k)| \\
&\geqslant |\phi''(r_k)| - C^3\,|\phi''(r_k)|^{(3/2)-3\epsilon}\,|\phi''(r_k)|^{-(1/2)+\epsilon} \\
&\geqslant \tfrac{1}{2}\,|\phi''(r_k)|, \qquad \text{for} \quad t \in I_{k,\epsilon},
\end{aligned}$$

provided $C_1^{-2\epsilon} C^3 \leqslant \frac{1}{2}$, and so

$$|\phi'(r_k + |\phi''(r_k)|^{-(1/2)+\epsilon})| = \left|\int_{r_k}^{r_k + |\phi''(r_k)|^{-1/2+\epsilon}} \phi''(t)\,dt\right| \geqslant \frac{1}{2}\,|\phi''(r_k)|^{(1/2)+\epsilon}.$$

Similarly $|\phi'| \geqslant \frac{1}{2}|\phi''(r_k)|^{(1/2)+\epsilon}$ at each endpoint of an $I_{k,\epsilon}$. Thus

$$|\phi'| \geqslant \min_{\substack{1 \leqslant k \leqslant m \\ 1 \leqslant \ell \leqslant n \\ 1 \leqslant i \leqslant 2}} \{\tfrac{1}{2}|\phi''(r_k)|^{(1/2)+\epsilon}, |\phi'(s_\ell)|, |\phi'(\tau_i)|\}$$

on $(\bigcup_k I_{k,\epsilon})^c$ by calculus. So

$$\left|\int_{(\bigcup_k I_{k,\epsilon})^c} e^{i\phi(s)}\psi(s)\,ds\right|$$
$$\leqslant C\left(\sum_{k=1}^{m}|\phi''(r_k)|^{-(1/2)-\epsilon} + \sum_{\ell=1}^{n}|\phi'(s_\ell)|^{-1} + \sum_{i=1}^{2}|\phi'(\tau_i)|^{-1}\right).$$

Finally,

$$\int_{I_{k,\epsilon}} e^{i\phi(s)}\psi(s)\,ds = |\phi''(r_k)|^{-(1/2)+\epsilon}\int_{-1}^{+1} e^{i\varphi(t)}\tilde{\psi}(t)\,dt,$$

where $\tilde{\psi}(t) = \psi(r_k + t\,|\phi''(r_k)|^{-(1/2)+\epsilon})$ and

$$\begin{cases} \varphi(t) = \phi(r_k + t\,|\phi''(r_k)|^{-(1/2)+\epsilon}), \\ \varphi'(0) = 0, \\ \varphi''(0) = \pm|\phi''(r_k)|^{2\epsilon}, \\ |\varphi'''(t)| = |\phi''(r_k)|^{-(3/2)+3\epsilon}\,|\phi'''(r_k + t\,|\phi''(r_k)|^{-(1/2)+\epsilon})| \leqslant C^3 & \text{by (37).} \end{cases}$$

Set $A = |\phi''(r_k)|^{2\epsilon}$ and $\varepsilon(t) = t^{-2}\{\varphi(t) - \varphi(0) - (\varphi''(0)\,t^2/2)\}$. Then we have

$$|\varepsilon(t)| = \left|\frac{t}{6}\varphi'''(c)\right| \leqslant C^3, \tag{38}$$

$$|\varepsilon'(t)| = \left|\frac{d}{dt}\left\{\int_0^t \frac{(1-(u/t))^2}{2}\varphi'''(u)\,du\right\}\right|$$
$$= \left|\int_0^t \left(1 - \frac{u}{t}\right)\frac{u}{t^2}\varphi'''(u)\,du\right|$$
$$\leqslant \sup_{0 \leqslant u \leqslant t}|\varphi'''(u)| \leqslant C^3.$$

Thus

$$\int_{I_{k,\epsilon}} e^{i\phi(s)}\psi(s)\,ds = |\phi''(r_k)|^{-1/2} A^{1/2} \int_{-1}^{+1} e^{i\{\varphi(0)+(A/2+\varepsilon(t))t^2\}}\tilde{\psi}(t)\,dt$$

$$= e^{i\varphi(0)}\,|\phi''(r_k)|^{-1/2} A^{1/2} \int_{-1}^{+1} e^{it^2((1/2)A+\varepsilon(t))}\tilde{\psi}(t)\,dt.$$

Now let

$$s = \sqrt{2}\,t\sqrt{\frac{A}{2}+\varepsilon(t)} = t\sqrt{A+2\varepsilon(t)}$$

$$ds = \left(\sqrt{A+2\varepsilon(t)} + \frac{t\varepsilon'(t)}{\sqrt{A+2\varepsilon(t)}}\right)dt = \frac{A+2\varepsilon(t)+t\varepsilon'(t)}{\sqrt{A+2\varepsilon(t)}}\,dt$$

so that

$$\int_{-1}^{+1} e^{it^2((1/2)A+\varepsilon(t))}\tilde{\psi}(t)\,dt = \int_{-\sqrt{A+2\varepsilon(-1)}}^{\sqrt{A+2\varepsilon(1)}} e^{i(s^2/2)}\,\frac{\tilde{\psi}(t)\sqrt{A+2\varepsilon(t)}}{A+2\varepsilon(t)+t\varepsilon'(t)}\,ds$$

$$= \tilde{\psi}(0)A^{-1/2}\int_{-A^{1/2}}^{A^{1/2}} e^{i(s^2/2)}\,ds + E,$$

where

$$|E| \leqslant \left|\int_{A^{1/2}}^{\sqrt{A+2\varepsilon(1)}} A^{-1/2}\,ds\right| + \left|\int_{-\sqrt{A+2\varepsilon(-1)}}^{-A^{1/2}} A^{-1/2}\,ds\right| + \int_{-A^{1/2}}^{A^{1/2}} \frac{|\tilde{\psi}(t)-\tilde{\psi}(0)|}{A^{1/2}}\,ds$$

$$+\tilde{\psi}(0)\int_{-A^{1/2}}^{A^{1/2}} \left|\frac{\sqrt{A+2\varepsilon(t)}}{A+2\varepsilon(t)+t\varepsilon'(t)} - \frac{1}{A^{1/2}}\right|\,ds$$

$$\leqslant CA^{-1} + CA^{-1} + C\,|\phi''(r_k)|^{-(1/2)+\epsilon}$$

$$+C\int_{-A^{1/2}}^{A^{1/2}} \frac{|\sqrt{A^2+2\varepsilon(t)A}-(A+2\varepsilon(t)+t\varepsilon'(t))|}{A^{3/2}}\,ds$$

$$\leqslant CA^{-1} = CA^{-1/2}\,|\phi''(r_k)|^{-\epsilon},$$

since $|\varepsilon(t)+t\varepsilon'(t)| \leqslant C$ by (38), and $|\phi''(r_k)|^{-(1/2)+\epsilon} \leqslant |\phi''(r_k)|^{-2\epsilon}$ since $\epsilon \leqslant \frac{1}{6}$. So

$$\int_{I_{k,\epsilon}} e^{i\phi(s)}\psi(s)\,ds = e^{i\varphi(0)}\,|\phi''(r_k)|^{-1/2}\,\tilde{\psi}(0)$$

$$\times \int_{-A^{1/2}}^{A^{1/2}} e^{i(s^2/2)}\,ds + e^{i\varphi(0)}\,|\phi''(r_k)|^{-1/2}\,A^{1/2}E$$

$$= e^{i\varphi(0)}\,|\phi''(r_k)|^{-1/2}\,\tilde{\psi}(0)\int_{-\infty}^{\infty} e^{i(s^2/2)}\,ds + O(|\phi''(r_k)|^{-(1/2)-\epsilon})$$

by the error estimate (39) for E together with $|\int_{A^{1/2}}^{\infty} e^{i(s^2/2)}\,ds| \leqslant CA^{-1/2}$. Finally, we have

$$\int_0^R e^{i(s^2/2)}\,ds = \int_0^R e^{i(\{(1+i)s\}^2/2)}\,d\{(1+i)s\} + O(R^{-1})$$

$$= (1+i)\int_0^R e^{-s^2}\,ds + O(R^{-1}),$$

by Cauchy's Theorem, and so

$$\int_{-\infty}^{\infty} e^{i(s^2/2)}\,ds = (1+i)\int_{-\infty}^{\infty} e^{-s^2}\,ds = \sqrt{2i}\,\sqrt{\pi} = \sqrt{2\pi i}.$$

Using $\varphi(0) = \phi(r_k)$ and $\tilde{\psi}(0) = \psi(r_k)$, we thus obtain

$$\int_{I_{k,\epsilon}} e^{i\phi(s)}\psi(s)\,ds = e^{i\phi(r_k)}\left(\frac{2\pi i}{\phi''(r_k)}\right)^{1/2}\psi(r_k) + O(|\phi''(r_k)|^{-(1/2)-\epsilon})$$

as required.

7. AVERAGE L^2 DECAY

In this section, we investigate the average L^2 decay of oscillatory integrals $\int e^{i\phi_t(s)}\psi(s)\,ds$ with phase $\phi_t(s)$ depending on a parameter t. Theorem 17 of the previous section indicates the special role played by the quantity $|\phi_t''|^{-1/2}$ evaluated at a critical point $r(t)$. Thus we begin with what amounts to a reverse $L^{2+2\mu} - L^2$ Hölder inequality for $|\phi_t''(r(t))|^{-1/2}$.

THEOREM 18. *Let* $\phi_t(s) = R(P(s) + tQ(s))$ *where* $P(s) = a_1 s + \cdots + a_{m+1}s^{m+1}$ *and* $Q(s) = b_1 s + b_{m+1}s^{m+1}$. *Let* $A = |a_1| + \cdots + |a_{m+1}|$ *and suppose* $\delta_1, \delta_2 > 0$ *satisfy*

$$A \geqslant 1, \qquad |b_1| \geqslant \delta_1 > 0, \qquad |b_k| \leqslant 1 \quad \text{for} \quad k = 1, 2, \ldots, m+1, \tag{40}$$

and if $a_1 Q(x) - b_1 P(x) = \sum_{k=2}^{m+1} c_k x^k$, *then*

$$\sum_{k=2}^{m+1} |c_k| > \delta_2 A, \tag{41}$$

(*i.e., P is not close to a multiple of Q*). *Let* $r(t)$ *denote a smooth real root of* ϕ'_t, $t_0 < t < t_1$, *where* $|r(t)| \leqslant (\delta_1/2m)$ *and* $\phi''_t(r(t)) \neq 0$ *for* $t_0 < t < t_1$. *Then for* $0 < \mu < 1/(2m-1)$,

$$\int_{t_0}^{t_1} \left| \frac{\phi''_t(r(t))}{R} \right|^{-(1+\mu)} dt \leqslant C_{m,\delta_1} (\delta_2 A)^{-\mu}. \tag{42}$$

Remark 8. The proof below shows that hypotheses (40) are sufficient for the equivalence

$$\int_{t_0}^{t_1} \left| \frac{\phi''_t(r(t))}{R} \right|^{-1} dt \approx C\delta_1^{-1} |r(t_1) - r(t_0)|,$$

so that the additional hypothesis (41) can be viewed as yielding the reverse Hölder inequality

$$\left\| \left| \frac{\phi''_t(r(t))}{R} \right|^{-1/2} \right\|_{L^{2+2\mu}} \leqslant C_{m,\delta_1} (c_2 A)^{-\mu} \left\| \left| \frac{\phi''_t(r(t))}{R} \right|^{-1/2} \right\|_{L^2}.$$

Proof. We have $0 = \phi'_t(r(t)) = R\{P'(r(t)) + tQ'(r(t))\}$ which implies that $-P'(r(t))/Q'(r(t)) = t$. So if we set $h(x) = -P'(x)/Q'(x)$, then $h(r(t)) = t$, $r(t) = h^{-1}(t)$, and $r'(t) = 1/h'(h^{-1}(t))$. Now

$$0 = \frac{d}{dt} (\phi'_t(r(t))) = \phi''_t(r(t))\, r'(t) + RQ'(r(t))$$

implies

$$\begin{aligned} \left| \frac{\phi''_t(r(t))}{R} \right| &= \left| \frac{Q'(r(t))}{r'(t)} \right| = |Q'(r(t))\, h'(h^{-1}(t))| \\ &\geqslant \frac{\delta_1}{2} |h'(h^{-1}(t))|, \end{aligned}$$

since $|Q'(s)| \geqslant |b_1| - sm \geqslant (\delta_1/2)$ for $|s| \leqslant (\delta_1/2m)$. Thus with $d_1 = \delta_1/2m$,

$$\int_{t_0}^{t_1} \left|\frac{\phi_t''(r(t))}{R}\right|^{-(1+\mu)} dt \leqslant C_{\delta_1} \int_{t_0}^{t_1} |h'(h^{-1}(t))|^{-(1+\mu)}\, dt$$

$$= C_{\delta_1} \int_{t_0}^{t_1} |h'(h^{-1}(t))|^{-\mu} \frac{dt}{h'(h^{-1}(t))},$$

$$= C_{\delta_1} \int_{r_0}^{r_1} |h'(r)|^{-\mu}\, dr \quad \text{with} \quad r = h^{-1}(t),$$

$$\frac{dr}{dt} = \frac{1}{h'(h^{-1}(t))},$$

$$\leqslant C_{\delta_1} \int_{-d_1}^{d_1} |h'(r)|^{-\mu}\, dr$$

since $|r(t)| \leqslant (\delta_1/2m) \leqslant 1$,

$$\leqslant C_{\delta_1} \int_{-d_1}^{d_1} |P'(r)\, Q''(r) - P''(r)\, Q'(r)|^{-\mu}\, dr,$$

since $|h'(x)| = |(d/dx)\,(-(P'(x)/Q'(x)))| = |P'(x)\, Q''(x) - P''(x)\, Q'(x)| \times |Q'(x)|^{-2}$ and $|Q'(x)| \geqslant (\delta_1/2) > 0$. Now if $P'(x)\, Q''(x) - P''(x)\, Q'(x) = \sum_{k=0}^{2m-1} d_k x^k$, then the proposition in Section 2 of [RiSt] shows that

$$\int_{t_0}^{t_1} \left|\frac{\phi_t''(r(t))}{R}\right|^{-(1+\mu)} dt \leqslant C \int_{-d_1}^{d_1} |P'(r)\, Q''(r) - P''(r)\, Q'(r)|^{-\mu}\, dr$$

$$\leqslant C \left(\sum_{k=0}^{2m-1} |d_k|\right)^{-\mu}$$

$$\leqslant C \left(\sum_{k=0}^{m-1} |a_1 b_{k+2} - b_1 a_{k+2}|\right)^{-\mu},$$

since for $0 \leqslant k \leqslant m-1$ we have

$$d_k = \sum_{i+j=k+3,\, i \geqslant 1,\, j \geqslant 2} (i a_i\, j(j-1)\, b_j - j(j-1)\, a_j i b_i)$$

$$= (k+2)(k+1)(a_1 b_{k+2} - b_1 a_{k+2}).$$

Finally, since $a_1 Q(x) - b_1 P(x) = \sum_{k=2}^{m+1} c_k x^k$ implies $c_{k+2} = a_1 b_{k+2} - b_1 a_{k+2}$, we have

$$\left(\sum_{k=0}^{m-1} |a_1 b_{k+2} - b_1 a_{k+2}|\right)^{-\mu} = \left(\sum_{k=2}^{m+1} |c_k|\right)^{-\mu} \leqslant C_m (\delta_2 A)^{-\mu},$$

by the hypothesis (41).

We shall actually need a variant of Theorem 18 involving an error term as follows.

THEOREM 19. *Let* $\phi_t(s)=R(P(s)+tQ(s))$ *where* $P(s)=a_1 s+\cdots+a_{m+1}s^{m+1}+E_P(s)$ *and* $Q(s)=b_1 s+b_{m+1}s^{m+1}+E_Q(s)$. *As in Theorem* 18, *let* $A=|a_1|+\cdots+|a_{m+1}|$ *and suppose* $\delta_1, \delta_2>0$ *satisfy*

$$A\geqslant 1, \qquad |b_1|\geqslant\delta_1>0, \qquad |b_k|\leqslant 1 \qquad \text{for} \quad k=1,2,\ldots,m+1,$$

and if $a_1 Q(x)-b_1 P(x)=\sum_{k=2}^{m+1} c_k x^k$, *then*

$$\sum_{k=2}^{m+1} |c_k|>\delta_2 A.$$

We suppose the error terms satisfy

$$\begin{aligned} \left|\frac{d^k}{ds^k} E_P(s)\right| &\leqslant C_3 s^{m+2-k}, \\ \left|\frac{d^k}{ds^k} E_Q(s)\right| &\leqslant C_3 s^{m+2-k}, \end{aligned} \tag{43}$$

for $0\leqslant k\leqslant m+2$. *Let* $r(t)$ *denote a smooth real root of* ϕ_t', $t_0<t<t_1$, *where* $|r(t)|\leqslant(\delta_1/2m)$ *and* $\phi_t''(r(t))\neq 0$ *for* $t_0<t<t_1$. *Then for* $0<\mu<1/(2m-1)$, *the reverse Hölder inequality* (42) *continues to hold but with the constant* C_m *now depending also on* C_3 *in* (43).

Proof. The proof of Theorem 18 applies without change up to the application of the result in [RiSt] to the polynomial $P'Q''-P''Q'$. This time we have, by (43),

$$P'(x)\,Q''(x)-P''(x)\,Q'(x)=\sum_{k=1}^{m-1} d_k x^k+O(x^m),$$

where as before, $d_k=(k+2)(k+1)(a_1 b_{k+2}-b_1 a_{k+2})$ for $0\leqslant k\leqslant m-1$. We can now invoke an extension of the proposition in Section 2 of [RiSt], given in Proposition 22 of the subsection on reverse Hölder inequalities below (with $f=P'Q''-P''Q'$ and $\ell=m-1$), to complete the proof just as in Theorem 18 above.

COROLLARY 20. *Let* ϕ_t *and* $r(t)$ *be as in Theorem* 19. *Then*

$$\text{(A)} \quad |\{t\colon |\phi_t''(r(t))|\leqslant BR^{\delta}\}|\leqslant CA^{-\mu}B^{1+\mu}R^{-(1-\delta)(1+\mu)}, \qquad \text{for} \quad 0<\delta\leqslant 1, \quad B>0, \tag{44}$$

(B) $$|\{t\colon \textit{condition } (!) \textit{ fails}\}| \leqslant CA^{(2/N)(1+\mu)-\mu}R^{-(1-(2/N))(1+\mu)}, \tag{45}$$

where (!) *is condition* (36) *in Theorem* 16 *with* $\phi=\phi_t$ *and* $r_k=r(t)$, *i.e.*,

$$(!)\ |\phi_t^{(N)}(t)|^{1/N}\leqslant C_N\,|\phi_t''(r(t))|^{1/2}, \qquad t\in\operatorname{supp}\psi. \tag{46}$$

Proof. Part (A) is an immediate consequence of Theorem 19 as follows:

$$(BR^{\delta-1})^{-(1+\mu)}\,|\{t\colon |\phi_t''(r(t))|\leqslant BR^{\delta}\}| \leqslant \int_{t_0}^{t_1}\left|\frac{\phi_t''(r(t))}{R}\right|^{-(1+\mu)}dt\leqslant C_m A^{-\mu}.$$

For part (B), we note that $\|\phi_t^{(N)}\|_\infty\leqslant CAR$ and so if (46) fails for some t, then $|\phi_t''(r(t))|\leqslant C(AR)^{2/N}$ and (A) applies with $B=A^{2/N}$ and $\delta=2/N$.

7.1. *Reverse Hölder Inequalities for Finite-Type Functions*

We begin with a simple weak type estimate for functions whose ℓth derivative is bounded below.

LEMMA 21. *Let I be an interval of length at most* 1 *and suppose that, for some $\ell\geqslant 1$, the function f satisfies*

$$|f^{(\ell)}(x)|\geqslant c_1>0, \qquad \textit{for}\quad x\in I.$$

Then

$$|\{x\in I\colon |f(x)|^{-1}>\lambda\}|\leqslant 4^{\ell}(c_1\lambda)^{-1/\ell}. \tag{47}$$

Proof. We prove the lemma by induction on ℓ. The case $\ell=1$ is an easy exercise (similar to the argument below, anyway), so suppose now that the lemma holds with $\ell-1$ in place of ℓ. Let $d\in I$ be the point where $|f^{(\ell-1)}(x)|$ achieves its minimum on I, and set

$$J=(d-\delta,d+\delta),$$

where $\delta>0$ will be chosen momentarily. By construction we have

$$|J|\leqslant 2\delta. \tag{48}$$

Since $|f^{(\ell-1)}|\geqslant c_1\delta$ on $I\backslash J$, which consists of at most two intervals of length not exceeding 1, we conclude from the induction assumption that

$$|\{x\in I\backslash J\colon |f(x)|^{-1}>\lambda\}|\leqslant 2[4^{\ell-1}(c_1\delta\lambda)^{-1/(\ell-1)}]. \tag{49}$$

Combining (48) and (49) now yields

$$|\{x \in I: |f(x)|^{-1} > \lambda\}| \leqslant 2\delta + 2[4^{\ell-1}(c_1\delta\lambda)^{-1/(\ell-1)}] \leqslant 4^{\ell}(c_1\lambda)^{-1/\ell},$$

upon choosing $\delta = (c_1\lambda)^{-1/\ell}$, so that

$$(c_1\delta\lambda)^{-1/(\ell-1)} = ((c_1\lambda)^{1-(1/\ell)})^{-1/(\ell-1)} = (c_1\lambda)^{-1/\ell}.$$

This completes the proof of the lemma.

We can now give the analogue, for functions of finite type, of the proposition in Section 2 of [RiSt].

PROPOSITION 22. *Suppose that I is an interval of length at most 1, and that*

$$0 < c_1 \leqslant \sum_{k=0}^{\ell} |f^{(k)}(x)| \leqslant \sum_{k=0}^{\ell+1} |f^{(k)}(x)| \leqslant C_2, \qquad \textit{for} \quad x \in I.$$

There is a constant C_ℓ depending only on ℓ such that

$$|\{x \in I: |f(x)|^{-1} > \lambda\}| \leqslant C_\ell \left(\frac{C_2}{c_1}\right)^{\ell} (c_1\lambda)^{-1/\ell}, \tag{50}$$

and in particular,

$$\int_I |f(x)|^{-\mu}\, dx \leqslant C_{\mu,\ell} \left(\frac{C_2}{c_1}\right)^{\ell} (c_1)^{-\mu}, \qquad \textit{for} \quad 0 < \mu < \frac{1}{\ell}. \tag{51}$$

Proof. Note that (51) is an immediate consequence of (50). We prove (50) by induction on ℓ. The case $\ell = 1$ is an easy exercise (similar to the argument below, anyway), so suppose now that the proposition holds with $\ell - 1$ in place of ℓ. Let $E = \{x \in [0, 1]: |f^{(\ell)}(x)| \geqslant (c_1/2)\}$. Then since $|f^{(\ell+1)}(x)| \leqslant C_2$, we can write

$$E \subset \bigcup_{i=1}^{N} I_i,$$

where $I_i \subset \{x \in I: |f^{(\ell)}(x)| \geqslant (c_1/4)\}$, and N is an integer at most $2C_2/c_1$. Now let $d_i \in I_i$ be a point where $|f^{(\ell-1)}(x)|$ achieves its minimum on I_i, and set

$$J_i = (d_i - \delta, d_i + \delta),$$

where $\delta > 0$ will be chosen momentarily. By construction we have

$$\left| \bigcup_{i=1}^{N} J_i \right| \leqslant 2N\delta. \tag{52}$$

Since $|f^{(\ell-1)}| \geqslant (c_1/4)\delta$ on $I_i \backslash J_i$, Lemma 21 applied to the (at most) two component intervals of each $I_i \backslash J_i$ shows that

$$\left| \left\{ x \in \bigcup_{i=1}^{N} I_i \Big\backslash \bigcup_{i=1}^{N} J_i \colon |f(x)|^{-1} > \lambda \right\} \right| \leqslant 2N \left[4^{\ell-1} \left(\frac{c_1}{4} \delta\lambda \right)^{-1/(\ell-1)} \right]. \tag{53}$$

Since $\sum_{k=0}^{\ell-1} |f^{(k)}| \geqslant \sum_{k=0}^{\ell} |f^{(k)}| - (c_1/2) \geqslant (c_1/2)$ on $I \backslash \bigcup_{i=1}^{N} I_i$, we conclude from the induction assumption applied to each of the (at most) $N+1$ intervals comprising $I \backslash \bigcup_{i=1}^{N} I_i$ that

$$\left| \left\{ x \in I \Big\backslash \bigcup_{i=1}^{N} I_i \colon |f(x)|^{-1} > \lambda \right\} \right| \leqslant (N+1) \left[C_{\ell-1} \left(\frac{C_2}{c_1} \right)^{\ell-1} \left(\frac{c_1}{2} \lambda \right)^{-1/(\ell-1)} \right]. \tag{54}$$

Combining (52), (53), and (54) now yields

$$\begin{aligned} |\{x \in I \colon |f(x)|^{-1} > \lambda\}| &\leqslant 2N\delta + 2N \left[4^{\ell-1} \left(\frac{c_1}{4} \delta\lambda \right)^{-1/(\ell-1)} \right] \\ &\quad + (N+1) \left[C_{\ell-1} \left(\frac{C_2}{c_1} \right)^{\ell-1} \left(\frac{c_1}{2} \lambda \right)^{-1/(\ell-1)} \right] \\ &\leqslant C_\ell \left(\frac{C_2}{c_1} \right)^{\ell} (c_1 \lambda)^{-1/\ell}, \end{aligned}$$

for suitable constants C_ℓ, upon choosing $\delta = (c_1 \lambda)^{-1/\ell}$. This completes the proof of the proposition.

7.2. *The Average Decay Estimate*

We can now give our average L^2 decay result. These estimates will be used to control error terms in the proof of Theorem 7 in the final section below.

THEOREM 23. *Let ϕ_t be as in Theorem 19, i.e., let $\phi_t(s) = R(P(s) + tQ(s))$ where $P(s) = a_1 s + \cdots + a_{m+1} s^{m+1} + E_P(s)$ and $Q(s) = b_1 s + b_{m+1} s^{m+1} + E_Q(s)$. Let $A = |a_1| + \cdots + |a_{m+1}|$ and suppose $\delta_1, \delta_2 > 0$ satisfy*

$$A \geqslant 1, \qquad |b_1| \geqslant \delta_1 > 0, \qquad |b_k| \leqslant 1 \quad \textit{for} \quad k = 1, 2, \ldots, m+1,$$

and if $a_1 Q(x) - b_1 P(x) = \sum_{k=2}^{m+1} c_k x^k$, *then*

$$\sum_{k=2}^{m+1} |c_k| > \delta_2 A.$$

Suppose the error terms satisfy

$$\begin{aligned} \left|\frac{d^k}{ds^k} E_P(s)\right| &\leqslant C_3 s^{m+2-k}, \\ \left|\frac{d^k}{ds^k} E_Q(s)\right| &\leqslant C_3 s^{m+2-k}, \end{aligned} \tag{55}$$

for $0 \leqslant k \leqslant m+2$, *and suppose* ψ *is supported in* $[\tau_1, \tau_2]$. *Then for* τ_1, τ_2 *sufficiently close to zero, we have*

$$\int_0^\infty \left|\int e^{i\phi_t(s)} \psi(s)\, ds\right|^2 dt \leqslant CR^{-1}. \tag{56}$$

Let $\{r_k(t)\}_{k=1}^m$ *denote the real roots of* ϕ_t' *in* (τ_1, τ_2) (*of course, m is also a function of t, but this plays no significant role, and we will suppress this dependence for the sake of convenience*). *Then, for some* $\epsilon > 0$, *we have*

$$\int_{\{t \in (0,\infty):\, |\phi_t''(r_k(t))| \leqslant R^{1-\epsilon} \text{ for some } k\}} \left|\int e^{i\phi_t(s)} \psi(s)\, ds\right|^2 dt \leqslant CR^{-1-\epsilon}. \tag{57}$$

If in addition we have $A \leqslant R^{(1/2)-6\epsilon}$, *then*

$$\int_0^\infty \left|\int e^{i\phi_t(s)} \psi(s)\, ds - \sum_{k=1}^m e^{i\phi_t(r_k(t))} \left(\frac{2\pi i}{\phi_t''(r_k(t))}\right)^{1/2} \psi(r_k(t))\right|^2 dt \leqslant CR^{-1-\epsilon}. \tag{58}$$

Proof. We begin by observing that if $\{s_\ell(t)\}_{\ell=1}^n$ are the real roots of ϕ_t'' in (τ_1, τ_2), then

$$\begin{aligned} \frac{d}{dt}\phi_t'(s_\ell(t)) &= \frac{d}{dt} R(P'(s_\ell(t)) + tQ'(s_\ell(t))) \\ &= R(P''(s_\ell(t)) + tQ''(s_\ell(t)))\, s_\ell'(t) + RQ'(s_\ell(t)) \\ &= \phi_t''(s_\ell(t))\, s_\ell'(t) + RQ'(s_\ell(t)) \\ &= RQ'(s_\ell(t)) \approx Rb_1 \approx R, \end{aligned}$$

provided τ_1 and τ_2 are sufficiently small. Similarly,

$$\frac{d}{dt}\phi_t'(\tau_j) = RQ'(\tau_j) \approx Rb_1 \approx R.$$

It follows that there exist γ_ℓ and η_j such that

$$\begin{aligned} |\phi_t'(s_\ell(t))| &\geqslant cR\,|t-\gamma_\ell|, &&\text{for} \quad s_\ell(t) \in (\tau_1, \tau_2), \\ |\phi_t'(\tau_j)| &\geqslant cR\,|t-\eta_j|, &&\text{for all} \quad t \in (0, \infty). \end{aligned} \tag{59}$$

Now, in order to prove (57), define

$$E = \left\{ \begin{array}{lll} t \in (0, \infty)\colon (46) \text{ holds with } r(t) = r_k(t) & \text{for all} & k; \\ |t-\gamma_\ell| \geqslant R^{\epsilon-1} \text{ and } |t-\eta_j| \geqslant R^{\epsilon-1} & \text{for all} & \ell, j; \text{ and} \\ |\phi_t''(r_k(t))| \leqslant R^{1-\epsilon} & \text{for some} & k. \end{array} \right\}$$

Since $\int e^{i\phi_t(s)}\psi(s)\,ds$ is bounded, inequality (45) in Corollary 20 yields

$$\begin{aligned} &\int_{\{t \in (0,\infty)\colon (46) \text{ fails with } r(t) = r_k(t) \text{ for some } k\}} \left| \int e^{i\phi_t(s)}\psi(s)\,ds \right|^2 dt \\ &\qquad \leqslant CR^{-(1-(2/N))(1+\mu)} A^{(2/N)(1+\mu)-\mu} \leqslant CR^{-1-\epsilon} \end{aligned}$$

if we choose N so large that $\mu > (2/N)(1+\mu)$ and $(1-(2/N))(1+\mu) > 1$. Since $\phi_t(s)$ is a polynomial of degree $m+1$ that is normalized by (40) and (41), we have

$$\left| \int e^{i\phi_t(s)}\psi(s)\,ds \right| \leqslant CR^{-1/(m+1)},$$

and so

$$\int_{\{t \in (0,\infty)\colon |t-\gamma_\ell| \leqslant R^{\epsilon-1}\}} \left| \int e^{i\phi_t(s)}\psi(s)\,ds \right|^2 dt \leqslant CR^{-2/(m+1)} R^{\epsilon-1} \leqslant CR^{-1-\epsilon},$$

if ϵ is chosen less than $1/(m+1)$. Altogether we've shown that

$$\int_{E^c \cap \{t \in (0,\infty)\colon |\phi_t''(r_k(t))| \leqslant R^{1-\epsilon} \text{ for some } k\}} \left| \int e^{i\phi_t(s)}\psi(s)\,ds \right|^2 dt \leqslant CR^{-1-\epsilon}.$$

On the other hand, if (46) holds with $r(t)=r_k(t)$ for all k, we can invoke Theorem 16 to obtain

$$\int_E \left| \int e^{i\phi_t(s)} \psi(s)\, ds \right|^2 dt$$

$$\leqslant C\left(\sum_{k=1}^{m} \int_E |\phi_t''(r_k(t))|^{-1}\, dt + \sum_{\ell=1}^{n} \int_E |\phi_t'(s_\ell(t))|^{-2}\, dt + \sum_{j=1}^{2} \int_E |\phi_t'(t_j)|^{-2}\, dt \right)$$

$$= C(\mathrm{I}+\mathrm{II}+\mathrm{III}),$$

where of course the integrations in I and II (but not in III) are further restricted to those t for which $r_k(t)$ and $s_\ell(t)$ respectively lie in (τ_1, τ_2). Now using Theorem 19, we obtain

$$\mathrm{I} \leqslant CR^{-1} \sum_{k=1}^{m} \int_E \left| \frac{\phi_t''(r_k(t))}{R} \right|^{-1} dt$$

$$\leqslant CR^{-1} \sum_{k=1}^{m} \left(\int \left| \frac{\phi_t''(r_k(t))}{R} \right|^{-(1+\mu)} \right)^{1/(1+\mu)} |E|^{\mu/(1+\mu)}\, dt$$

$$\leqslant CR^{-1} C_m R^{-\epsilon\mu},$$

since $|E| \leqslant CR^{-\epsilon\mu}$ by (44) in Corollary 20 with $B=1$ and $\delta = 1-\epsilon$. Since $|t-\gamma_\ell| \geqslant R^{\epsilon-1}$ on E, we have from (59),

$$\mathrm{II} \leqslant C \sum_{\ell=1}^{n} \int_{\{t:\, |t-\gamma_\ell| \geqslant R^{\epsilon-1}\}} |t-\gamma_\ell|^{-2}\, dt \leqslant CR^{-2}R^{1-\epsilon} = CR^{-1-\epsilon},$$

and similarly for term III.

Thus far we've proven (57). The first inequality (56) follows just as above except that we do not include the restriction "$|\phi_t''(r_k(t))| \leqslant R^{1-\epsilon}$ *for some* k" in the definition of E. Thus the measure of E is no longer small, and in the estimate for term I we simply use $\int |(\phi_t''(r_k(t))/R)|^{-1}\, dt \leqslant C$ without applying Hölder's inequality.

Finally, we turn to the third inequality (58). Consider the set

$$F = \{ t \in (0, \infty) : |\phi_t''(r_k(t))| > R^{1-\epsilon} \text{ for all } k \}.$$

From the assumption $A \leqslant R^{(1/2)-6\epsilon}$, we obtain for $t \in F$,

$$|\phi'''(t)|^{1/3} \leqslant (CAR)^{1/3} \leqslant CR^{(1/2)-2\epsilon} \leqslant C(R^{1-\epsilon})^{(1/2)-\epsilon} \leqslant C\, |\phi_t''(r_k(t))|^{(1/2)-\epsilon},$$

for all t and $k=1, 2, \ldots, m$. Thus condition (37) in Theorem 17 holds with ϕ_t in place of ϕ and $r_k(t)$ in place of $r(t)$. So applying Theorem 17 yields

$$\int_F \left| \int e^{i\phi_t(s)} \psi(s)\, ds - \sum_{k=1}^{m} e^{i\phi_t(r_k(t))} \left(\frac{2\pi i}{\phi_t''(r_k(t))} \right)^{1/2} \psi(r_k(t)) \right|^2 dt$$

$$\leqslant C \left(\sum_{k=1}^{m} \int_F |\phi_t''(r_k(t))|^{-1-\epsilon}\, dt + \sum_{\ell=1}^{n} \int |\phi_t'(s_\ell(t))|^{-2}\, dt + \sum_{j=1}^{2} \int |\phi_t'(t_j)|^{-2}\, dt \right)$$

$$= C(\mathrm{I} + \mathrm{II} + \mathrm{III}).$$

Now terms II and III are dominated by $CR^{-1-\epsilon}$ just as in the proof of (57) above. For term I, we use the fact that $|\phi_t''(r_k(t))| > R^{1-\epsilon}$ for $t \in F$ to obtain

$$I \leqslant (R^{1-\epsilon})^{-\epsilon} \sum_{k=1}^{m} \int_F |\phi_t''(r_k(t))|^{-1}\, dt \leqslant CR^{-1-\epsilon+\epsilon^2},$$

upon using the consequence $\int |(\phi_t''(r_k(t))/R)|^{-1}\, dt \leqslant C$ of Theorem 19 noted above. Of course, the integral over F^c is handled by the method of proof of (57). This completes the proof of Theorem 23.

8. MAXIMAL THEOREMS FOR NON-ISOTROPIC OPERATORS

We recall our main theorem for $\mathcal{M}'$ on mixed homogeneous surfaces, but in the setting of $\mathbb{R}^{n+1}$ rather than $\mathbb{R}^n$, as we will be expressing the surface as a graph.

THEOREM 24. *Suppose $\Phi(x)$ is mixed homogeneous of degree $(a_1, \ldots, a_n)$, with $a_j > 1$, namely,*

$$\Phi(\lambda^{1/a_1} x_1, \ldots, \lambda^{1/a_n} x_n) = \lambda \Phi(x), \qquad \lambda > 0, \quad x \in \mathbb{R}^n.$$

Suppose further that

$$\Phi(\omega)^{-1} \in L^p(\mathbb{S}^{n-1}), \qquad 0 < p \leqslant \min\left\{\frac{n}{m}, \frac{1}{2}\right\},$$

and $\sum = \{x : \Phi(x) = 1\}$ is of finite type with polynomial bounds, namely,

$$\sum_{2 \leqslant |\beta| \leqslant \ell} \left| \frac{\partial^{|\beta|}}{\partial y^\beta} \Phi(x) \right| \geqslant c\, |x|^{-M}, \tag{60}$$

for some $M \geqslant 0, \ell \geqslant 2$, *and where* $\beta = (\beta_1, ..., \beta_{n-1})$ *is a multi-index, and* $(y_1, ..., y_{n-1})$ *is a coordinate system orthogonal to* $\nabla\Phi(x)$ *at* x. *Let* $\mathscr{M}'$ *be defined as in* (27) *above with* S *given as the graph of* $\Phi + c_0$, *and with* $0 \leqslant \beta_1 = \beta_2 = \cdots = \beta_{n-1} < \beta_n$. *Then* $\mathscr{M}'$ *is bounded on* $L^p(\mathbb{R}^{n+1})$, *i.e.,* (8) *holds, for* $p > 1/\rho$. *Moreover, the constant* C_p *in* (8) *is at most* $C'_p(1 + |c_0|)^{1/p}$ (*in the case* $\beta_1 = 0$, *our proof yields an additional factor* $\log(1 + |c_0|)$).

Conversely, if (8) *holds for a given* p *and* $c_0 \neq 0$, *then* $p > m/n$ *and* $\Phi(\omega)^{-1} \in L^{1/p}(\mathbb{S}^{n-1})$.

Proof. The converse assertion is proved just as in Theorem 2 of [IoSa1], since the only issue is the behaviour of $\mathscr{M}'f(x, x_{n+1})$ for small x_{n+1} when f is a function of x_{n+1} blowing up appropriately at 0, and in this respect, $\mathscr{M}'$ behaves the same as $\mathscr{M}$. The point here is that the tangent plane at the origin is horizontal, and thus doesn't rotate while dilating.

Turning to the main assertion, we first note that it suffices to consider the case where the cutoff function $\psi(r)$ is supported in the interval $[\frac{1}{2}, 2]$. Indeed, with this done, one writes $\psi = \sum_{k=0}^{\infty} \psi_k$ where $\psi_k(r) = \varphi(2^k r)$, $k \geqslant 1$, and φ is supported in $[\frac{1}{2}, 2]$. With $\mathscr{M}'_k$ denoting the maximal operator corresponding to ψ_k, we rescale as at the beginning of Section 3, and obtain that the $\mathscr{M}'_k$ are bounded on $L^p(\mathbb{R}^{n+1})$ with constant $C2^{(km/p)-kn}$. Note that the factor $2^{km/p}$ arises since the resealed maximal operator has c_0 replaced by $2^{km}c_0$. If $p > m/n$, Minkowski's inequality finishes the proof.

After perhaps making a change of scale $t \to t^\alpha$, we may assume that in the definition of $\mathscr{M}'$, $\beta_1 = ... = \beta_n = \beta$, while $\beta_{n+1} = 1 + \beta$. For $\alpha > 0$, we define

$$\begin{aligned} \mathscr{M}'_\alpha f(x, x_{n+1}) &= \sup_{t>0} \left| \int f(x - t^\beta y, x_{n+1} - t^{1+\beta}(\Phi(y) + c_0))\, \Phi(y)^\alpha\, \psi(y)\, dy \right| \\ &= \sup_{t>0} |f * \tau_t^\alpha(x)|, \end{aligned}$$

where $\widehat{\tau^\alpha}(\xi, \lambda) = \int_{\mathbb{R}^n} e^{i\{x \cdot \xi + \lambda(\Phi(x) + c_0)\}} |\Phi(x)|^\alpha \psi(x)\, dx$. As in [IoSa1], it suffices to show that $\mathscr{M}'_\alpha$ is bounded on $L^2(\mathbb{R}^{n+1})$ for $\alpha + \rho > \frac{1}{2}$, and with norm at most $C(1 + |c_0|)^{1/2}$. Indeed, applying Hölder's inequality, we get

$$|\mathscr{M}'f| \leqslant (\mathscr{M}'_\alpha |f|^r)^{1/r} \left(\int |\Phi(y)|^{-\alpha(r'/r)}\, \psi(y)\, dy \right)^{1/r'} \leqslant C_{\alpha, r} (\mathscr{M}'_\alpha |f|^r)^{1/r},$$

provided $\alpha(r'/r) < \rho$, i.e., $r > (\alpha + \rho)/\rho$. Fix $p > 1/\rho \geqslant 2$ and set $r = p/2$. Then $\frac{1}{2} - \rho < \rho(r-1)$ and thus we can choose α in $(\frac{1}{2} - \rho, \rho(r-1))$, which yields both $\alpha + \rho > \frac{1}{2}$ and $r > (\alpha + \rho)/\rho$. Then, if we have shown that $\mathscr{M}'_\alpha$ is bounded on $L^2(\mathbb{R}^{n+1})$ with norm at most $C(1+|c_0|)^{1/2}$ for $\alpha+\rho>\frac{1}{2}$,

$$\left(\int |\mathcal{M}'f|^p\right)^{1/p} \leqslant C\left(\int (\mathcal{M}'_\alpha |f|^r)^2\right)^{1/p} \leqslant C\left((1+|c_0|)\int |f|^p\right)^{1/p}.$$

The L^2 bound in turn follows from Theorem 15 provided we verify that $\widehat{\tau^\alpha}$ is bounded and

$$\begin{aligned} \left\{\int_1^2 |\widehat{\tau^\alpha}(t^\beta\xi, t^{1+\beta}\lambda)|^2\,dt\right\}^{1/2} &\leqslant C(1+|\xi|+|\lambda|)^{-(1/2)-\epsilon}, \\ \left\{\int_1^2 |\nabla\widehat{\tau^\alpha}(t^\beta\xi, t^{1+\beta}\lambda)|^2\,dt\right\}^{1/2} &\leqslant C(1+|c_0|)(1+|\xi|+|\lambda|)^{-(1/2)-\epsilon}. \end{aligned} \tag{61}$$

Now set $1/m=(1/n)((1/a_1)+\cdots+(1/a_n))$, $|x|_a=(|x_1|^{a_1}+\cdots+|x_n|^{a_n})^{1/m}$ and $\tilde{S}^{n-1}=\{\omega\in\mathbb{R}^n\colon |\omega|_a=1\}$. Define $\lambda^b\circ x=(\lambda^{b_1}x_1,\ldots,\lambda^{b_n}x_n)$ for a multi-index $b=(b_1,\ldots,b_n)$ so that in particular we have $\Phi(\lambda^{1/a}\circ x)=\lambda\Phi(x)$. With the change of variables $r=|x|_a$ and $\omega=(|x|_a^{-m})^{1/a}\circ x$, we have $|\omega|_a=(\sum_{j=1}^n |x_j r^{-m/a_j}|^{a_j})^{1/m}=1$, and so if the cutoff function ψ depends only on $r=|x|_a$, we obtain (upon omitting the harmless factor $e^{i\lambda c_0}$)

$$\begin{aligned} \widehat{\tau^\alpha}(\xi,\lambda) &= \int_{\mathbb{R}^n} e^{i\{x\cdot\xi+\lambda\Phi(x)\}}\psi(x)\,\Phi(x)^\alpha\,dx \\ &= \int_{\mathbb{R}^n} e^{i\{x\cdot\xi+\lambda\Phi((|x|_a^{-m})^{1/a}\circ x)|x|_a^m\}}\psi(x)\,\Phi(x)^\alpha\,dx \\ &= \int_{\tilde{S}^{n-1}}\left[\int_0^\infty e^{i\{\sum_{j=1}^n \omega_j\xi_j r^{m/a_j}+\lambda\Phi(\omega)r^m\}} r^{n-1+m\alpha}\psi(r)\,dr\right]\Phi(\omega)^\alpha\,d\omega \\ &= \int_{\tilde{S}^{n-1}} H^\alpha(\omega,\xi,\lambda)\,\Phi(\omega)^\alpha\,d\omega, \end{aligned}$$

where

$$H^\alpha(\omega,\xi,\lambda)=\int_0^\infty e^{i\{\sum_{j=1}^n \omega_j\xi_j r^{m/a_j}+\lambda\Phi(\omega)r^m\}} r^{n-1+m\alpha}\psi(r)\,dr. \tag{62}$$

Clearly, $\widehat{\tau^\alpha}$ is bounded. Turning to (61), we restrict attention to τ^α, since the argument for $\nabla\tau^\alpha$ is the same, except that applying $\partial/\partial\lambda$ to $\widehat{\tau^\alpha}$ brings down a factor of $\Phi(x)+c_0$ into the integrand, resulting in the extra factor

$(1+|c_0|)$. In the case $|\lambda| \leqslant c\,|\xi|$, for c small enough, $|\nabla_x(x\cdot\xi+\lambda\Phi(x))| = |\xi+\lambda\nabla\Phi(x)| \geqslant c\,|\xi|$ and integration by parts yields $|\widehat{\tau^\alpha}(\xi,\lambda)| \leqslant C_N(1+|\xi|+|\lambda|)^{-N}$ for all $N\geqslant 0$, and (61) follows immediately. So we now assume that $|\lambda| > c\,|\xi|$ and write

$$\begin{aligned}\int_1^2 |\widehat{\tau^\alpha}(t^\beta\xi, t^{1+\beta}\lambda)|^2\,dt &= \int_1^2 \left|\int_{S^{n-1}} H^\alpha(\omega, t^\beta\xi, t^{1+\beta}\lambda)\,\Phi(\omega)^\alpha\,d\omega\right|^2 dt\\ &\leqslant \int_1^2 \left|\int_{\{\Phi(\omega)<\lambda^{-m\varepsilon}\}} H^\alpha(\omega, t^\beta\xi, t^{1+\beta}\lambda)\,\Phi(\omega)^\alpha\,d\omega\right|^2 dt\\ &\quad+\int_1^2 \left|\int_{\{\Phi(\omega)>\lambda^{-m\varepsilon}\}} H^\alpha(\omega, t^\beta\xi, t^{1+\beta}\lambda)\,\Phi(\omega)^\alpha\,d\omega\right|^2 dt\\ &= \mathrm{I}+\mathrm{II}.\end{aligned}$$

To handle term I, we use the estimate

$$\int_1^2 |H^\alpha(\omega, t^\beta\xi, t^{1+\beta}\lambda)|^2\,dt \leqslant C\,|\lambda\Phi(\omega)|^{-1}. \tag{63}$$

To see this in the special case $\beta=0$, let $\eta\in C_c^\infty(\mathbb{R}_+)$ satisfy $\eta\geqslant 0$ and $\eta(t)=1$ for $1\leqslant t\leqslant 2$. Then

$$\begin{aligned}\int |H^\alpha(\omega,\xi,t\lambda)|^2\,\eta(t)\,dt &= \int H^\alpha(\omega,\xi,t\lambda)\,\overline{H^\alpha(\omega,\xi,t\lambda)}\,\eta(t)\,dt\\ &= \iint\left[\int e^{i\{\sum_{j=1}^n \omega_j\xi_j(r^{m/a_j}-s^{m/a_j})+t\lambda\Phi(\omega)(r^m-s^m)\}}\eta(t)\,dt\right]\\ &\quad\times r^{n-1+m\alpha}\psi(r)\,s^{n-1+m\alpha}\psi(s)\,dr\,ds.\end{aligned}$$

Now the inner integral in square brackets equals

$$\int\left\{\left(\frac{1-(\partial^2/\partial t^2)}{1+|\lambda\Phi(\omega)|^2\,|r^m-s^m|^2}\right)e^{i\{\sum_{j=1}^n \omega_j\xi_j(r^{m/a_j}-s^{m/a_j})+t\lambda\Phi(\omega)(r^m-s^m)\}}\right\}\eta(t)\,dt,$$

which, upon integrating by parts, is dominated by

$$\int\frac{|(1-(\partial^2/\partial t^2))\,\eta(t)|}{1+|\lambda\Phi(\omega)|^2\,|r^m-s^m|^2}\,dt \leqslant C(1+|\lambda\Phi(\omega)|^2\,|r-s|^2)^{-1}.$$

Thus

$$\int |H^\alpha(\omega, \xi, t\lambda)|^2 \eta(t)\, dt \leqslant C \iint (1 + |\lambda\Phi(\omega)|^2 |r-s|^2)^{-1} \psi(r)\, \psi(s)\, dr\, ds$$
$$= C\, |\lambda\Phi(\omega)|^{-1},$$

which is (63) as required.

Unfortunately, this simple argument doesn't work in the case $\beta \neq 0$, since the t derivative of the phase function no longer has the special form above. Instead, we must apply (56) of Theorem 23. For this, recall from (62) that

$$H^\alpha(\omega, t^\beta \xi, t^{1+\beta}\lambda) = \int_1^\infty e^{i\{t^\beta \sum_{j=1}^n \omega_j \xi_j r^{m/a_j} + t^{1+\beta}\lambda\Phi(\omega)\, r^m\}} r^{n-1+m\alpha} \psi(r)\, dr$$
$$= \int_0^\infty e^{i\phi_t(r)} r^{n-1+m\alpha} \psi(r)\, dr,$$

where

$$\phi_t(r) = \lambda\Phi(\omega) \left\{ t^\beta \sum_{j=1}^n \left(\frac{\omega_j \xi_j}{\lambda\Phi(\omega)} \right) r^{m/a_j} + t^{1+\beta} r^m \right\}.$$

At this point we must group together like powers of r. Note that the hypothesis $a_j > 1$ implies that the coefficient of r^m, within the braces above, is exactly $t^{1+\beta}$. In general, the coefficient of r^{m/a_k} is $t^\beta((\sum_{j:\, a_j = a_k} \omega_j \xi_j)/(\lambda\Phi(\omega)))$. For convenience, we will assume that the a_j are all distinct, the general argument being the same with $\sum_{j:\, a_j = a_k} \omega_j \xi_j$ in place in place of $\omega_k \xi_k$.

Now if $|\lambda\Phi(\omega)| \geqslant C \sum_{j=1}^n |\omega_j \xi_j|$, then

$$\left| \frac{d}{dr} \phi_t(r) \right| \geqslant c\, |\lambda\Phi(w)|$$

on supp ψ. So integration by parts yields $|H^\alpha(\omega, t^\beta\xi, t^{1+\beta}\lambda)| \leqslant C_N(1 + |\lambda\Phi(\omega)|)^{-N}$ and the left side of (63) is simply

$$\int_1^2 |H^\alpha(\omega, t^\beta \xi, t^{1+\beta}\lambda)|^2\, dt \leqslant C_N \int_1^2 (1 + |\lambda\Phi(\omega)|)^{-2N}\, dt \leqslant C\, |\lambda\Phi(\omega)|^{-1}.$$

On the other hand, if $|\lambda\Phi(\omega)| \leqslant C \sum_{j=1}^n |\omega_j \xi_j|$, then we write

$$\phi_t(r) = \lambda\Phi(\omega) \left\{ t^\beta \sum_{j=1}^n \left(\frac{\omega_j \xi_j}{\lambda\Phi(\omega)} \right) r^{m/a_j} + t^{1+\beta} r^m \right\}$$
$$= R\{ t^\beta \vec{B} \cdot \vec{\sigma}(r) + t^{1+\beta} \vec{e}_{n+1} \cdot \vec{\sigma}(r) \},$$

where $R = \lambda\Phi(\omega)$, $\vec{B} = ((\omega_1 \xi_1/\lambda\Phi(\omega)),..., (\omega_n \xi_n/\lambda\Phi(\omega)), 0)$ and $\vec{\sigma}(r) = (r^{m/a_1}, r^{m/a_2},..., r^{m/a_n}, r^m)$.

Let $\vec{T}_1, \vec{T}_2,..., \vec{T}_{n+1}$ denote the unit tangent, principal normal, etc., to the curve $\vec{\sigma}$ (see Subsection 8.2 on local canonical form below). If ρ denotes the rotation taking $\vec{T}_1,..., \vec{T}_{n+1}$ to $\vec{e}_1,..., \vec{e}_{n+1}$, then with $\vec{V} = \rho\vec{e}_{n+1}$ we have

$$\begin{aligned}\phi_t(r) &= R(t^\beta \vec{B} + t^{1+\beta}\vec{e}_{n+1}) \cdot \vec{\sigma}(r) = R(t^\beta \rho\vec{B} + t^{1+\beta}\rho\vec{e}_{n+1}) \cdot \rho\vec{\sigma}(r) \\ &= R(t^\beta \rho\vec{B} + t^{1+\beta}\vec{V}) \cdot \vec{\Gamma}(s),\end{aligned}$$

where $\vec{\Gamma}(s)$ is the local canonical form of $\vec{\sigma}$ at some $r_0 \in \operatorname{supp} \psi$, and

$$|\vec{V} \cdot \vec{e}_1| = |\vec{e}_{n+1} \cdot \vec{T}_1| \geqslant c,$$

as a simple computation involving $\vec{\sigma}$ shows. Here $\vec{\Gamma}(s) = (\Gamma_1(s),..., \Gamma_n(s))$ satisfies

$$\Gamma_j(s) = \frac{k_1 k_2 \cdots k_{j-1}}{j!} s^j + O(s^{j+1}),$$

where k_1 is the curvature, k_2 is the torsion, etc. An elementary computation shows that the k_j are all nonzero for the curve $\vec{\sigma}(r) = (r^{m/a_1}, r^{m/a_2},..., r^{m/a_n}, r^m)$ provided $a_1 > a_2 > \cdots > a_n > 1$. See Subsection 8.2 below for these facts.

If we now set

$$\begin{aligned}P(s) &= \rho\vec{B} \cdot \vec{\Gamma}(s), \\ Q(s) &= \vec{V} \cdot \vec{\Gamma}(s),\end{aligned}$$

then the hypotheses of Theorems 18, 19, and 23 are satisfied. Indeed, $\|\vec{B}\| \geqslant C^{-1}$ implies that A in Theorem 18 is bounded below, and

$$\begin{aligned}Q(s) = \vec{V} \cdot \vec{\Gamma}(s) &= \sum_{j=1}^{n} (\vec{V} \cdot \vec{e}_j)\, \Gamma_j(s) \\ &= \sum_{j=1}^{n} (\vec{V} \cdot \vec{e}_j) \left(\frac{k_1 k_2 \cdots k_{j-1}}{j!} s^j + O(s^{j+1}) \right)\end{aligned}$$

implies $|b_1| = |\vec{V} \cdot \vec{e}_1| \geqslant c$ and $|b_k| \leqslant C$ for $k \geqslant 2$. Hypothesis (41) is a consequence of the fact that r^m is not in the linear span of the monomials $\{r^{m/a_1}, r^{m/a_2},..., r^{m/a_n}\}$ since $a_j > 1$. Thus the hypotheses of Theorem 19 hold in the case $\beta = 0$. However, the conclusion persists for $\beta > 0$ since

$$0 = \phi_t'(r(t)) = Rt^\beta \{P'(r(t)) + tQ'(r(t))\}$$

implies that

$$t = -\frac{P'(r(t))}{Q'(r(t))},$$

and

$$\frac{d}{dt} t^{-\beta}\phi_t'(r(t)) = RQ'(r(t)) + t^{-\beta}\phi_t''(r(t))\, r'(t)$$

implies that

$$\left|\frac{\phi_t''(r(t))}{R}\right| = t^{\beta}\left|\frac{Q'(r(t))}{r'(t)}\right|,$$

and the additonal factor of t^{β} causes no harm since $t \in [1, 2]$. Thus (56) of Theorem 23 applies, and after passing from $\vec{\Gamma}(s)$ back to $\vec{\sigma}(r)$, we obtain (63) as claimed.

Using Hölder's inequality, we can now dominate term I by

$$\begin{aligned}
\mathrm{I} &\leqslant \int_1^2 \left(\int_{\mathfrak{S}^{n-1}} \Phi(\omega)^{-\rho}\, d\omega\right) \int_{\mathfrak{S}^{n-1} \cap \{\Phi(\omega) < \lambda^{-m\varepsilon}\}} |H^{\alpha}(\omega, t^{\beta}\xi, t^{1+\beta}\lambda)|^2\, \Phi(\omega)^{2\alpha+\rho}\, d\omega\, dt,\\
&\leqslant C \int_{\mathfrak{S}^{n-1} \cap \{\Phi(\omega) < \lambda^{-m\varepsilon}\}} \left\{\int_1^2 |H^{\alpha}(\omega, t^{\beta}\xi, t^{1+\beta}\lambda)|^2\, dt\right\} \Phi(\omega)^{2\alpha+\rho}\, d\omega,\\
&\leqslant C \int_{\mathfrak{S}^{n-1} \cap \{\Phi(\omega) < \lambda^{-m\varepsilon}\}} (\lambda\Phi(\omega))^{-1}\, \Phi(\omega)^{2\alpha+\rho}\, d\omega, \qquad \text{by } (63),\\
&\leqslant C\lambda^{-1} \int_{\mathfrak{S}^{n-1}} (\lambda^{-m\epsilon})^{2(\alpha+\rho-(1/2))}\, \Phi(\omega)^{-\rho}\, d\omega,\\
&\leqslant C\lambda^{-1-2m\epsilon(\alpha+\rho-(1/2))} = C\lambda^{-1-\epsilon'},
\end{aligned}$$

since $\alpha + \rho > \frac{1}{2}$.

Finally, we turn to estimating term II. First we claim that

$$\int_1^2 \left| H^{\alpha}(\omega, t^{\beta}\xi, t^{1+\beta}\lambda) - \sum_k e^{i\phi_t(r_k(t))} r_k(t)^{m\alpha+n-1}\, \psi(r_k(t)) \left(\frac{2\pi i}{\phi_t''(r_k(t))}\right)^{1/2} \right|^2 dt$$
$$\leqslant C\, |\lambda\Phi(\omega)|^{-1-\epsilon}, \tag{64}$$

where $\phi_t(r) = t^{\beta} \sum_{j=1}^{n} \omega_j \xi_j r^{m/a_j} + t^{1+\beta}\lambda\Phi(\omega)\, r^m$ and $r_k(t)$ is a root of $t^{-\beta}\phi_t'(r) = \sum_{j=1}^{n} \omega_j \xi_j (m/a_j)\, r^{(m/a_j)-1} + t\lambda\Phi(\omega)\, mr^{m-1}$. Of course, $r_k(t) = r_{k,\omega}(t)$ and

$\phi_t = \phi_{t,\omega}$ are actually functions of t, ω, ξ, and λ. For this, recall from above that

$$H^\alpha(\omega, \xi, t\lambda) = \int_0^\infty e^{i\phi_t(r)} r^{n-1+m\alpha} \psi(r)\, dr,$$

where

$$\phi_t(r) = \lambda\Phi(\omega) \left\{ t^\beta \sum_{j=1}^n \left(\frac{\omega_j \xi_j}{\lambda\Phi(\omega)} \right) r^{m/a_j} + t^{1+\beta} r^m \right\}.$$

Now if $|\lambda\Phi(\omega)| \geqslant C \sum_{j=1}^n |\omega_j \xi_j|$, then

$$\left| \frac{d}{dr} \phi_t(r) \right| \geqslant c\, |\lambda\Phi(w)|$$

on supp ψ and $\psi(r_k(t)) = 0$ for all k. So integration by parts yields $|H^\alpha(\omega, t^\beta\xi, t^{1+\beta}\lambda)| \leqslant C_N(1 + |\lambda\Phi(\omega)|)^{-N}$ and since $\psi(r_k(t)) = 0$ for all k, the left side of (64) is simply

$$\int_1^2 |H^\alpha(\omega, t^\beta\xi, t^{1+\beta}\lambda)|^2\, dt \leqslant C \int_1^2 (1 + |\lambda\Phi(\omega)|)^{-2N}\, dt \leqslant C\, |\lambda\Phi(\omega)|^{-1-\epsilon}.$$

On the other hand, if $|\lambda\Phi(\omega)| \leqslant C \sum_{j=1}^n |\omega_j \xi_j|$, then as in the proof of (63) above,

$$\phi_t(r) = R\{ t^\beta \vec{B} \cdot \vec{\sigma}(r) + t^{1+\beta} \vec{e}_{n+1} \cdot \vec{\sigma}(r) \},$$

where $R = \lambda\Phi(\omega)$, $\vec{B} = ((\omega_1\xi_1/\lambda\Phi(\omega)),..., (\omega_n\xi_n/\lambda\Phi(\omega)), 0)$ and $\vec{\sigma}(r) = (r^{m/a_1}, r^{m/a_2},..., r^{m/a_n}, r^m)$. This time we also note that

$$C \leqslant \|\vec{B}\| \leqslant C\, |\Phi(\omega)|^{-1} \leqslant C\lambda^{m\epsilon}, \tag{65}$$

since $\lambda > c\,|\xi|$ and since $|\Phi(\omega)| > \lambda^{-m\epsilon}$ in the range of integration for term II.

Then with ρ, $\vec{V}$, and $\vec{\Gamma}$ as in the proof of (63), we set

$$P(s) = \rho\vec{B} \cdot \vec{\Gamma}(s),$$
$$Q(s) = \vec{V} \cdot \vec{\Gamma}(s).$$

As before, the hypotheses of Theorems 19 and 23 are satisfied. This time however, by (65) again, $A \approx \|\rho\vec{B}\| \leqslant C\lambda^{m\epsilon}$, and since $R = \lambda\Phi(\omega) \geqslant \lambda^{1-m\epsilon}$, we have

$$A \leqslant CR^{m\epsilon/(1-m\epsilon)} \leqslant CR^{(1/2)-6\epsilon}$$

for ϵ sufficiently small. Thus (58) of Theorem 23 applies, and after passing from $\vec{\Gamma}(s)$ back to $\vec{\sigma}(r)$, we obtain (64) as claimed.

We can now begin to estimate term II as follows:

$$\begin{aligned}
\mathrm{II} \leqslant & \int_1^2 \left| \int_{\{\Phi(\omega) > \lambda^{-m\epsilon}\} \cap \{|\phi''_{t,\omega}(r_{k,\omega}(t))| > |\lambda\Phi(\omega)|^{1-\epsilon}\, \forall k\}} \left\{ H^\alpha(\omega, t^\beta\xi, t^{1+\beta}\lambda) - \sum_k [\cdots] \right\} \Phi(\omega)^\alpha \, d\omega \right|^2 dt \\
& + \left| \int_{\{\Phi(\omega) > \lambda^{-m\epsilon}\} \cap \{|\phi''_{t,\omega}(r_{k,\omega}(t)| > |\lambda\Phi(\omega)|^{1-\epsilon}\, \forall k\}} \left\{ \sum_k [\cdots] \right\} \Phi(\omega)^\alpha \, d\omega \right|^2 dt \\
& + \int_1^2 \left| \int_{\{\Phi(\omega) > \lambda^{-m\epsilon}\} \cap \{|\phi''_{t,\omega}(r_{k,\omega}(t)| \leqslant |\lambda\Phi(\omega)|^{1-\epsilon}\, \text{some } k\}} H^\alpha(\omega, t^\beta\xi, t^{1+\beta}\lambda)\, \Phi(\omega)^\alpha \, d\omega \right|^2 dt \\
= & \ \mathrm{III} + \mathrm{IV} + \mathrm{V}.
\end{aligned}$$

Using (64), we can dispense with III immediately. With

$$E = \{\omega : \Phi(\omega) > \lambda^{-m\epsilon}\} \cap \{\omega : |\phi''_{t,\omega}(r_{k,\omega}(t)| > |\lambda\Phi(\omega)|^{1-\epsilon}\, \forall k\}, \tag{66}$$

we have, using Hölder's inequality,

$$\begin{aligned}
\mathrm{III} & \leqslant \int_1^2 \left(\int \Phi(\omega)^{-\rho} \, d\omega \right) \int_E \left| H^\alpha(\omega, t^\beta\xi, t^{1+\beta}\lambda) - \sum_k [\cdots] \right|^2 \Phi(\omega)^{2\alpha+\rho} \, d\omega \, dt \\
& \leqslant C \int_{\{\Phi(\omega) > \lambda^{-m\epsilon}\}} \lambda^{-1-\epsilon} \Phi(\omega)^{2(\alpha+\rho-(1/2)-(\epsilon/2))} \Phi(\omega)^{-\rho} \, d\omega \leqslant C\lambda^{-1-\epsilon},
\end{aligned}$$

by (64) with $R = |\lambda\Phi(\omega)|$ if $\alpha + \rho \geqslant \frac{1}{2} + (\epsilon/2)$.

Using (57) of Theorem 23, we can easily estimate term V as follows:

$$\begin{aligned}
\mathrm{V} & \leqslant \int_1^2 \left(\int \Phi(\omega)^{-\rho} \, d\omega \right) \int_{\{\Phi(\omega) > \lambda^{-m\epsilon}\} \cap \{|\phi''_{t,\omega}(r_{k,\omega}(t)| \leqslant |\lambda\Phi(\omega)|^{1-\epsilon}\}} |H^\alpha(\omega, t^\beta\xi, t^{1+\beta}\lambda)|^2 \\
& \quad \times \Phi(\omega)^{2\alpha+\rho} \, d\omega \, dt \\
& \leqslant \int_{\{\Phi(\omega) > \lambda^{-m\epsilon}\}} \left\{ \int_{\{t : |\phi''_{t,\omega}(r_{k,\omega}(t)| \leqslant |\lambda\Phi(\omega)|^{1-\epsilon}\}} |H^\alpha(\omega, t^\beta\xi, t^{1+\beta}\lambda)|^2 \, dt \right\} \Phi(\omega)^{2\alpha+\rho} \, d\omega \\
& \leqslant \int_{\{\Phi(\omega) > \lambda^{-m\epsilon}\}} C \, |\lambda\Phi(\omega)|^{-1-\epsilon} \, \Phi(\omega)^{2\alpha+\rho} \, d\omega
\end{aligned}$$

by (57) with $R=|\lambda\Phi(\omega)|$. Thus

$$\mathrm{V}\leqslant\int\lambda^{-1-\epsilon}\Phi(\omega)^{2(\alpha+\rho-(1/2)-(\epsilon/2))}\,\Phi(\omega)^{-\rho}\,d\omega\leqslant C\lambda^{-1-\epsilon},$$

provided $\alpha+\rho\geqslant(1/2)+(\epsilon/2)$.

So it remains to investigate

$$\mathrm{IV}=\int_1^2\left|\sum_k\int_E e^{i\phi_t(r_k(t))}r_k(t)^{n-1+m\alpha}\,\psi(r_k(t))\left(\frac{2\pi i}{\phi_t''(r_k(t))}\right)^{1/2}\Phi(\omega)^\alpha\,d\omega\right|^2 dt.$$

with E as in (66), i.e.,

$$E=\{\omega\colon\Phi(\omega)>\lambda^{-m\epsilon}\}\cap\{\omega\colon|\phi''_{t,\omega}(r_{k,\omega}(t))|>|\lambda\Phi(\omega)|^{1-\epsilon}\;\forall k\}.$$

Note that if we pass from the sphere $\tilde{S}^{n-1}$ to the level set Σ via the change of variables

$$\eta=\Phi(\omega)^{-1/a}\circ\omega\equiv\left(\frac{\omega_1}{\Phi(\omega)^{1/a_1}},\ldots,\frac{\omega_n}{\Phi(\omega)^{1/a_n}}\right),$$

and set

$$\tilde{\phi}_t(r)=t^\beta\sum_{j=1}^n\eta_j\xi_j r^{m/a_j}+t^{1+\beta}\lambda r^m,$$

then with $\tilde{r}_k(t)$ denoting a root of $\tilde{\phi}'_t$, we have

$$\begin{aligned}\phi_t(r)&=\tilde{\phi}_t(\Phi(\omega)^{1/m}r),\\ \phi'_t(r)&=\Phi(\omega)^{1/m}\,\tilde{\phi}'_t(\Phi(\omega)^{1/m}\,r),\\ \phi''_t(r)&=\Phi(\omega)^{2/m}\,\tilde{\phi}''_t(\Phi(\omega)^{1/m}\,r),\\ r_k(t)&=\Phi(\omega)^{-1/m}\,\tilde{r}_k(t),\end{aligned}\tag{67}$$

and we get

$$\begin{aligned}\mathrm{IV}&=\int_1^2\left|\sum_k\int_E e^{i\tilde{\phi}_t(\tilde{r}_k(t))}\tilde{r}_k(t)^{m\alpha+n-1}\,\psi(\Phi(\omega)^{-1/m}\,\tilde{r}_k(t))\right.\\ &\quad\times\left.\left(\frac{2\pi i}{\tilde{\phi}''_t(\tilde{r}_k(t))}\right)^{1/2}\Phi(\omega)^{-n/m}\,d\omega\right|^2 dt\\ &=\int_1^2\left|\sum_k\int_{\Sigma_{\epsilon,t}} e^{i\tilde{\phi}_t(\tilde{r}_k(t))}\tilde{r}_k(t)^{m\alpha+n-1}\,\psi(|\eta|_a\,\tilde{r}_k(t))\right.\\ &\quad\times\left.\left(\frac{2\pi i}{\tilde{\phi}''_t(\tilde{r}_k(t))}\right)^{1/2}\frac{d\sigma(\eta)}{|\nabla\Phi(\eta)|}\right|^2 dt\end{aligned}$$

where

$$|\eta|_a = \Phi(\omega)^{-1/m}, \qquad \sum_{\epsilon,\, t} = \left\{\eta \in \sum : |\eta|_a \leqslant \lambda^{\epsilon},\ |\tilde{\phi}_t''(\tilde{r}_k(t))| > \lambda^{1-\epsilon}\, |\eta|_a^{2-m(1-\epsilon)}\right\},$$

and of course $\sum$ is the level set $\{\eta \in \mathbb{R}^n : \Phi(\eta) = 1\}$.

It suffices to consider a single k in the sum above, so for convenience in notation we suppress the subscript k and the tildes over ϕ_t and r_k, but indicate the dependence of ϕ_t and r on η by writing $\phi_{t,\eta}$ and $r_\eta(t)$. Thus by expanding the square, we can rewrite term IV as

$$\begin{aligned}\int_1^2 \int_{\Sigma_{\epsilon,t}} \int_{\Sigma_{\epsilon,t}} & e^{i\{\phi_{t,\eta}(r_\eta(t)) - \phi_{t,\zeta}(r_\zeta(t))\}} (r_\eta(t)\, r_\zeta(t))^{n-1+m\alpha}\, \psi(|\eta|_a\, r_\eta(r)) \\ & \times \psi(|\zeta|_a r_\zeta(t))(2\pi i)[\phi''_{t,\eta}(r_\eta(t))\, \phi''_{t,\zeta}(r_\zeta(t))]^{-1/2} \frac{d\sigma(\eta)}{|\nabla\Phi(\eta)|} \frac{d\sigma(\zeta)}{|\nabla\Phi(\zeta)|}\, dt,\end{aligned} \tag{68}$$

since $\Phi(\omega)^{-n/m}\, d\omega = (\,d\sigma(\eta)/|\nabla\Phi(\eta)|)$ (see e.g., (28)(c) in [IoSa1]). We now observe that the phase function $\varphi(t, \eta, \zeta) = \phi_{t,\eta}(r_\eta(t)) - \phi_{t,\zeta}(r_\zeta, (t))$ in (68) satisfies

$$\begin{aligned}\frac{\partial}{\partial t}\{t^{-\beta}\varphi(t, \eta, \zeta)\} &= t^{-\beta}\phi'_{t,\eta}(r_\eta(t))\, r'_\eta(t) + \lambda r_\eta(t)^m \\ &\quad - \{t^{-\beta}\phi'_{t,\zeta}(r_\zeta(t))\, r'_\zeta(t) + \lambda r_\zeta(t)^m\} \\ &= \lambda(r_\eta(t)^m - r_\zeta(t)^m).\end{aligned}$$

At this time we invoke the following consequence of the finite-type assumption on Σ:

$$\sum_{1 \leqslant |\alpha| \leqslant \ell} \left|\frac{\partial^{|\alpha|}}{\partial y^\alpha}\{r_\eta(t)^m\}\right| \geqslant c\, |\eta|_a^{-N} \geqslant c\lambda^{-N\epsilon} \tag{69}$$

for some large N. See Lemma 25 in the subsection on finite type below for a proof of (69). It follows from (69) that

$$\sum_{1 \leqslant |\alpha| \leqslant \ell} \left|\frac{\partial^{|\alpha|}}{\partial y^\alpha} \frac{\partial}{\partial t}\{t^{-\beta}\varphi(t, \eta, \zeta)\}\right| \geqslant c\lambda^{1-N\epsilon},$$

and hence that

$$\sum_{1 \leqslant |\alpha| \leqslant \ell} \left|\frac{\partial^{|\alpha|}}{\partial y^\alpha} \frac{\partial}{\partial t} \varphi(t, \eta, \zeta)\right| + \sum_{1 \leqslant |\alpha| \leqslant \ell} \left|\frac{\partial^{|\alpha|}}{\partial y^\alpha} \varphi(t, \eta, \zeta)\right| \geqslant c\lambda^{1-N\epsilon}. \tag{70}$$

Indeed, if we set $f(t, y) = t^{-\beta}\varphi(t, \eta, \zeta)$, then

$$\frac{\partial^{|\alpha|}}{\partial y^\alpha}\frac{\partial}{\partial t}\{t^\beta f(t, y)\} = t^\beta \frac{\partial^{|\alpha|}}{\partial y^\alpha}\frac{\partial}{\partial t} f(t, y) + \beta t^{\beta-1}\frac{\partial^{|\alpha|}}{\partial y^\alpha} f(t, y).$$

Let ν be a large constant to be determined later. Let $\{\mu_j\} \subset \Sigma_\epsilon$ be a maximal collection of points whose pairwise distances apart are at least $\lambda^{-\nu\epsilon}$. Then $\bigcup_j B(\mu_j, \lambda^{-\nu\epsilon})$ covers $\Sigma_{\epsilon, t}$ where $B(\mu_j, \lambda^{-\nu\epsilon})$ denotes the ball centered at μ_j of radius $\lambda^{-\nu\epsilon}$. Let $\{\rho_j(\eta)\}_j$ denote a partition of unity subordinate to $\{B(\mu_j, 2\lambda^{-\nu\epsilon})\}_j$ and let $\{\eta_j(t)\}_j$ be a partition of unity on $[1, 2]$ with η_j supported in $[t_j - \lambda^{-\nu\epsilon}, t_j + \lambda^{-\nu\epsilon}]$. Now decompose integration over $[1, 2] \times \sum_{\epsilon, t} \times \sum_{\epsilon, t}$ in (68) into a sum of at most $\lambda^{M\epsilon}$ pieces of the form

$$\begin{aligned} L_{i, j, k} = \int_1^2 \int_{\Sigma_{\epsilon, t}} \int_{\Sigma_{\epsilon, t}} & e^{i\{\phi_{t,\eta}(r_\eta(t)) - \phi_{t,\zeta}(r_\zeta(t))\}} (r_\eta(t)\, r_\zeta(t))^{n-1+m\alpha}\, \psi(|\eta|_a r_\eta(t)) \\ & \times \psi(|\zeta|_a r_\zeta(t))(2\pi i)[\phi''_{t,\eta}(r_\eta(t))\, \phi''_{t,\zeta}(r_\zeta(t))]^{-1/2} \\ & \times \eta_i(t)\, \rho_j(\eta)\, \rho_k(\zeta) \frac{d\sigma(\eta)}{|\nabla\Phi(\eta)|} \frac{d\sigma(\zeta)}{|\nabla\Phi(\zeta)|}\, dt. \end{aligned} \tag{71}$$

We claim that

$$|\phi''_{t,\eta}(r_\eta(t))| \approx |\phi''_{t_i,\mu_j}(r_{\mu_j}(t_i))| \geqslant c\lambda^{1-c\epsilon}, \qquad \text{for} \quad \eta_i(t)\,\rho_j(\eta) \neq 0. \tag{72}$$

Indeed, from (67) we have

$$\begin{aligned} |\phi''_{t,\eta}(r_\eta(t))| &= |\phi''_{t,\omega}(r_{t,\omega}(t))\, |\Phi(\omega)^{-2/m} > |\lambda\Phi(\omega)|^{1-\epsilon}\, \Phi(\omega)^{-2/m} \\ &= \lambda^{1-\epsilon}\Phi(\omega)^{1-(2/m)-\epsilon} > \lambda^{1-\epsilon}(\lambda^{-m\epsilon})^{1-(2/m)-\epsilon} > \lambda^{1-\epsilon(m+1)} \end{aligned}$$

on the set E. Moreover, $0 \equiv \phi'_{t,\eta}(r_\eta(t))$ implies

$$0 \equiv \nabla_\eta[\phi'_{t,\eta}(r_\eta(t))] = \left(t^\beta \xi_j \frac{m}{a_j} r_\eta(t)^{(m/a_j)-1}\right)_{j=1}^n + \phi''_{t,\eta}(r_\eta(t))\, \nabla_\eta r_\eta(t),$$

which yields

$$\|\nabla_\eta r_\eta(t)\| = |\phi''_{t,\eta}(r_\eta(t))|^{-1} \left\| t^\beta \xi \frac{m}{a} r_\eta(t)^{(m/a)-1} \right\| \leqslant C\lambda^{c\epsilon} \tag{73}$$

since $|\phi''_{t,\eta}(r_\eta(t))| \geqslant c\lambda^{1-c\epsilon}$ on $\Sigma_{\epsilon,t}$ and $r_\eta(t) \approx |\eta|_a^{-1}$ since $\psi(|\eta|_a r_\eta(t)) \neq 0$. Thus we now have

$$|\nabla_\eta\{(\phi''_{t,\eta}(r_\eta(t)))^2\}| = 2\,|\phi''_{t,\eta}(r_\eta(t))| \left|\nabla_\eta\left\{\sum_{j=1}^n t^\beta \eta_j \xi_j \frac{m}{a_j}\left(\frac{m}{a_j}-1\right)\right.\right.$$

$$\left.\left.\times\, r_\eta(t)^{(m/a_j)-2} + t^{1+\beta}\lambda m(m-1)\, r_\eta(t)^{m-2}\right\}\right|$$

$$\leqslant C\lambda\,\|\lambda\,|\eta|_a^{3-m}\,\nabla_\eta r_\eta(t)\| \leqslant C\lambda^{2+c\epsilon},$$

and if we choose v much larger than c, we obtain

$$|(\phi''_{t,\eta}(r_\eta(t)))^2 - (\phi''_{t_i,\mu_j}(r_{\mu_j}(t_i)))^2|$$

$$\leqslant C\lambda^{2+c\epsilon}\lambda^{-v\epsilon} = C\lambda^{2-(v-c)\epsilon} \leqslant \tfrac{1}{2}\,|\phi''_{t_i,\mu_j}(r_{\mu_j}(t_i))|^2,$$

which yields (72).

Using the finite type condition (70),we can now invoke Proposition 5 on p. 317 of [St2] to obtain the following estimate for $L_{i,j,k}$:

$$|L_{i,j,k}| \leqslant C(c\lambda^{1-N\epsilon})^{-1/\ell}\,|\phi''_{t_i,\mu_j}(r_{\mu_j}(t_i))|^{-1/2}\,|\phi''_{t_i,\mu_k}(r_{\mu_k}(t_i))|^{-1/2}$$

$$\times (r_{\mu_j}(t_i)\, r_{\mu_k}(t_i))^{n-1+m\alpha}\,(\psi(|\mu_j|_a\, r_{\mu_j}(t_i))\,\psi(|\mu_k|_a\, r_{\mu_k}(t_i)).$$

More precisely, we use the implicit function theorem as in [IoSa1] to write Σ as the graph of a function Ψ on the support of $\eta_i(t)\,\rho_j(\eta)\,\rho_k(\zeta)$, and then use (70) to establish the hypotheses needed for the proposition in [St2]. Using (72) and (73), and then changing variables back to the sphere $\tilde{S}^{n-1}$, we obtain

$$|L_{i,j,k}| \leqslant Cc\lambda^{-(1/\ell)+N\epsilon}\int_{t_i}^{t_i+\lambda^{-v\epsilon}}\int_{Q_j} r_{j,v}(t)^{n-1+m\alpha}\,\psi(r_{j,v}(t))$$

$$\times\, |\phi''_{t,v}(r_{j,v}(t)|^{-1/2}\,\Phi(v)^\alpha\,dv \int_{Q_k} r_{k,\omega}(t)^{n-1+m\alpha}\,\psi(r_{k,\omega}(t))$$

$$\times\, |\phi''_{t,\omega}(r_{k,\omega}(t)|^{-1/2}\,\Phi(\omega)^\alpha\,d\omega\,dt$$

where Q_j is a ball containing the support of $\rho_j(v)$, and $[t_i, t_i+\lambda^{-v\epsilon}]$ contains the support of $\eta_i(t)$. Applying Hölder's inequality and using $\Phi^{-1} \in L^p(\tilde{S}^{n-1})$ yields

$$|L_{i,j,k}| \leqslant C\lambda^{-(1/\ell)+N_0\epsilon} \int_{t_i}^{t_i+\lambda^{-\nu\epsilon}} \int_{Q_j} |\phi''_{t,v}(r_{j,v}(t))|^{-1}\, \Phi(v)^{2\alpha+\rho}\, \psi(r_{k,v}(t))\, dv\, dt$$

$$+ C\lambda^{-(1/\ell)+N_0\epsilon} \int_{t_i}^{t_i+\lambda^{-\nu\epsilon}} \int_{Q_k} |\phi''_{t,\omega}(r_{k,\omega}(t))|^{-1}\, \Phi(\omega)^{2\alpha+\rho}$$

$$\times \psi(r_{k,\omega}(t))\, d\omega\, dt.$$

Finally, summing in i, j, k gives

$$\mathrm{IV} \leqslant \sum_{i,j,k} |L_{i,j,k}| \leqslant C\lambda^{-(1/\ell)+N_0\epsilon} \sum_{i,j,k} \int_{t_i}^{t_i+\lambda^{-\nu\epsilon}} \int_{Q_k} |\phi''_{t,\omega}(r_{k,\omega}(t))|^{-1}$$

$$\times \Phi(\omega)^{2\alpha+\rho}\, \psi(r_{k,\omega}(t))\, d\omega\, dt$$

$$= C\lambda^{-1-(1/\ell)+N_1\epsilon} \sum_k \int_1^2 \int_{Q_k} \left| \frac{\phi''_{t,\omega}(r_{k,\omega}(t))}{\lambda\Phi(\omega)} \right|^{-1}$$

$$\times \Phi(\omega)^{2\alpha+\rho-1}\, \psi(r_{k,\omega}(t))\, d\omega\, dt$$

$$\leqslant C\lambda^{-1-(1/\ell)+N_1\epsilon} \sum_k \int_{Q_k} \Phi(\omega)^{2\alpha+\rho-1}\, d\omega,$$

by Theorem 18,

$$\leqslant C\lambda^{-1-(1/\ell)+N_2\epsilon}, \qquad \text{if} \quad \alpha+\rho>\tfrac{1}{2}.$$

Thus we obtain $\mathrm{IV} \leqslant C\lambda^{-1-\epsilon'}$ as required if we choose ϵ so small that $N_2\epsilon < (1/\ell)$. This completes the proof of Theorem 24 (equivalently, of Theorem 7).

8.1. *Finite Type*

In this subsection, we prove the result on finite type which implies the crucial inequality (69) above.

LEMMA 25. *Let $r_\eta(t)$ be as above and set $s_\eta(t) = (r_\eta(t))^m$. If Σ is finite type ℓ with constant $\delta > 0$, i.e.,*

$$\sum_{1 \leqslant |\alpha| \leqslant \ell} \left| \frac{\partial^{|\alpha|}}{\partial y^\alpha} \vec{v} \cdot \eta \right| \geqslant \delta,$$

for all unit vectors $\vec{v}$, then

$$\sum_{1 \leqslant |\alpha| \leqslant \ell} \left| \frac{\partial^{|\alpha|}}{\partial y^\alpha} s_\eta(t) \right| \geqslant c\delta.$$

Proof. If $\sum_{1 \leqslant |\alpha| \leqslant \ell} |(\partial^{|\alpha|}/\partial y^{\alpha})\, s_\eta(t)| < c\delta$, then (see p. 343 of [St3]),

$$\sum_{k=1}^{\ell} \left| \left(\frac{\partial}{\partial y}\right)^k s_\eta(t) \right| < c\delta, \qquad \text{for all directions } \; y.$$

Now we have (after dividing out by t^β),

$$0 \equiv \sum_{j=1}^{n} \xi_j \eta_j \frac{m}{a_j} r_\eta(t)^{m/a_j} + \lambda t m r_\eta(t)^m = \sum_{j=1}^{n} \xi_j \eta_j \frac{m}{a_j} s_\eta(t)^{1/a_j} + \lambda t m s_\eta(t),$$

and so

$$\begin{aligned} 0 &\equiv \left(\frac{\partial}{\partial y}\right)^k \left\{ \sum_{j=1}^{n} \xi_j \eta_j \frac{m}{a_j} s_\eta(t)^{1/a_j} + \lambda t m s_\eta(t) \right\} \\ &= \sum_{j=1}^{n} \xi_j \frac{m}{a_j} \left(\sum_{\sigma+\tau=k} \left(\frac{\partial}{\partial y}\right)^\sigma \eta_j \left(\frac{\partial}{\partial y}\right)^\tau s_\eta(t)^{1/a_j} \right) + \lambda m \left(\frac{\partial}{\partial y}\right)^k s_\eta(t) \\ &= \sum_{j=1}^{n} \xi_j \frac{m}{a_j} \left\{ \left(\frac{\partial}{\partial y}\right)^k \eta_j \right\} s_\eta(t)^{1/a_j} + O(c\delta\, |\xi|), \end{aligned}$$

upon noting that $s_\eta(t) \in \operatorname{supp} \psi$, and so is bounded away from zero. Thus we can write

$$\begin{aligned} 0 &= \sum_{j=1}^{n} w_j \left(\frac{\partial}{\partial y}\right)^k \eta_j + O(c\delta\, |\xi|) = \left(\frac{\partial}{\partial y}\right)^k (\vec{w} \cdot \eta) + O(c\delta\, |\xi|) \\ &= \|w\| \left(\frac{\partial}{\partial y}\right)^k (\vec{v} \cdot \eta) + O(c\delta |\xi|), \end{aligned}$$

where $w_j = \xi_j (m/a_j)\, s_\eta(t)^{1/a_j}$ and $\vec{v} = \vec{w}/\|\vec{w}\|$ is a unit vector and $\|\vec{w}\| \approx |\xi|$. Thus

$$\sum_{k=1}^{\ell} \left| \left(\frac{\partial}{\partial y}\right)^k (\vec{v} \cdot \eta) \right| \leqslant \frac{C}{\|\vec{w}\|} c\delta\, |\xi| \leqslant C\, c\delta < \delta,$$

for c sufficiently small, contradicting the hypothesis since, once again by p. 343 in [St3],

$$\sum_{1 \leqslant |\alpha| \leqslant \ell} \left| \frac{\partial^{|\alpha|}}{\partial y^\alpha} \vec{v} \cdot \eta \right| \approx \sup \sum_{k=1}^{\ell} \left| \left(\frac{\partial}{\partial y}\right)^k (\vec{v} \cdot \eta) \right|$$

where the sup is taken over all directions y. This completes the proof of the lemma.

8.2. *Local canonical form*

The purpose of this subsection is to briefly review the local canonical form as applied to curves of the form $\vec{\alpha}(r) = (r^{b_1}, r^{b_2}, \ldots, r^{b_n})$, where $b_1 > b_2 > \cdots > b_n > 0$. But first we begin with an arbitrary smooth curve $\vec{\sigma_0}(s)$, parameterized by arc length s, so that

$$k_0 = \|\vec{\sigma_0}'(s)\| = 1.$$

Now define

$$\vec{\sigma_1} = \frac{1}{k_0} \vec{\sigma_0}'$$

and

$$k_1 = \|\vec{\sigma_1}'\|.$$

Now $\vec{\sigma_1} \cdot \vec{\sigma_1} = 1$ implies $\vec{\sigma_1}' \cdot \vec{\sigma_1} = 0$, so that if $k_1 \neq 0$ we can define

$$\vec{\sigma_2} = \frac{1}{k_1} \vec{\sigma_1}',$$

with the result that $\{\vec{\sigma_1}, \vec{\sigma_2}\}$ is an orthonormal set.

Now $\vec{\sigma_2} \cdot \vec{\sigma_2} = 1$ implies $\vec{\sigma_2}' \cdot \vec{\sigma_2} = 0$, and $\vec{\sigma_2} \cdot \vec{\sigma_1} = 0$ implies $\vec{\sigma_2}' \cdot \vec{\sigma_1} = -\vec{\sigma_2} \cdot \vec{\sigma_1}' = -k_1$, so that $\vec{\sigma_2}' + k_1 \vec{\sigma_1}$ is perpendicular to both $\vec{\sigma_1}$ and $\vec{\sigma_2}$. Thus if

$$k_2 = \|\vec{\sigma_2}' + k_1 \vec{\sigma_1}\|$$

is nonzero, we can define

$$\vec{\sigma_3} = \frac{1}{k_2} (\vec{\sigma_2}' + k_1 \vec{\sigma_1}),$$

so that $\{\vec{\sigma_1}, \vec{\sigma_2}, \vec{\sigma_3}\}$ is an orthonormal set.

Continuing in this way, if we have already defined $k_1, k_2, \ldots, k_{m-1}$ and $\{\vec{\sigma_1}, \vec{\sigma_2}, \ldots, \vec{\sigma_m}\}$ by

$$k_j = \|\vec{\sigma_j}' + k_{j-1} \overrightarrow{\sigma_{j-1}}\|, \qquad 2 \leqslant j < m,$$

$$\overrightarrow{\sigma_{j+1}} = \frac{1}{k_j} (\vec{\sigma_j}' + k_{j-1} \overrightarrow{\sigma_{j-1}}), \qquad 2 \leqslant j < m,$$

so that $\{\vec{\sigma_1}, \vec{\sigma_2}, \ldots, \vec{\sigma_m}\}$ is orthonormal, then $\vec{\sigma_m} \cdot \vec{\sigma_m} = 1$ implies $\vec{\sigma_m}' \cdot \vec{\sigma_m} = 0$, and $\vec{\sigma_m} \cdot \vec{\sigma_j} = 0$ implies $\vec{\sigma_m}' \cdot \vec{\sigma_j} = -\vec{\sigma_m} \cdot \vec{\sigma_j}' = -\vec{\sigma_m} \cdot (k_j \overrightarrow{\sigma_{j+1}} - k_{j-1} \overrightarrow{\sigma_{j-1}})$, which vanishes for $j<m-1$, and equals $-k_{m-1}$ for $j=m-1$. Thus if we define

$$k_m = \|\overrightarrow{\sigma_m}' + k_{m-1}\overrightarrow{\sigma_{m-1}}\|,$$

$$\overrightarrow{\sigma_{m+1}} = \frac{1}{k_m}(\overrightarrow{\sigma_m}' + k_{m-1}\overrightarrow{\sigma_{m-1}}),$$

we obtain that $\{\overrightarrow{\sigma_1}, \overrightarrow{\sigma_2}, ..., \overrightarrow{\sigma_{m+1}}\}$ is orthonormal.

The curve $\overrightarrow{\sigma_0}$ in $\mathbb{R}^n$ is said to be nondegenerate provided $k_1 k_2 \cdots k_{n-1} \neq 0$, and if we fix s_0 and then translate, flip, and rotate coordinates so that $\overrightarrow{\sigma_1}(s_0), \overrightarrow{\sigma_2}(s_0), ..., \overrightarrow{\sigma_n}(s_0)$ are the coordinate directions $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_n}$ with origin $\overrightarrow{\sigma_0}(s_0)$, then $\overrightarrow{\sigma_0}$ expressed in these new coordinates is referred to as the local canonical form of $\overrightarrow{\sigma_0}$ at s_0. Note that k_1 is the curvature and k_2 is the torsion of $\overrightarrow{\sigma_0}$. If we expand $\overrightarrow{\sigma_0}$ by Taylor's formula in the new coordinates t, the local canonical form is essentially $(t, (k_1/2)\,t^2, (k_1 k_2/6)\,t^3, ..., ((k_1 k_2 \cdots k_{n-1})/n!)\,t^n)$ plus higher order terms.

We turn now to the special curves $\vec{\sigma}(r) = ((r^{b_1+1}/(b_1+1)), (r^{b_2+1}/(b_2+1)), ..., (r^{b_n+1}/(b_n+1)))$, where $b_1 > b_2 > \cdots > b_n > -1$. Then $\vec{\sigma}'(r) = (r^{b_1}, r^{b_2}, ..., r^{b_n})$ which yields

$$\overrightarrow{\sigma_1} = \frac{\vec{\sigma}'(r)}{ds/dr} = \frac{(r^{b_1}, r^{b_2}, ..., r^{b_n})}{\sqrt{r^{2b_1} + r^{2b_2} + \cdots + r^{2b_n}}}.$$

Continuing, we compute

$$\frac{d}{ds}\overrightarrow{\sigma_1} = \frac{(d/dr)\,\overrightarrow{\sigma_1}}{ds/dr} = \frac{(b_1 r^{b_1-1}, ..., b_n r^{b_n-1})}{r^{2b_1} + \cdots + r^{2b_n}}$$
$$- \frac{(2b_1 r^{2b_1-1} + \cdots + 2b_n r^{2b_n-1})}{2(r^{2b_1} + \cdots + r^{2b_n})^2}(r^{b_1}, r^{b_2}, ..., r^{b_n}),$$

which has first and last components

$$\frac{(b_1 - b_2)\, r^{b_1+2b_2-1} + \cdots + (b_1 - b_n)\, r^{b_1+2b_n-1}}{(r^{2b_1} + \cdots + r^{2b_n})^2},$$

and

$$\frac{(b_n - b_1)\, r^{b_n+2b_1-1} + \cdots + (b_n - b_{n-1})\, r^{b_n+2b_{n-1}-1}}{(r^{2b_1} + \cdots + r^{2b_n})^2},$$

respectively. Since $b_1 > b_2 > \cdots > b_n$, we see that these components are nonvanishing on $(0, \infty)$. Thus the first and last components of $\overrightarrow{\sigma_2}$ are nonvanishing, and similar calculations show that the same holds for $\overrightarrow{\sigma_3}, ..., \overrightarrow{\sigma_n}$.

We conclude that the curve $\bar{\sigma}(r) = ((r^{b_1+1}/(b_1+1)), (r^{b_2+1}/(b_2+1)), ..., (r^{b_n+1}/(b_n+1)))$ is nondegenerate on $(0, \infty)$. Note that if two of the exponents were equal, then $\bar{\sigma}$ would be contained in a hyperplane and thus be degenerate.

8.3. *Parametric Surfaces*

In this last subsection, we turn to proving the main result for $\mathscr{M}'$ on parametric surfaces of codimension 1 and 2, which we recall for convenience in the setting of $\mathbb{R}^{n+1}$.

THEOREM 26. *Suppose an ℓ-dimensional surface S, $\ell = n$ or $n-1$ (but $\ell \geqslant 2$), is given parametrically as*

$$S = \{(\Phi_1(x), ..., \Phi_n(x), \Phi(x) + c_0) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^\ell\}, \tag{74}$$

where $\Phi(x)$ is P-homogeneous of degree m, and $\Phi_j(x)$ is P-homogeneous of degree $m_j \neq m$. Suppose further that

(i) *There is $0 < \rho \leqslant \min\{(\ell/m), \frac{1}{2}\}$ such that*

$$\Phi(\omega)^{-1} \in L^\rho(\mathbb{S}^{\ell-1}).$$

(ii) *The image of*

$$\sum = \{x : \Phi(x) = 1\}$$

under the map $\mathscr{R}$ is of finite type with polynomial bounds.

(iii) *For each $v \in \mathbb{S}^{n-1}$,*

$$\operatorname{rank}\left[\frac{\partial^2}{\partial x_i \partial x_j}\left\{\sum_{k=1}^n v_k \Phi_k(x)\right\}\right]_{1 \leqslant i, j \leqslant \ell} \geqslant 2$$

whenever

$$\nabla_x \left\{\sum_{k=1}^n v_k \Phi_k(x)\right\} = 0.$$

Let $\mathscr{M}'$ be defined as in (7) above with $0 \leqslant \beta_1 = \beta_2 = \cdots = \beta_n < \beta_{n+1}$. Then $\mathscr{M}'$ is bounded on $L^p(\mathbb{R}^{n+1})$, i.e., (8) holds, for $p > (1/\rho)$. Moreover, the constant C_p in (8) is at most $C_p'(1 + |c_0|)^{1/p}$ (in the case $\beta_1 = 0$, our proof yields an additional factor $\log(1 + |c_0|)$).

Proof. Just as in the proof of Theorem 7, the mixed homogeneity of the functions $\Phi_1, ..., \Phi_n, \Phi$ permits a rescaling argument and reduction to the case where the cutoff function $\psi(r)$ is supported in $[\frac{1}{2}, 2]$. This time, however, the scaling factor is $2^{(km/p)-\ell k}$, and the terms sum provided $p > \ell/m$, accounting for the restriction $0 < p \leqslant \min\{(\ell/m), \frac{1}{2}\}$ in (i). With $\{T_r\}_{r>0}$ denoting the dilation group, let $\|x\| = \inf\{t > 0: T_{t^{-1}}x \in \mathbb{S}^{\ell-1}\}$. Then the "polar coordinate" change of variable $x = T_r\omega$ where $r = x$ and $\omega \in \mathbb{S}^{\ell-1}$ has Jacobian $cr^{\ell-1} dr d\omega$ (since trace P is ℓ) and results in the following formula for $\widehat{\tau^\alpha}$:

$$\widehat{\tau^\alpha}(\xi, \lambda) = \int_{\mathbb{R}^\ell} e^{i\{\mathscr{R}(x)\cdot\xi + \lambda\Phi(x)\}}\psi(x)\,\Phi(x)^\alpha\,dx$$

$$= \int_{\mathbb{S}^{\ell-1}} \left[\int_0^\infty e^{i\{\mathscr{R}(T_r\omega)\cdot\xi + \lambda\Phi(\omega)r^m\}} r^{\ell-1+m\alpha}\psi(r)\,dr\right] \Phi(\omega)^\alpha\,d\omega.$$

In the case $|\lambda| \geqslant c|\xi|$, we verify (61) as before, using the fact that $\mathscr{R}(T_r\omega)\cdot\xi = \sum_{j=1}^n \Phi_j(\omega)\,\xi_j r^{m_j}$ and r^m is not in the linear span of the r^{m_j} since $m_j \neq m$. Hypothesis (i) enters exactly as before, while hypothesis (ii) enters in proving the finite type condition (69) for the roots $r_\eta(t)^m$. More precisely, we compute that for the new phase function

$$\begin{aligned}\tilde{\phi}_t(r) &= t^\beta \xi\cdot\mathscr{R}(T_r\eta) + t^{1+\beta}\lambda r^m \\ &= t^\beta\xi\cdot[r^{m_1}\Phi_1(\eta),..., r^{m_n}\Phi_n(\eta)] + t^{1+\beta}\lambda r^m,\end{aligned}$$

we have

$$\begin{aligned}t^{-\beta} r\tilde{\phi}'_t(r) &= \xi\cdot[m_1 r^{m_1}\Phi_1(\eta),..., m_n r^{m_n}\Phi_n(\eta)]\,PT_r\eta + mt\lambda r^m, \\ &= [m_1 r^{m_1}\xi_1,..., m_n r^{m_n}\xi_n]\cdot\mathscr{R}(\eta) + mt\lambda r^m. \end{aligned} \tag{75}$$

Denote by $F(y, r)$ the function on the right side of (75) but with η replaced by a coordinate patch $y \in \mathbb{R}^{n-1}$ (as in [IoSa1]), and similarly let $r(y)$ denote a smoothly varying root, i.e., $F(y, r(y)) = 0$. Now (69) follows easily from the chain rule as in the proof of Lemma 25 above, if we assume that $F(y, r)$ is of finite type in the y-variable, uniformly in r. Hypothesis (ii) is precisely what is needed for this.

Finally, hypothesis (iii) is used to obtain decay of order -1 for $\widehat{\tau^\alpha}$ in the case $|\lambda| \leqslant c\,|\xi|$, for c sufficiently small. Indeed, it suffices to verify the rank condition for $\sum_{k=1}^n \nu_k\Phi_k(x) + \epsilon\Phi(x)$ in place of $\sum_{k=1}^n \nu_k\Phi_k(x)$, and a continuity argument establishes this for sufficiently small $\epsilon > 0$.

We end the paper with an example of a parametric surface S of codimension 1 to which Theorem 9 (or more precisely Remark 3) applies, so that $\mathscr{M}'$ is bounded on L^2, yet the decay of the Fourier transform of surface-carried measure is strictly worse than $-\frac{1}{2}$. This is in contrast to our conjecture that if S is the graph of a mixed homogeneous function satisfying the hypotheses of Theorem 9, then the decay of the Fourier transform of surface-carried measure is $-\frac{1}{2}-\epsilon$, and $\mathscr{M}$ is bounded on L^2.

EXAMPLE. We construct a three-dimensional surface S in $\mathbb{R}^4$ given parametrically as

$$S=\{(\Phi_1(x), \Phi_2(x), \Phi_3(x), 1+\Phi_4(x))\in\mathbb{R}^4: x\in\mathbb{R}^3, \tfrac{1}{2}\leqslant |x|\leqslant 2\},$$

where Φ_1, Φ_2, Φ_3, and Φ_4 are homogeneous in the usual sense of degrees 1, 2, 3, and 4 respectively. If we let $x=r\omega=r(\omega_1, \omega_2, \omega_3)$ where $\omega_3=\sqrt{1-\omega_1^2-\omega_2^2}$, and if A is a large positive constant, then we define for ω_1, ω_2 small

$$\begin{aligned}
\Phi_1(r\omega)&=(8A+8\omega_1^k-\omega_2)r,\\
\Phi_2(r\omega)&=(-6A-6\omega_1^k+3\omega_2)r^2,\\
\Phi_3(r\omega)&=(-3\omega_2)r^3,\\
\Phi(r\omega)&=(A+\omega_1^k+\omega_2)r^4.
\end{aligned}\tag{76}$$

To see how these definitions arise, consider the phase function

$$\begin{aligned}
\phi_t&=\xi\cdot\mathscr{R}(x)+\lambda t\Phi(x)\\
&=\xi_1\Phi_1(\omega)\,r+\xi_2\Phi_2(\omega)\,r^2+\xi_3\Phi_3(\omega)\,r^3+\lambda t\Phi(\omega)\,r^4.
\end{aligned}$$

Set $t=1$ and $\xi_1=\xi_2=\xi_3=\lambda$ so that

$$\phi_1(r)=\lambda[\Phi_1(\omega)\,r+\Phi_2(\omega)\,r^2+\Phi_3(\omega)\,r^3+\Phi(\omega)\,r^4].$$

Now $r=1$ is a repeated critical point of $\phi_1(r)$ if and only if both

$$\begin{aligned}
0&=\phi_1'(1)=\lambda[\Phi_1(\omega)+2\Phi_2(\omega)+3\Phi_3(\omega)+4\Phi(\omega)],\\
0&=\phi_1''(1)=\lambda[2\Phi_2(\omega)+6\Phi_3(\omega)+12\Phi(\omega)],
\end{aligned}$$

in other words,

$$\begin{aligned}
\Phi_1(\omega)&=3\Phi_3(\omega)+8\Phi(\omega),\\
\Phi_2(\omega)&=-3\Phi_3(\omega)-6\Phi(\omega).
\end{aligned}$$

Note also that

$$\phi_1(1)=\lambda[\Phi_3(\omega)+3\Phi(\omega)]$$

and

$$\phi'''_1(1) = \lambda[6\Phi_3(\omega) + 24\Phi(\omega)],$$

so that by stationary phase,

$$\begin{aligned}
&\int e^{i\{\xi \cdot \mathscr{R}(x) + \lambda\Phi(x)\}}\psi(|x|)\,dx \\
&= \int_{\mathbb{S}^2}\int_0^\infty e^{i\phi_1(r)}\psi(r)\,r^2\,dr\,d\omega \\
&\approx \int_{\mathbb{S}^2} e^{i\phi_1(1)}\psi(1)[\phi'''_1(1)]^{-1/3}\,d\omega \\
&= [6\lambda]^{-1/3}\int_{\mathbb{S}^2} e^{i[\Phi_3(\omega)+3\Phi(\omega)]}[\Phi_3(\omega)+4\Phi(\omega)]^{-1/3}\,d\omega,
\end{aligned}$$

provided $r = 1$ is a repeated critical point of $\phi_1(r)$. With the choices made above in (76), this is indeed the case and we have

$$\phi_1(1) = \lambda[\Phi_3(\omega) + 3\Phi(\omega)] = \lambda[3A + 3\omega_1^k],$$

and so

$$\begin{aligned}
&\int e^{i\{\xi \cdot \mathscr{R}(x) + \lambda\Phi(x)\}}\,\psi(|x|)\,dx \\
&\approx [6\lambda]^{-1/3}\int_{\mathbb{S}^2} e^{i\lambda[3A+3\omega^{k}_1]}[4A + 4\omega_1^k + \omega_2]^{-1/3}\,d\omega \\
&\approx C\lambda^{-1/3}\int_{\mathbb{S}^2} e^{i\lambda[3A+3\omega^{k}_1]}d\omega \\
&\approx C\lambda^{-(1/3)-(1/k)}.
\end{aligned}$$

Thus decay of the Fourier transform of surface-carried measure is strictly worse than $-\frac{1}{2}$ provided we choose $k > 6$.

On the other hand, we now verify the hypotheses of Theorem 9, or rather the weakened form in Remark 3. Clearly (i) holds since $\Phi(\omega)^{-1} \in L^p(\mathbb{S}^2)$ for all $p > 0$ if A is large enough. As for (ii), we have

$$\Sigma = \{x : \Phi(x) = 1\} = \{x = r\omega : A + \omega_1^k + \omega_2 = r^{-4}\}.$$

Now the map $\mathscr{R} = (\Phi_1, \Phi_2, \Phi_3)$ takes spheres centered at the origin into planes since for each fixed r, $\Phi_3(r\omega)$ is a linear combination of $\Phi_1(r\omega)$ and

$\Phi_2(r\omega)$. It follows that Σ is then mapped into a surface of finite type, so (ii) holds.

Finally, we show that the weakened hypothesis (iii) in Remark 3 holds: there is $\epsilon > 0$ such that $|\hat{\sigma}(\xi', \xi_4)| \leqslant C|\xi|^{-(1/2)-\epsilon}$ for ξ in the cone $\mathscr{C} = \{(\xi', \xi_4) \in \mathbb{R}^4 : |\xi_4| \leqslant c|\xi'|\}$ for c small. We do this by first showing that

$$\operatorname{rank}\left[\frac{\partial^2}{\partial x_i \partial x_j}\left\{\sum_{k=1}^{3} v_k \Phi_k(x)\right\}\right]_{1 \leqslant i, j \leqslant 3} \geqslant 1 \tag{77}$$

whenever

$$\nabla_x \left\{\sum_{k=1}^{3} v_k \Phi_k(x)\right\} = 0.$$

For this, we write

$$\begin{aligned} v \cdot \mathscr{R}(x) &= \sum_{k=1}^{3} v_k \Phi_k(x) \\ &= (8v_1 r - 6v_2 r^2)(A + \omega_1^k) + (-v_1 r + 3v_2 r^2 - 3v_3 r^3)\,\omega_2. \end{aligned}$$

We now assume, in order to derive a contradiction, that both $\nabla(v \cdot \mathscr{R}(x))$ and $\nabla^2(v \cdot \mathscr{R}(x))$ vanish at some point, which we assume occurs when $r = 1$ (the general case is similar). A calculation yields

$$\begin{aligned} 0 &= \frac{\partial}{\partial \omega_1}(v \cdot \mathscr{R}(x))|_{r=1} = 2(4v_1 - 3v_2)\,k\omega_1^{k-1} \\ 0 &= \frac{\partial}{\partial \omega_2}(v \cdot \mathscr{R}(x))|_{r=1} = -v_1 + 3v_2 - 3v_3 \\ 0 &= \frac{\partial^2}{\partial r^2}(v \cdot \mathscr{R}(x))|_{r=1} = 6v_2(-2A - 2\omega_1^k + \omega_2) - 18v_3\omega_2 \\ 0 &= \frac{\partial^2}{\partial r \partial \omega_2}(v \cdot \mathscr{R}(x))|_{r=1} = -v_1 + 6v_2 - 9v_3. \end{aligned} \tag{78}$$

The second and fourth equations in (78) yield

$$v_1 = \tfrac{3}{2} v_2, \qquad v_3 = \tfrac{1}{2} v_2. \tag{79}$$

We consider the cases $\omega_1 \neq 0$ and $\omega_1 = 0$ separately. If $\omega_1 \neq 0$, then the first equation in (78) yields $v_1 = \frac{3}{4} v_2$, which together with (79) implies that $\nu=(0,0,0)$, a contradiction. If, on the other hand, $\omega_1=0$, then the third equation in (78) yields

$$0=6\nu_2(-2A+\omega_2)-18\nu_3\omega_2,$$

which together with (79) implies that

$$\omega_2=\frac{12\nu_2 A}{6(\nu_2-3\nu_3)}=\frac{2\nu_2 A}{-(1/2)\nu_2}=-4A,$$

which is a contradiction for $A>\frac{1}{4}$. This establishes our assertion regarding (77), and this guarantees a decay of at least $-\frac{1}{2}$. The additional $-\epsilon$ decay arises from the fact that there is finite type in the remaining directions.

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