The Erdős distinct distance problem

Eyvindur Ari Palsson

Department of Mathematics
Virginia Tech

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CoronaVirus Lecture Series
University of Rochester

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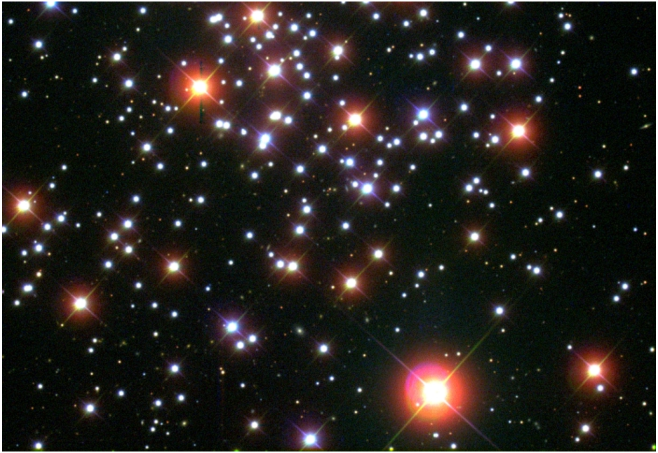

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by Julia Garibaldi (Author), Alex Iosevich (Author), Steven Senger (Author)

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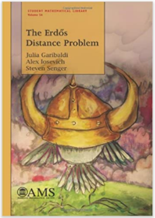

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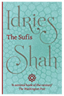

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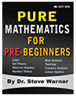

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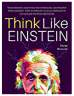

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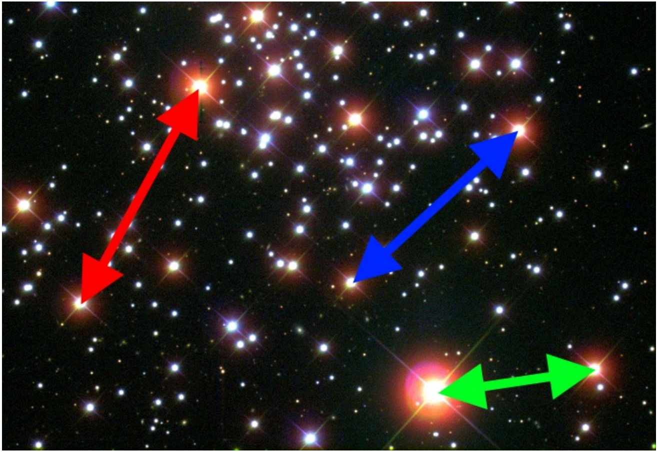
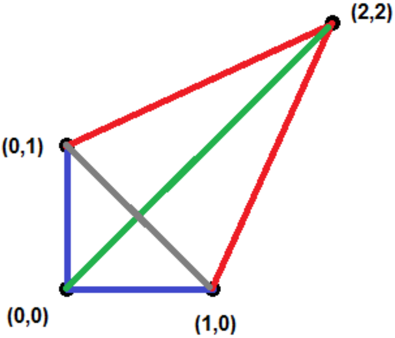
Distinct patterns



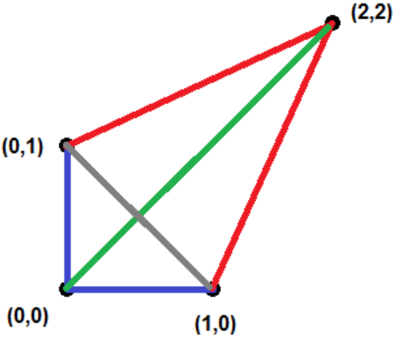
- How many distinct distances are there?

Example



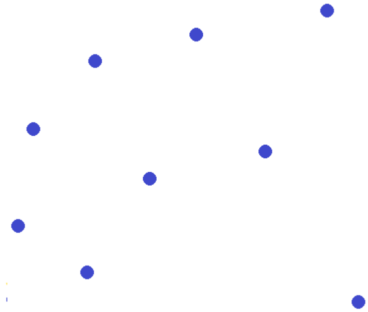
- Distances: 1, 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{5}$, $\sqrt{8}$
- Distinct distances: 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8}$

Example



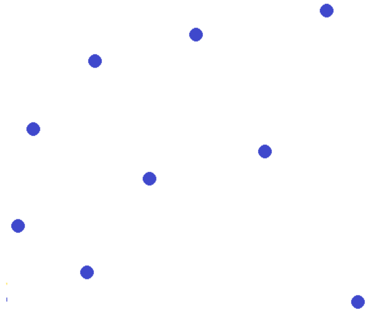
- Distances: 1, 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{5}$, $\sqrt{8}$
- Distinct distances: 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8}$
- How many distinct distances are there in general?

Least upper bounds



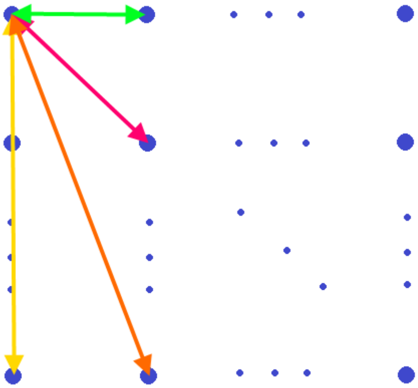
- N points in the plane.
- Upper bound $\binom{N}{2} = \frac{N(N-1)}{2} \sim N^2$.

Least upper bounds



- N points in the plane.
- Upper bound $\binom{N}{2} = \frac{N(N-1)}{2} \sim N^2$.
- If randomly selected obtain $\binom{N}{2} \sim N^2$.

Greatest lower bounds on a $\sqrt{N} \times \sqrt{N}$ lattice?



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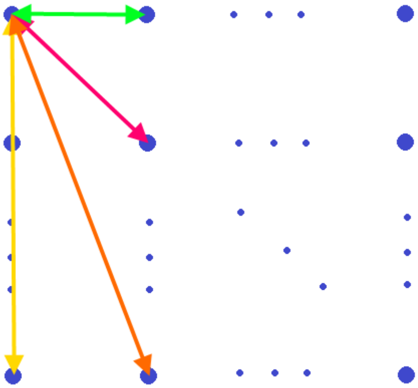
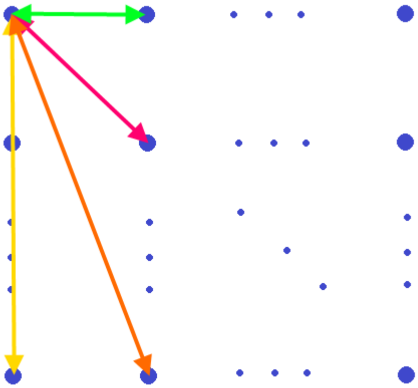


- Distinct distances squared start out with $1^2 = 1$, $(\sqrt{2})^2 = 2$ and go up to $(\sqrt{N})^2 + (\sqrt{N})^2 = 2N$.

Greatest lower bounds on a $\sqrt{N} \times \sqrt{N}$ lattice?



- Distinct distances squared start out with $1^2 = 1$, $(\sqrt{2})^2 = 2$ and go up to $(\sqrt{N})^2 + (\sqrt{N})^2 = 2N$.
- Asymptotically, up to constants, $\frac{N}{\sqrt{\log(N)}}$ distinct distances.

Erdős distinct distance problem (1946)

- What is the least number of distinct distances determined by N points in the plane?
- Conjecture $\frac{N}{\sqrt{\log(N)}}$ as $N \to \infty$.

First result by Erdős

Theorem (Erdős)

Given N points in the plane there exists a point that determines at least $\sqrt{N}$ distinct distances.

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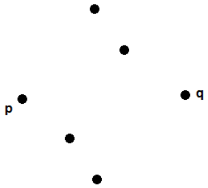
Proof.



- Consider points p, q

First result by Erdős

Theorem (Erdős)

Given N points in the plane there exists a point that determines at least $\sqrt{N}$ distinct distances.

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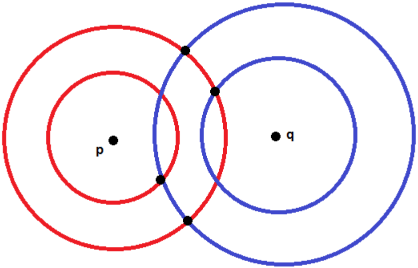
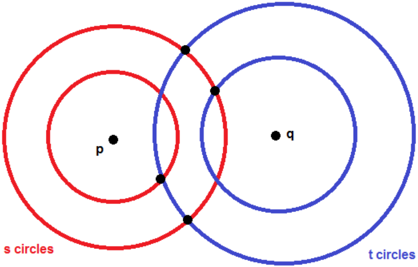


- s number of circles around p, t number of circles around q.

First result by Erdős

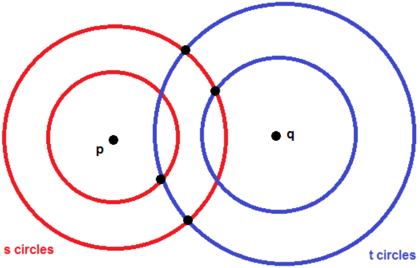
Proof.



- All points except p and q on intersection of circles.

First result by Erdős

Proof.



- All points except p and q on intersection of circles.
- Circles intersect in at most 2 points so at most $2st$ intersections.

First result by Erdős

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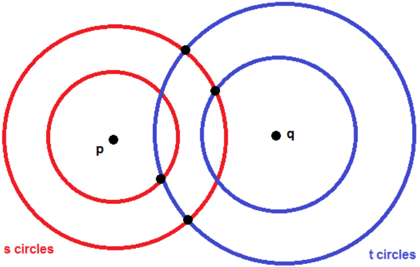
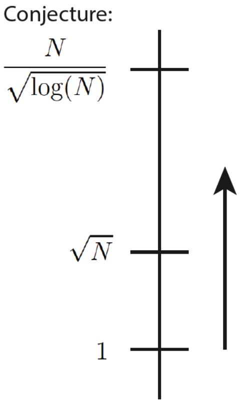


- All points except p and q on intersection of circles.
- Circles intersect in at most 2 points so at most $2st$ intersections.
- Thus $2st \geq N - 2$ so the bigger of s and t is at least $\sqrt{\frac{N-2}{2}}$.

Progress on the conjecture in the plane

- Have $\frac{N}{\sqrt{\log(N)}}$ for the lattice. Is it possible to have fewer distinct distances?

- $\frac{N}{\log(N)}$ (Guth, Katz 2010)
- $N^{0.864...}$ (Katz, Tardos 2004)
- $N^{0.8634...}$ (Tardos 2003)
- $N^{0.8571}$ (Solymosi, Toth 2001)
- $N^{0.8}$ (Szekely 1993)
- $\frac{N^{0.8}}{\log(N)}$ (Chung, Szemeredi, Trotter 1992)
- $N^{0.7143...}$ (Chung 1984)
- $N^{0.66...}$ (Moser 1952)
- $N^{0.5}$ (Erdős 1946)



Conjecture and results in higher dimensions

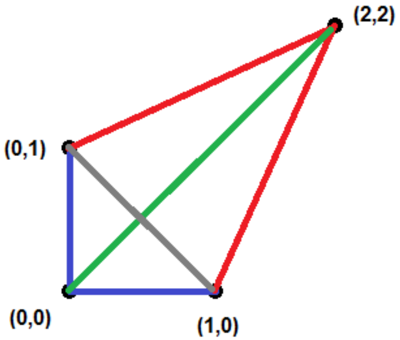
- For dimensions $d \geq 3$ the conjecture is $N^{\frac{2}{d}}$.
- Best result in $\mathbb{R}^3$ is $N^{\frac{3}{5}}$ by Solymosi and Vu from 2008.
- Best result in $\mathbb{R}^d$, $d \geq 4$, is
  - $N^{\frac{2d+2}{d^2+2d-2}}$ if d even.
  - $N^{\frac{2d+2}{d^2+2d-5/3}}$ if d odd.

by Solymosi and Vu from 2008.

Quote by Erdős

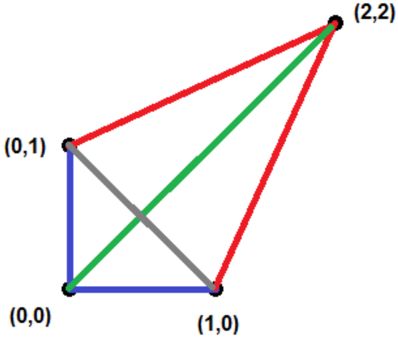
"My most striking contribution to geometry is, no doubt, my problem on the number of distinct distances. This can be found in many of my papers on combinatorial and geometric problems."

Motivating example



- Distances: 1, 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{5}$, $\sqrt{8}$
- Unit distances: 1, 1

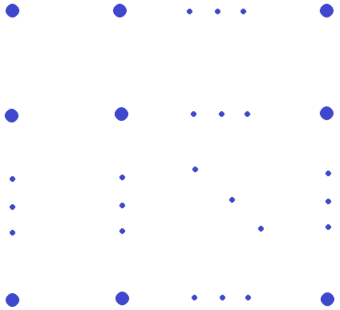
Motivating example



- Distances: 1, 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{5}$, $\sqrt{8}$
- Unit distances: 1, 1
- How many unit distances are there in general?

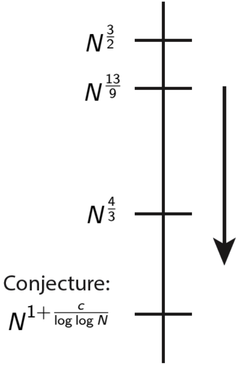
Erdős unit distance problem (1946)

- At most how many times can the unit distance occur among N points in the plane?
- Conjecture $N^{1+\frac{c}{\log\log N}}$ for some constant $c > 0$ as $N \to \infty$.



Progress on the conjecture in the plane

- $N^{\frac{3}{2}}$ (Erdős 1946)
- $o(N^{\frac{3}{2}})$ (Jozsa, Szemeredi 1975)
- $N^{\frac{13}{9}}$ (Beck, Spencer 1984)
- $N^{\frac{4}{3}}$ (Spencer, Szemeredi, Trotter 1984)



Unit distance almost implies the distinct distance one

- For N points in the plane let
  - $u_2(N)$ maximum number of unit distances.
  - $v_2(N)$ minimum number of distinct distances.
- $u_2(N)v_2(N) \geq \binom{N}{2} \sim N^2$

Unit distance almost implies the distinct distance one

- For N points in the plane let
  - $u_2(N)$ maximum number of unit distances.
  - $v_2(N)$ minimum number of distinct distances.
- $u_2(N)v_2(N) \geq \binom{N}{2} \sim N^2$
- If $u_2(N) \lesssim N^{1+\epsilon}$ then get

$$v_2(N) \gtrsim \frac{N^2}{N^{1+\epsilon}} = N^{1-\epsilon}$$

Conjecture and results in higher dimensions

- Conjecture $N^{\frac{4}{3}} \log\log N$ in $\mathbb{R}^3$ by Erdős in 1960.
- Best result $N^{\frac{295}{197}+\epsilon}$ in $\mathbb{R}^3$ by Zahl in 2018.
- Trivial in $\mathbb{R}^d$, $d \geq 4$, due to the Lenz example.
  - Place $\left\lfloor \frac{N}{2} \right\rfloor$ points on the circle $x_1^2 + x_2^2 = \frac{1}{2}$, $x_3 = \ldots = x_d = 0$.
  - Place remaining points on the circle $x_3^2 + x_4^2 = \frac{1}{2}$, $x_1 = x_2 = x_5 = \ldots = x_d = 0$.
  - All points on first circle are unit distance from any point on the second circle and vice versa.
  - Get up to constants N^2 unit distances.

The Szemeredi-Trotter incidence theorem

- $\mathcal{P}$ set of points, $\mathcal{L}$ set of lines in $\mathbb{R}^2$.
- An incidence is a pair $(p, \ell) \in \mathcal{P} \times \mathcal{L}$ such that p is on ℓ.
- Denote by $I(\mathcal{P}, \mathcal{L})$ the number of incidences in $\mathcal{P} \times \mathcal{L}$

Theorem (Szemeredi, Trotter 1983)

Let $\mathcal{P}$ be a set of m points and let $\mathcal{L}$ be a set of n lines, both in $\mathbb{R}^2$. Then

$$I(\mathcal{P}, \mathcal{L}) \lesssim m^{2/3} n^{2/3} + m + n$$

The Spencer-Szemeredi-Trotter result

Theorem (Spencer, Szemeredi, Trotter 1984)

Let $\mathcal{P}$ be a set of m points and let Γ be a set of n circles, both in $\mathbb{R}^2$. Then

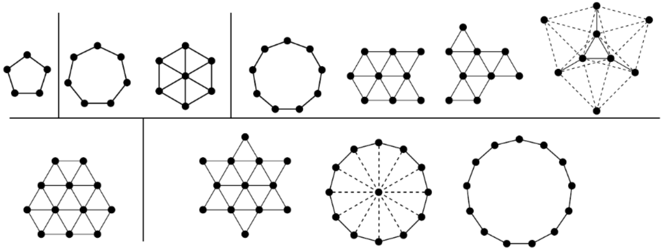
$$I(\mathcal{P}, \Gamma) \lesssim m^{2/3} n^{2/3} + m + n$$

- Consider N points in the plane and draw a unit circle around each for a total of N circles.
- Total of $I(\mathcal{P}, \mathcal{C}) \lesssim N^{\frac{4}{3}}$ unit distances.

Optimal point sets determining few distinct distances

$g_d(N) =$ min number of distinct distances among N points in $\mathbb{R}^d$

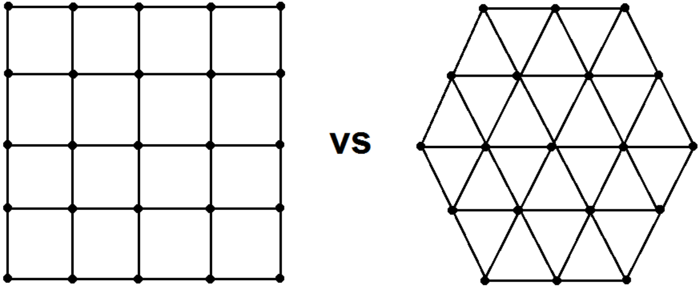
N	1	2	3	4	5	6	7	8	9	10	11	12	13
$g_2(N)$	0	1	1	2	2	3	3	4	4	5	5	5	6



SETS WITH k DISTINCT DISTANCES, $2 \leq k \leq 6$, AND MAXIMUM NUMBER OF POINTS

Research Problems In Discrete Geometry

Structure of optimal sets



Conjecture (Erdős 1988)

Optimal sets exist in a triangular lattice for N big enough.

Distances with specified multiplicities

- Does there exist a set with 3 points and distances d_1, d_2, d_2?
  Note that 3 points form $\binom{3}{2} = 3$ distances.

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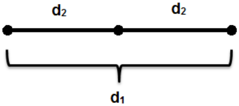

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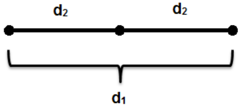
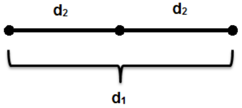


- Does there exist a set with 4 points and distances $d_1, d_2, d_2, d_3, d_3, d_3$? Note that 4 points form $\binom{4}{2} = 6$ distances.

Distances with specified multiplicities

- Does there exist a set with 3 points and distances d_1, d_2, d_2? Note that 3 points form $\binom{3}{2} = 3$ distances.



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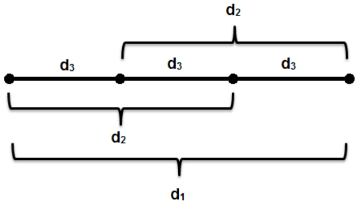
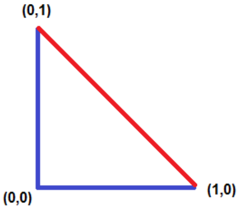


- What if no three points are allowed to be on a single line and no four points allowed to be on a circle?



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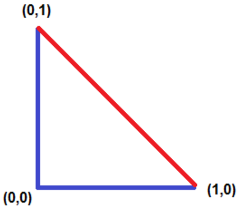

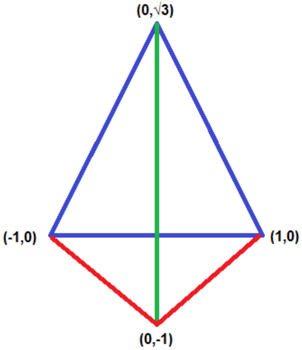

Crescent configurations

Definition (Crescent Configuration)

We say N points are in *crescent configuration* in the plane if they lie in general position and determine $N-1$ distinct distances, such that for every $1 \leq i \leq N-1$ there is a distance that occurs exactly i times.

- Note $1+2+\ldots+(N-1) = \frac{N(N-1)}{2} = \binom{N}{2}$ so all possible distances are specified.

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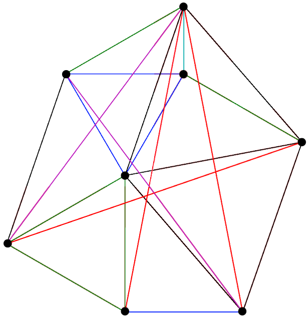
- Note $1 + 2 + \ldots + (N-1) = \frac{N(N-1)}{2} = \binom{N}{2}$ so all possible distances are specified.

Definition (General Position)

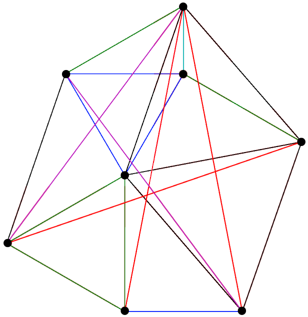
We say that N points are in *general position* in the plane if no three points lie on the same line and no four points lie on the same circle.

Crescent configuration for 8 points in the plane



- The construction above is due to Palásti.

Crescent configuration for 8 points in the plane



- The construction above is due to Palásti.
- No construction is known for 9 points in the plane.
- Erdős conjectured that for a sufficiently large N it was impossible to find crescent configurations.

Crescent configurations exist in high enough dimensions

Theorem (Burt, Goldstein, Manski, Miller, Palsson, Suh)

For all $N \geq 3$, there exists a set of N points in a crescent configuration in $\mathbb{R}^{N-2}$.

- In $\mathbb{R}^3$ we take *general position* to mean that no 4 points lie on the same plane and no 5 points lie on the same sphere.

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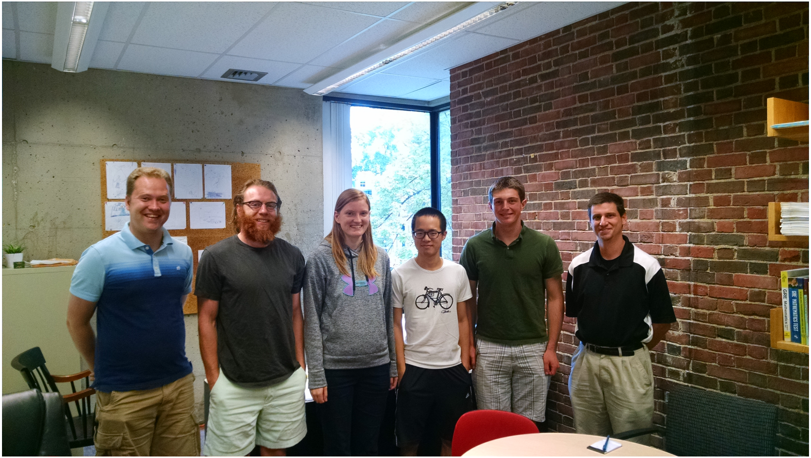
- In $\mathbb{R}^3$ we take *general position* to mean that no 4 points lie on the same plane and no 5 points lie on the same sphere.
- Theorem shows that a 5 point crescent configuration exists in $\mathbb{R}^3$.
- Can you find bigger crescent configurations in $\mathbb{R}^3$?

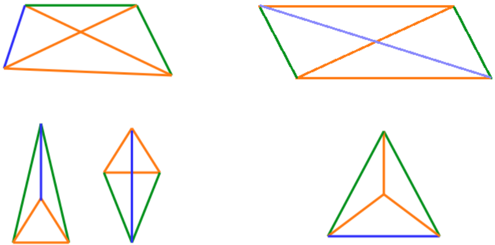
SMALL group 2015



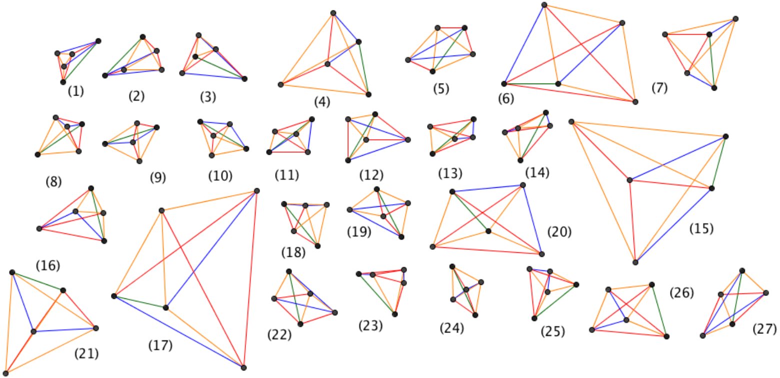
Only 4 crescent configurations on 4 points

Theorem (Durst, Hlavacek, Huynh, Miller, Palsson)

There are precisely 4 crescent configurations on 4 points up to graph isomorphism.



Many crescent configurations on 5 points



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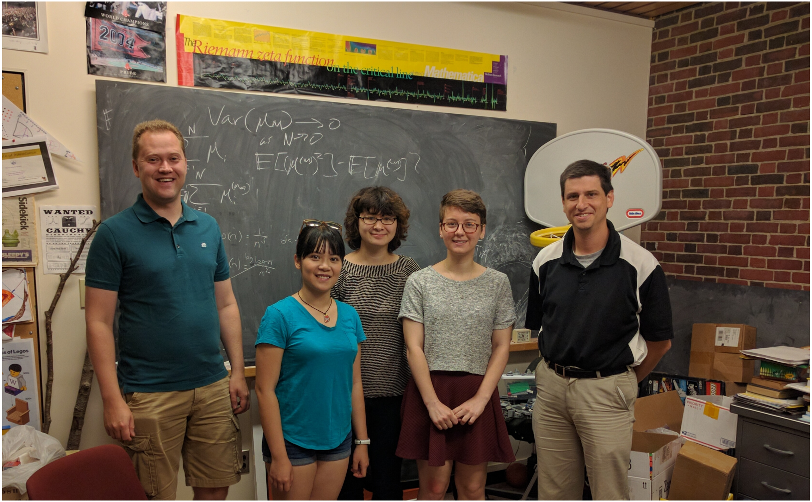

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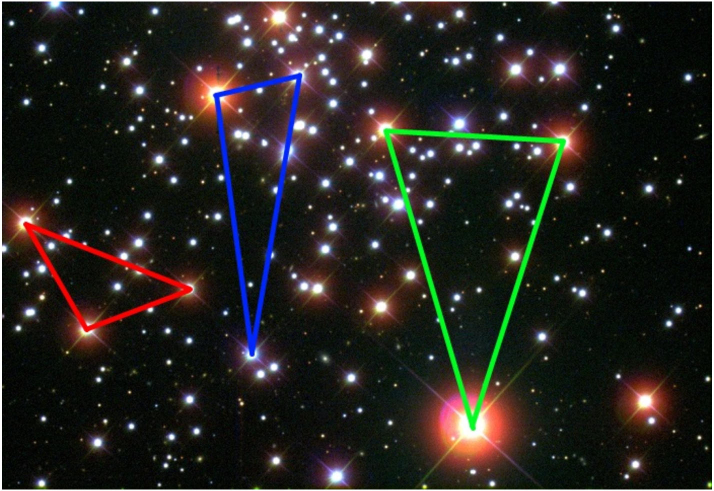
- Considered the question for general norms $\|\cdot\|$ in $\mathbb{R}^2$.
- Many constructions and/or results for particular norms.
  - $\|x\|_2 = (x_1^2 + x_2^2)^{\frac{1}{2}}$
  - $\|x\|_1 = |x_1| + |x_2|$
  - $\|x\|_\infty = \max(|x_1|, |x_2|)$
  - $\|x\|_p = (x_1^p + x_2^p)^{\frac{1}{p}}$
- The students: Sara Fish, Dylan King, Catherine Wahlenmayer

Higher order patterns: Triangles



Triangles in the plane

- Consider triangles obtained from N points in the plane.
- Least upper bounds $\binom{N}{3} \sim N^3$
- Greatest lower bounds
  - N^2 (Rudnev)
  - $\frac{N^2}{\log(N)}$ (Brass, Moser, Pach and Guth, Katz)
  - $N^{\frac{12}{7}}$ under additional constraints (Greenleaf, Iosevich)
  - $N^{\frac{5}{3}}$ (Brass, Moser, Pach and Szemerédi, Trotter)

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- What is the structure of the optimal sets for triangles?

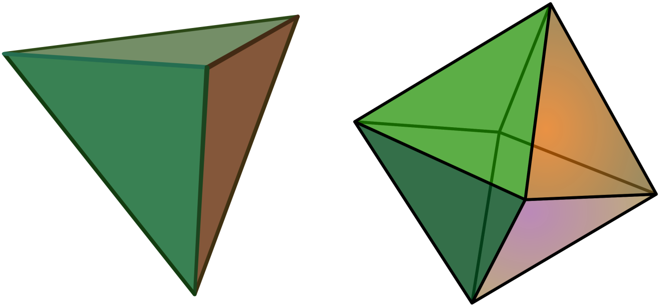
Optimal point sets determining few distinct triangles

$t_d(N)$ = min number of distinct triangles among N points in $\mathbb{R}^d$

N	2	3	4	5	6	7	8	...	d	d+1	d+2	...	2d
$t_2(N)$	0	1	1	2	3	?	?	...	?	?	?	...	?
$t_3(N)$	0	1	1	2	2	?	?	...	?	?	?	...	?
$t_4(N)$	0	1	1	1	2	2	2	...	?	?	?	...	?
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t_d(N)$	0	1	1	1	1	1	1	...	1	1	2	...	2

Joint with Hazel Brenner, James Depret-Guillaume, Alyssa Epstein, Adam Lott, Steven J. Miller, Steven Senger and Robert Stuckey.

Key objects in $\mathbb{R}^3$



Wikipedia

- Will the other Platonic solids, the cube, the dodecahedron and the icosahedron, also play a role?

Questions?

Thank you!

For more info see my website
https://intranet.math.vt.edu/people/palsson/