Resolutions of spheres

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The groups stabilize along the diagonals.
Freudenthal Suspension Theorem

**Theorem (Freudenthal)**

*For sufficiently large $n$ the suspension map*

\[
\text{Map}(S^k, \Sigma^n X) \xrightarrow{\Sigma} \text{Map}(S^{k+1}, \Sigma^{n+1} X)
\]

*is an isomorphism.*

**Remark**

Recall that $\Sigma S^n = S^{n+1}$.

**Corollary**

*For $n > k + 1$*

\[
\pi_{k+n}(S^n) \cong \pi_{k+n+1}(S^{n+1}).
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Resolutions of spheres

Definition

For $n > k + 1$, the group

$$\pi_k S := \pi_{n+k} S^n$$

is called the $k$-th stable homotopy group of spheres or the $k$-th stable stem.
A spectrum $F$ is

- A collection of spaces $\{F_n\}_{n=0,1,2,...}$
- With structure maps $\Sigma F_n \to F_{n+1}$

**Example**

Given any space $X$ we can construct a spectrum $\tilde{X}$ by taking

- $\tilde{X}_n = \Sigma^n X$
- $\Sigma^{n+1} X = \Sigma \tilde{X}_n \xrightarrow{\sim} \tilde{X}_{n+1} = \Sigma^{n+1} X$

**Example**

Let $S_n = S^n$, the $n$-sphere. The spectrum $S$ is called the sphere spectrum.
Homotopy Groups

Definition

Given a spectrum $E$, define its homotopy groups as

$$\pi_k E := \lim_{n} \pi_{n+k} E_n$$

Remark

The $k$-th stable stem

$$\pi_k S$$

is the $k$-th homotopy group of the sphere spectrum.
\( \pi_n(S) \) are very hard to compute.
All stable homotopy groups are finite, so we can investigate them one torsion at a time.
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There is the spectrum $S_{(p)}$, called the $p$-local sphere.
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$p$-torsion part of $\pi_n(S) = \pi_n(S_{(p)})$
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## Algebra vs. Topology

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They fit into a **chromatic tower**

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\]

The tower converges.

### Theorem (Chromatic Convergence Theorem)

For nice spectra

\[
X = \lim_{n} \ L_{E(n)}X
\]
We can study the homotopy groups not only one prime at a time but also one chromatic layer $L_{E(n)}$ at a time.

This is called Chromatic Homotopy Theory.
Resolutions of spheres

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but \( L_{E(k)}S_p \) are made out of smaller building blocks called \( L_{K(m)}S_pS \) which are more approachable.
Resolutions of spheres

- $\pi_n(L_{E(k)}S(p))$ are very hard to compute
- but $L_{E(k)}S(p)$ are made out of smaller building blocks called $L_{K(m)}S(p)S$ which are more approachable.
- We approach them with the help of Lubin-Tate theory of deformations of formal group laws.
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Morava theory

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for $F$ a subgroup of $\mathbb{G}_n$ we can form homotopy fixed points spectra $E_n^{hF}$. 
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for \( F \) a subgroup of \( \mathbb{G}_n \) we can form homotopy fixed points spectra \( E_n^{hF} \).

\( E_n^{h\mathbb{G}_n} = L_{K(n)} S(p) \).
Theorem (Hopkins-Mahowald-Sadofsky)

For $n = 1$ and prime 2 there is the fiber sequence

$$L_{K(1)} S \rightarrow KO\mathbb{Z}_2 \rightarrow KO\mathbb{Z}_2,$$

which is equivalent to

$$E_1^{hG_1} \rightarrow E_1^{hC_2} \rightarrow E_1^{hC_2}$$
Theorem (Goerss-Henn-Mahowald-Rezk)

For $n = 2$ and prime 3 there exists the tower of fibrations

$$L_{K(2)} S \rightarrow X_3 \rightarrow X_2 \rightarrow X_1 \rightarrow E_2^{hG_{24}}$$

where the fibers are homotopy fixed points spectra with respect to various finite subgroups of Morava Stabilizer Group.

Conjecture (Work in Progress)

For $n = 2$ and prime 2 there exists the tower of fibrations

$$E_2^{hG_2^1} \rightarrow X_2 \rightarrow X_1 \rightarrow E_2^{hG_{24}}$$

where the fibers are homotopy fixed points spectra with respect to various finite subgroups of Morava Stabilizer Group and the top of the tower is ”half” of $L_{K(2)} S$. 