Towards a Resolution of the $K(2)$-local Sphere at the Prime 2.

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Homotopy Groups of Spheres

Consider the sphere spectrum $S$.

**Question**

How do we compute $\pi_* S$?

**Answer**

We choose appropriate localizations so that the problem becomes approachable.
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Chromatic Homotopy Theory

- Fix a prime $p$.
- The Johnson-Wilson theories $\{E(n)\}_{n=0,1,...}$ allow to filter the category of $p$-local spectra.
- Localizations with respect to Johnson-Wilson theories form chromatic tower

$$\ldots \to L_{E(2)}X \to L_{E(1)}X \to L_{E(0)}X.$$

Chromatic Convergence Theorem (Hopkins, Ravenel)

For a finite $p$-local spectrum $X$

$$X = \holim_n L_{E(n)}X.$$
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**Morava $K$-theory**

Let $K(n)$ denote $n$-th Morava $K$-theory.

**Theorem (Ravenel, Hovey-Strickland)**

*There is a homotopy pullback diagram:*

$$
\begin{array}{ccc}
L_{E(n)}X & \longrightarrow & L_{K(n)}X \\
\downarrow & & \downarrow \\
L_{E(n-1)}X & \longrightarrow & L_{E(n-1)L_{K(n)}X}.
\end{array}
$$

So we can concentrate on computing $\pi_* L_{K(n)}S$. We do it with the help of Morava $E$-theory and Morava Stabilizer Group.
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**Morava $E$-theory**

**Theorem (Morava, Goerss-Hopkins-Miller, Devinatz)**

- For each $n$ there exists a spectrum $E_n$, called the $n$-th Morava $E$-theory,
- and a group $\mathbb{G}_n$, called the Morava Stabilizer group.
- $\mathbb{G}_n$ acts on $E_n$.
- for $H$ a closed subgroup of $\mathbb{G}_n$ we can form homotopy fixed points spectra $E_n^{hH}$.
- $E_n^{h\mathbb{G}_n} = L_{K(n)} S^0$.
- For any closed subgroup $H$ of $\mathbb{G}_n$ there is a spectral sequence

$$E_2^{s,t} = H^*(H,(E_n)_*) \Longrightarrow \pi_* E_n^{hH}.$$
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**Known results, $n=1$, $p=2$**

**Theorem (Adams, Baird, Ravenel)**

*For $n = 1$ and $p = 2$ there is the fiber sequence*

\[ L_{K(1)}S^0 \rightarrow KO\mathbb{Z}_2 \rightarrow KO\mathbb{Z}_2, \]

*which is equivalent to*

\[ E_1^{hG_1} \rightarrow E_1^{hC_2} \rightarrow E_1^{hC_2}. \]
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**Known results, $n=2$**

Using the spectral sequence

$$E_2^{*,*} = H^*_c(G_2, (E_2)_*) \implies \pi_* L_{K(2)} S^0 :$$

- At $p \geq 5$ Shimomura and Yabe computed $\pi_* L_{K(2)} S^0$.  
- At $p = 3$ $G_2$ contains $C_3$ Shimomura and Wang computed $\pi_* L_{K(2)} S^0$.  
- At $p = 2$ $G_2$ contains $Q_8$ Shimomura and Wang computed the second page of the spectral sequence.
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Different approach

Plan

Try to build the $K(2)$-local sphere spectrum out of $E_2^{hH_i}$ for finite subgroups $H_i$ of $G_2$. Work with the subgroup $G_1^2$ of $G_2$, such that there is a fiber sequence

$$L_{K(2)}S^0 \rightarrow E^{hG_2^1} \rightarrow E^{hG_2^1}.$$
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Known results, $n=2$, $p=3$

**Theorem (Goerss, Henn, Mahowald, Rezk)**

There exists a resolution in the $K(2)$-local category at the prime 3

$$E^{hG_2^1} \rightarrow E^{hG_{24}} \rightarrow \Sigma^8 E^{hSD_{16}} \rightarrow \Sigma^{40} E^{hSD_{16}} \rightarrow \Sigma^{48} E^{hG_{24}}$$

which can be realized to a tower of fibrations:

$$\begin{align*}
\Sigma^{45} E^{hG_{24}} & \rightarrow E^{hG_2^1} \\
\Sigma^{38} E^{hSD_{16}} & \rightarrow X_2 \\
\Sigma^{7} E^{hSD_{16}} & \rightarrow X_1 \\
E^{hG_{24}} & \rightarrow E^{hG_{24}}
\end{align*}$$
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**Tower Spectral Sequence**

Given a tower of fibrations with limit $Z$ and fibers $F_i$:

\[
\begin{array}{c}
\uparrow & \uparrow & \uparrow \\
Z & X_n & \ldots & X_0 \\
\downarrow & \downarrow & \downarrow \\
F_{n+1} & F_n & F_0
\end{array}
\]

there exists a spectral sequence

\[
E_1^{s,t} = \pi_{t-s}F_s \Rightarrow \pi_{t-s}Z.
\]
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**Known results, $n=2$, $p=2$**

**Theorem (Goerss, Henn, Mahowald, Rezk)**

There exists a resolution in the $K(2)$-local category at the prime 2

$$E^{hS^1_2} \rightarrow E^{hG_{24}} \rightarrow E^{hC_6} \rightarrow E^{hC_6} \rightarrow X$$

which can be realized to a tower of fibrations:

$$
\begin{array}{ccc}
\Sigma^{-3}X & \rightarrow & E^{hS^1_2} \\
\downarrow & & \\
\Sigma^{-2}E^{hC_6} & \rightarrow & X_2 \\
\downarrow & & \\
\Sigma^{-1}E^{hC_6} & \rightarrow & X_1 \\
\downarrow & & \\
E^{hG_{24}} & \rightarrow & E^{hG_{24}}
\end{array}
$$
New results, n=2, p=2

Theorem (B.)

*In the tower of fibrations:*

\[
\begin{align*}
\Sigma^{-3} X & \longrightarrow E^{hS^1_2} \\
\Sigma^{-2} E^{hC_6} & \longrightarrow X_2 \\
\Sigma^{-1} E^{hC_6} & \longrightarrow X_1 \\
E^{hG_{24}} & \longrightarrow E^{hG_{24}} \\
\end{align*}
\]

\[\pi_* X = \pi_* \sum^{48} E^{hG_{24}}.\]
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**Idea of the proof**

**Theorem (Henn)**

There exists a resolution in the $K(2)$-local category

$$E^{hS^1_2} \rightarrow E^{hG_{24}} \vee E^{hG_{24}} \rightarrow E^{hC_6} \vee E^{hC_4} \rightarrow E^{hC_2} \rightarrow E^{hC_6}$$

which can be realized to a tower of fibrations:

$$
\begin{align*}
\Sigma^{-3} E^{hC_6} & \rightarrow E^{hS^1_2} \\
\Sigma^{-2} E^{hC_2} & \rightarrow X_2 \\
\Sigma^{-1}(E^{hC_6} \vee E^{hC_4}) & \rightarrow X_1 \\
E^{hG_{24}} \vee E^{hG_{24}} & \rightarrow E^{hG_{24}}
\end{align*}
$$
New Results, n=2, p=2

Lemma

$\Delta^{2+8i}$ is a homotopy class in $\pi_* X$.

Theorem (Folklore, Hopkins-Mahowald)

If $\Delta^{2+8i}$ is a homotopy class in $\pi_* X$, then $\pi_* X = \pi_* \Sigma^{48} E^{hG_{24}}$. 
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**Work in progress**

**Conjecture**

$\mathcal{X}$ is homotopy equivalent to $\Sigma^{48} E^{hG_{24}}$. 