Resolving the $K(2)$-local Sphere

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Manifolds, $K$-theory and Related Topics
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How do we compute stable homotopy groups of spheres?

We choose appropriate localizations of the stable category so that the problem becomes approachable.
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Morava $K$-theory

- Concentrate on computing $\pi_* L_{K(n)} S$ for all primes $p$ and all $n$.
- Computational tools: Morava $E$-theory and Morava Stabilizer Group.
Morava $E$-theory

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$G_n$ acts on $E_n$. 

Also can form $E_{hH}^n$ for subgroups $H < G_n$. 

$E^* = H^*(H, (E_n)^*) = \Rightarrow \pi_* E_{hH}^n$. 

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- Also can form $E_n^{hH}$ for subgroups $H < \mathcal{G}_n$
- $E_2^{*,*} = H^*(H, (E_n)_*) \Longrightarrow \pi_* E_n^{hH}$. 
Theorem (Adams, Baird, Ravenel)

For \( n = 1 \) and \( p = 2 \), \( G_1 \cong \mathbb{Z}_2^\times \cong C_2 \times \mathbb{Z}_2 \) and there is the fiber sequence

\[
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- For $n = 1$ and $p > 2$ $G_1 \cong \mathbb{Z}_p^\times \cong C_{p-1} \times \mathbb{Z}_p$ and there is the fiber sequence

$$E_1^{hG_1} \cong L_{K(1)}S^0 \rightarrow E_1^{hc_{p-1}} \rightarrow E_1^{hc_{p-1}}.$$
Known results, n=2

- There exists a fiber sequence
  \[ L_{K(2)}S^0 \rightarrow E_2^{h\mathbb{G}_2^1} \rightarrow E_2^{h\mathbb{G}_2^1}. \]

- \( \mathbb{G}_2^1 \) is a subgroup of \( \mathbb{G}_2 \).
Known results, $n=2$, $p=3$

Theorem (Goerss, Henn, Mahowald, Rezk)

There exists a resolution in the $K(2)$-local category at the prime 3

$$E^{hG_{2}} \xrightarrow{} E^{hG_{24}} \xrightarrow{} \Sigma^{8} E^{hSD_{16}} \xrightarrow{} \Sigma^{40} E^{hSD_{16}} \xrightarrow{} \Sigma^{48} E^{hG_{24}}$$

which can be realized to a tower of fibrations:

$$\Sigma^{45} E^{hG_{24}} \xrightarrow{} E^{hG_{2}}$$

$$\Sigma^{38} E^{hSD_{16}} \xrightarrow{} X_{2}$$

$$\Sigma^{7} E^{hSD_{16}} \xrightarrow{} X_{1}$$

$$E^{hG_{24}} \xrightarrow{} E^{hG_{24}}$$
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Tower Spectral Sequence

Given a tower of fibrations with limit $Z$ and fibers $F_i$

\[
\begin{array}{c}
Z \\
\uparrow \\
F_{n+1}
\end{array} \xrightarrow{} \begin{array}{c} X_n \\
\uparrow \\
F_n
\end{array} \xrightarrow{} \ldots \xrightarrow{} \begin{array}{c} X_0 \\
\uparrow \\
F_0
\end{array}
\]

there exists a spectral sequence

\[
E_1^{s,t} = \pi_{t-s} F_s \Rightarrow \pi_{t-s} Z.
\]
New results, n=2, p=2

Theorem (B.)

There exists a resolution in the $K(2)$-local category at the prime 2

$$E^{hS_2^1} \to E^{hG_{24}} \to E^{hC_6} \to E^{hC_6} \to X$$

where $\pi_* X = \pi_* \Sigma^{48} E^{hG_{24}}$ and which can be realized to a tower of fibrations:

$$\Sigma^{-3} X \to E^{hS_2^1}.$$

$$\Sigma^{-2} E^{hC_6} \to X_2$$

$$\Sigma^{-1} E^{hC_6} \to X_1$$

$$E^{hG_{24}} \to E^{hG_{24}}.$$