A Research Statement

My research lies within the field of sub-Riemannian geometry and how it relates to subelliptic partial differential equations. This is a wide area that naturally intersects with several complex variables, differential geometry, control theory and neurobiology, amongst many others. My work focuses on two distinct but related threads within this field: submanifolds and mean curvature in sub-Riemannian geometry and the tangential Cauchy-Riemann operators on CR manifolds. [1]

Loosely speaking, sub-Riemannian geometry is the study of geometries that arise from situations when movement in some directions is prohibitively expensive. Unlike Riemannian geometry the infinitesimal length defining metric is defined not the entire tangent bundle, but instead on a sub-distribution of allowable directions. This results in different notions of length, area and volume. Many classical geometric questions are still wide open or have qualitatively different solutions in this setting. It is not know if all distance minimizing curves must be smooth. There are examples of non-smooth minimal surfaces. Except for the first Heisenberg group, the identity of isoperimetric minimizers is still a matter of conjecture. This thread of my work has primarily concerned the study of the analog of mean curvature in the sub-Riemannian. In [6] and [7], we developed a generalization of some fundamental notions of Riemannian geometry, such as the frame bundle, canonical connection and second fundamental form, to a wide class of sub-Riemannian structures. Using these tools, we were able to characterize critical points for the horizontal minimal and isoperimetric surface variational problems in terms of subelliptic PDE. This allowed us to establish the analogous links between the variational problems, nonlinear subelliptic PDEs and the horizontal mean curvature of submanifolds. In [5] we explored applications of these results on minimal surfaces to biological models of visual processing and image completion arising in computer science and neurobiology. Future work will focus on applying these methods to study the properties of sub-Riemannian minimal and CMC surfaces. Questions of interest include: what are the regularity properties of these surfaces? Is there a classification of compact stable CMC surfaces in the Heisenberg groups? What are the isoperimetric minimizers in the Heisenberg groups and other Carnot groups? Are there general monotonicity results or Sobolev-type estimates on sub-Riemanninan minimal surfaces? How do symmetries of the ambient geometric structure relate to properties of minimal and CMC surfaces? My previous work has developed a computationally efficient framework in which to study these problems.

In the complex setting, the natural objects of sub-Riemannian geometry are the Cauchy-Riemann (CR) manifolds. These are abstract models of the boundaries of domains in the theory of several complex variables. The complex structure of the ambient complex space does not descend completely to the boundary. The rotation operator J induced from multiplication by i does not preserve the tangent bundle as one tangent direction must always be rotated out of the surface. What results is a complex structure defined on a sub-distribution. Associated to these manifolds is the tangential Cauchy-Riemann complex, $\bar{\partial}_b$. The natural Laplacian for this complex is the prototype linear sub-elliptic equation. My Ph. D. thesis [1] and subsequent paper [3] focused on the existence and boundary regularity of solutions to the associated Neumann problem for the Heisenberg ball. This problem is complicated by both non-coercive boundary conditions and characteristic boundary points. Using a geometric approach exploiting a spectral decomposition on the leaves of a foliation by spheres, I established sharp, global estimates in non-isotropic L^2 Sobolev spaces. In [2] and [4], I generalized these ideas to a wider class of foliated CR manifolds and constructed both sharp estimates and counter-examples to global

[1] Details of existing and planned work are contained in subsequent sections, either attached or downloadable from www.math.rochester.edu/people/faculty/hladky

hypoellipticity. Future work will address the issues of characteristic boundaries and further weakening the assumptions of [4]. Characteristic boundary points occur where the domain boundary is tangent to the horizontal sub-distribution and these are present in generic domains. The dimension of the space of well-behaved boundary operators jumps at characteristic points which makes a priori estimates much harder to derive. In the approach developed in my previous work, the study of characteristic points essentially becomes the study of asymptotic behavior of eigenvalues on the leaves of a foliation as they approach a degenerate leaf. The success of this idea in the model case of the Heisenberg ball, [3], suggests that this analytic framework is well suited to the study of this problem.

A more detailed research statement expanding on both research threads can be downloaded at:

http://www.math.rochester.edu/people/faculty/hladky/Research.pdf

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B Sub-Riemannian Geometry

B.1 Introduction to sub-Riemannian geometry and horizontal mean curvature

Riemannian geometry describes situations where all directions, while maybe not being precisely equal, are at least all valid options. In sub-Riemannian geometry some directions are forbidden. Consider for example a unicycle rider moving around the Euclidean plane. The rider may move forward or backwards in the direction which they happen to be facing, they may turn on the spot, or they may try to combine these. What they cannot do is slide sideways. The notion of distance for this rider is no longer straight-forward Euclidean distance as their orientation must also be considered. The quickest path between two configurations is generally not going to be a straight line as this would of necessity waste time turning on the spot at each end.

More technically, a sub-Riemannian manifold is a triple $(M, V_0, \langle\cdot,\cdot\rangle)$ consisting of a smooth manifold M, a smooth subbundle $V_0 \subset TM$ (called horizontal) and an inner product $\langle\cdot,\cdot\rangle$ on V_0. Associated to this structure is the Carnot-Carathéodory distance metric

$$d_{cc}(x,y) = \inf\left\{\int \langle\,\dot\gamma\,,\,\dot\gamma\,\rangle^{1/2} dt : \gamma \in \mathscr{A}, \gamma(0) = x, \gamma(1) = y\right\}$$

where $\mathscr{A}$ is the collection of absolutely continuous curves with $\dot\gamma \in V_0$ everywhere. If there are no such connecting curves, $d_{cc}(x,y)$ is set to $+\infty$.

For our unicycle rider, the space of all configurations becomes $\mathbb{R}^2_{x,y} \times \mathbb{S}^1_\theta$. The horizontal distributions are spanned by rotation on the spot $\frac{\partial}{\partial\theta}$ and forward movement $\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$. A little thought should convince you that it is possible to move from any configuration to any other along allowable paths. So the Carnot-Carathéodory distance between configurations is always finite.

Set $V_{j+1} = V_j + [V_j, V_0]$ for $j \geq 0$. The sub-Riemannian structure has *step size* $r+1$ at a point $p \in M$ if r is the least integer such that $(V_r)_p = T_pM$. The distribution V_0 is *bracket-generating* if every point of M has finite step size. For the unicycle space $[\frac{\partial}{\partial\theta}, \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}] = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}$ and so $V_1 = TM$ and M has step size 1 at every point.

Example B.1 (Carnot Groups). Let G be a connected, simply connected nilpotent Lie group whose Lie algebra admits a grading

$$\mathfrak{g} = \mathfrak{v}_0 \oplus \cdots \oplus \mathfrak{v}_r$$

with $\mathfrak{v}_k = [\mathfrak{v}_0, \mathfrak{v}_{k-1}]$ for $1 \leq k \leq r$ and $[\mathfrak{v}_0, \mathfrak{v}_r] = 0$. Setting V_0 to be the span of the left invariant vector fields generated by $\mathfrak{v}_0$ and choosing a left invariant inner product on V_0 yields a natural sub-Riemannian structure. The Carnot group G has constant step size $r+1$.

Carnot groups are the natural tangent spaces to the general sub-Riemannian manifolds, [4]. □

A second class of examples occurs naturally in several complex variables and complex geometry. This class will be discussed in more detail in the other section of my research statement.

Example B.2 (Pseudohermitian Manifolds). Let (M^{2n+1}, η) be a contact structure, i.e. an odd dimensional manifold together with a non-vanishing 1-form η, together with a smooth bundle map $J \colon V_0 \to V_0$ where

- $V_0 = \mathrm{Ker}(\eta) \subset TM$,
- $J^2 = -1$,
- $N_J = 0$ where $N_J(X,Y) := [X, JY] + [JX, Y] - J([X,Y] + [JX, JY])$.

The pseudohermitian structure is strictly pseudoconvex if the Levi form

$$L_\eta(X,Y) = d\eta(X, JY)$$

is positive definite on V_0. In this case, (M, V_0, L) is sub-Riemannian structure with constant step size 1.

Pseudohermitian manifolds are closely related to Cauchy-Riemannn manifolds and real hypersurfaces in complex manifolds. There are many connections between natural operators on these spaces and the theory of holomorphic functions in several complex variables ([5], [11],[23], [33], [32], [34],[36], [37].) □

Possibly the most studied sub-Riemannian geometry lies in the intersection of these examples classes:

Example B.3 (Heisenberg Groups). The nth Heisenberg group is the space $\mathbb{H}^n = \mathbb{R}^{2n}_{x,y} \times \mathbb{R}_t$ together with the horizontal distribution V_0 spanned by the global orthonormal frame

$$X_j = \partial_{x^j} - \frac{y^j}{2}\partial_t, \quad Y_j = \partial_{y^j} + \frac{x^j}{2}\partial_t \qquad 1 \le j \le n.$$

The group structure of $\mathbb{H}^n$ is given by

$$(x_1, y_1, t_1) \cdot (x_2, y_2, t_2) = \left(x_1 + x_2, y_1 + y_2, t_1 + t_2 + \frac{1}{2}\sum_{k=1}^{n} \left(x_1^j y_2^j - x_2^j y_1^j \right) \right).$$

With this group structure $\mathbb{H}^n$ is a Lie group and the frame of V_0 consists of left invariant vector fields. Its Lie algebra $\mathfrak{h}^n$ is the simplest non-abelian example:

$$[X_i, Y_j] = -\delta_{ij}\partial_t, \qquad [X_i, X_j] = [Y_i, Y_j] = [\partial_t, X_j] = [\partial_t, Y_j] = 0.$$

The Heisenberg groups acquire additional importance as a proto-type example of pseudohermitian manifolds. The additional structure is defined by

$$\eta = dt + \sum_{j=1}^{n} \left(\frac{y^j}{2} dx^j - \frac{x^j}{2} dy^j \right), \quad JX_j = -Y_j, JY_j = X_j.$$

□

Sub-Riemannian spaces occur in analysis as the natural spaces on which to study solutions to subelliptic PDE. Much of this work was pioneered by the school of Stein, where the tools of harmonic analysis were adapted to such settings (see, for example [21], [22], [45], [49]). In the last few decades, a vast amount of effort has gone into developing the theory of subelliptic equations along these lines, with particular emphasis on using the geometry of the underlying spaces to derive information about solutions, inequalities of Poincaré, Sobolev or isoperimetric type and quasiconformal maps ([1], [8], [9], [10], [24], [29], [31], [46]).

Recent interest in subelliptic PDEs has led to a focus of attention on the submanifold geometry of sub-Riemannian manifolds. In particular, there has been substantial effort to understand the existence and properties of natural variational problems and how the solutions are related to subelliptic equations.

The size of a submanifold with respect to the Carnot-Carathéodory can be described using the notion of Hausdorff measure. This is however difficult to work with. Fortunately for most examples of sub-Riemannian geometries, there presence is a natural choice of Riemannian metric g extending $\langle \cdot, \cdot \rangle$. With this metric there is an equivalent, simpler notion: the horizontal perimeter of Σ, defined by

$$P(\Sigma) = \int_\Sigma |(\nu_g)_0| \, \nu_g \lrcorner dV_g = \int_\Sigma \nu \lrcorner dV_g \tag{1}$$

where ν_g is the Riemannian unit normal vector and ν is the horizontal unit normal vector.

If $\Sigma = \partial\Omega$ is the boundary of a domain, then this is the sub-Riemannian equivalent of the de Giorgi [20] perimeter measure

$$P(\partial\Omega) = \sup_{\|X\|_\infty \leq 1, X\in C^\infty(V_0)} \int_\Omega \text{div } X \, dV_g$$

The situation is complicated by the presence of *characteristic points*, i.e. points where the horizontal distribution V_0 is tangent to the hypersurface. The collection of characteristic points is denoted $C(\Sigma)$ can be viewed as subset of Σ where $(\nu_g)_0 = 0$ and ν is undefined.

The questions that I am most interested in this field are the following:

Question 1 (Minimal Surface Problem)**.** Given a fixed boundary can the hypersurfaces spanning the boundary with least perimeter measure be geometrically characterized? What properties do these surfaces have?

Question 2 (Isoperimetric Problem)**.** Given a fixed volume, what are the closed hypersurfaces bounding this volume of minimal perimeter? What properties do these surfaces have?

In the Riemannian setting, these questions are classical and the solutions are well-understood. A necessary condition for a solution surface is to have zero mean curvature and constant mean curvature respectively. Furthermore, the stability of these surfaces has been explored in depth. For example, it has been long known that the solution to the isoperimetric problem in Euclidean space is the round sphere.

Until recently, in sub-Riemannian geometry these questions were much less understood. A necessary condition for Question 1 had been found by various authors in a variety of restricted categories of sub-Riemannian geometries (see [12], [13], [17] , [27], [40]) whereas work on Question 2 had largely been restricted to the Heisenberg groups. Here it was conjectured by Pansu ([38], [39]) that the isoperimetric domains are

$$|t| < \phi(r) = \frac{L^2\pi}{8} - \frac{L^2}{4}\arctan\left(\frac{r}{\sqrt{L^2 - r^2}}\right) + \frac{r}{4}\sqrt{L^2 - r^2}, \quad 0 \leq r < L \tag{2}$$

where $r^2 = \sum_{j=1}^{2}\left(x_j^2 + y_j^2\right)$ and L is a positive constant, together with their group translates under symmetry constraints. In $\mathbb{H}^1$, Question 2 has recently been solved by Ritoré and Rosales ([43]) for C^2 domains, but their methods are particular to 3 dimensions and do not generalize. In higher dimensions there are positive results supporting the conjecture for surfaces with radial symmetry ([35], [44]) and bigraphs over circular domains ([18]).

In [28], Pauls and I, introduced a general framework for studying these problems. The technique was to refine the notion of the frame bundle for a wide class of sub-Riemannian manifolds (vertically rigid or VR spaces) in such a way as to reflect the sub-Riemannian structure. Then, using a specially adapted connection, they were able to employ the formal variation techniques of Bryant, Griffith and Grossman [7] to classify solutions to both questions in terms of non-linear subelliptic equations as follows: for a C^2 hypersurface in a sub-Riemannian manifold with horizontal unit normal ν and chosen tangent, orthonormal, horizontal frame $e_1, \dots e_k$, the *horizontal second fundamental form* is defined by

$$II_0 = \begin{pmatrix} \langle \nabla_{e_1}\nu, e_1\rangle & \dots & \langle \nabla_{e_1}\nu, e_k\rangle \\ \vdots & \vdots & \vdots \\ \langle \nabla_{e_k}\nu, e_1\rangle & \dots & \langle \nabla_{e_k}\nu, e_k\rangle \end{pmatrix} \tag{3}$$

where $\nu, e_1, \dots, e_k$ is an orthonormal frame for V_0. The connection used can be either the Levi-Cevita connection for a Riemannian extension or an adapted connection of the type developed by Pauls and myself. The *horizontal mean curvature*, H is then defined by

$$H = \text{trace}(II_0). \tag{4}$$

At characteristic points, the horizontal mean curvature is defined using limits if they exist.

The solutions to the posed questions are given by the following theorems:

Theorem B.4. *Any C^2 minimal surface in a VR space must satisfy $H = 0$.*

Theorem B.5. *Any C^2 solution to the isoperimetric problem in a VR space must have locally constant mean curvature away from characteristic points.*

The first of these theorems had already been established in differing forms by a variety of authors in special cases ([13], [17] , [27], [40]). However, the category of VR spaces includes all the examples currently in the literature and many more, for instance both of the examples from the introduction are VR spaces. The second result was previously only known in simple cases or with added symmetry restrictions ([6], [18], [35], [44]).

In [26], we extended these results to the non-vertically rigid case at the expense of the introduction of a torsion term. The theorems remain the same if H is replaced by the divergence of the horizontal unit normal, but this is no longer equivalent to the trace of the horizontal second fundamental form.

When the horizontal distribution has dimension 2, both minimal and isoperimetric surfaces can be geometrically described. If $\Sigma \subset M$ is a smooth hypersurface, then away from characteristic points, $T\Sigma \cap V_0$ is one-dimensional and hence integrable. The non-characteristic portions of Σ are then foliated by horizontal curves. The subelliptic PDEs describing minimal and isoperimetric surfaces reduce to the statements that these curves are geodesics and constant curvature for an adapted connection respectively. In the simpler examples, these geodesics can be explicitly computed, allowing for very specific descriptions. Observations of this type had been made for several individual examples, before being tied together in some generality by Pauls and myself [28].

The presence of characteristic points and their affect on solutions is also only crudely understood. For C^2 surfaces, it can be shown the (Riemannian) Hausdorff dimension of the characteristic locus is at most one less than the dimension of the hypersurface. When the characteristic locus realizes this maximal dimension, there is an constraint on $C(\Sigma)$, in addition to $H = 0$ or $H = c$, for a hypersurface to be a critical point for perimeter measure. If the characteristic locus is smaller than this maximal size, then there is no additional constraint. This has been explored in detail for pseudohermitian manifolds in [14] and $\mathbb{H}^1$ in [43]. The case of general sub-Riemannian manifolds is studied by myself and Pauls in [26].

B.2 Proposed Research

My area of interest is in the interaction between geometry and subelliptic PDEs. The many and deep connections between Riemannian geometry and elliptic equations have proved greatly beneficial to both geometry and analysis. The study of the minimal and isoperimetric problems above, provides an excellent prototype model for exploring similar relations between a non-linear subelliptic equation and the geometry of sub-Riemannian manifolds. I propose to investigate these connections and explore some applications in both computing and the biological sciences. The expectation is that this research will provide insight not only into sub-Riemannian geometry but also into the general properties of subelliptic equations.

B.2.1. Regularity and geometric properties of minimal and isoperimetric surfaces.

The existing work of the Pauls and myself, [28], allows us to characterize both minimal and isoperimetric surfaces as solutions to non-linear subelliptic PDEs. But there are still many questions as to what properties these solutions share. The regularity of such surfaces is currently very poorly understood, especially in higher dimensions. With horizontal dimension 2, the geometric characterization demonstrates a lack of isotropic regularity. Although the horizontal foliating geodesics are known to be smooth, the surfaces are not generically smooth in directions transverse to these curves. Indeed there are specific examples of solutions that fail to be C^1, [41].

The first component of my proposed program of research is to address these questions from a variety of view points.

(a) The framework used by Pauls and myself can be used to construct second variation formulas applicable to both the minimal and isoperimetric surface problems. The condition of stability, i.e. $\frac{d^2}{dt^2}_{|t=0} P_0(\Sigma_t) \geq 0$ imposes strong constraints on a hypersurface. This has been used previously to provide analogues of the classical Bernstein problem in $\mathbb{H}^1$, [19]. I hope to apply the second variation formulas in the higher Heisenberg groups to classify stable, constant mean curvature C^2 hypersurfaces, or at least establish strong geometric restrictions on such surfaces. The hope is that the only possible such hypersurfaces will prove to be example (2). The classical analogue to this approach can be found in [2] and [3], where the authors show that compact, stable constant mean curvature spaces in Riemannian manifolds of constant sectional curvature must be umbilic hence geodesic spheres.

(b) There has been progress recently on understanding horizontal perimeter measure for submanifolds of higher codimension. The general framework of [28] should prove suitable for studying the resulting variational problems.

(c) Noethers theorem (see for example [7]) provides a connection between the symmetries of the underlying manifold and conservation laws for minimal surfaces. In the Euclidean case, this provides a powerful tool for obtaining information. In the more symmetric sub-Riemannian manifolds, such as the Heisenberg groups or other Carnot groups, the I expect this approach to be fruitful in obtaining additional geometric information. For example, there is a natural notion of dilation in Carnot groups, which can be used derive a Minkowski formula relating the horizontal perimeter measure and volume of compact constant curvature surfaces [26].

(d) In Riemannian geometry there is a close connection between minimal surfaces and harmonic maps and the existence of isothermal coordinates provides quick and powerful regularity results. The expectation is that these ideas can be adapted to present minimal surfaces as the images of some linear sub-Laplacian between manifolds. This sub-Laplacian will be subelliptic but the regularity properties of such operators have been extensively explored (see for example, [21], [30], [48]). Heuristically, this approach suggests there should be better regularity results in higher dimensions; horizontal dimension 2 is exactly where the the sub-Laplacian for a hypersurface becomes fully degenerate. It is conjectured that in the higher Heisenberg groups, there will be smooth regularity for minimal and isoperimetric surfaces at least away from the characteristic locus. This should essentially be a consequence of the fact that any frame for the restriction of horizontal distribution to the hypersurface will satisfy Hörmander's condition [30].

(e) Sub-Riemannian metrics can be viewed as limits of sequences of Riemannian extensions. On the Riemannian manifolds existing elliptic machinery can utilized. Solutions to the sub-Riemannian problem can then be explored by studying the convergence properties of the Riemannian solutions. Initial explorations of this idea can be found in [40] or [47]. Existing examples and previous work by Pauls ([40], [42]) suggest it is reasonable to hope for horizontal estimates sufficient to allow regularity within nonisotropic Sobolev spaces.

B.2.2. General submanifold theory

The framework developed by Pauls and myself in [28] to answer the minimal and isoperimetric problems should also allow for a general study of submanifolds in sub-Riemannian geometry.

In particular, the moving frame and frame bundle theory could be developed in conjunction with the horizontal fundamental form to yield general analogues of the structural equations of Riemannian submanifolds, such as the Gauss and Codazzi-Mainardi equations.

B.2.3. Applications in neurobiology and computer science

A recent biological model of the visual cortex provides a direct application for research on minimal surfaces. Experiments on the first layer of the visual cortex (V1) have suggested that it contains simple cells that are sensitive to brightness gradients of a retinal image. Intuitively, these cells record the direction of the intensity level set through the point the cell represents. Thus mathematically V1 views an image as surface in $\mathbb{R}^2_{(x,y)} \times \mathbb{S}^1_\theta$, position and orientation. Furthermore there is chemical communication between cells. Early evidence suggested that this communication occurred only between cells of the same location, i.e. in the ∂_θ direction. However, improved experimental techniques has provided evidence of long range communication in the spatial directions. Specifically there is communication between cells if the spatial vector between them is parallel to the direction determined by the orientation θ. This leads to the consideration of the two dimensional distribution

$$X_1 = \cos\theta \partial_x + \sin\theta \partial_y$$
$$X_2 = \partial_\theta.$$

The manifold $\mathbb{R}^2_{(x,y)} \times \mathbb{S}^1_\theta$ together with this distribution and the metric making X_1 and X_2 orthonormal is our old friend the unicycle space. But in the literature, it is better known as the "rototranslation space".

Using this model, Citti and Sarti ([16], and with Manfredini [47], [15]) explored the visual occlusion problem, i.e. the question of how to fill in missing or occluded visual data of an image. They showed that an appropriate model is to solve the minimal surface equation with appropriate Dirichlet data in the rototranslation space. Since the rototranslation space has horizontal dimension 2 this can be approached by a careful study of the horizontal geodesics to an adapted connection. The study of this topic was begun in [27]. However there is still much work to be done:

(a) The construction of various examples of solutions to the Dirichlet problem in the rototranslation space and examples. Emphasis will be placed on non-smooth solutions and solutions with characteristic points. This should facilitate a deeper understanding of the various pathologies that can arise.

(b) As the biological understanding of the workings of the visual cortex improve, there it will be possible and necessary to refine the model incorporating the functions of other cortex layers. Already there is work in the biology literature concerning a feed-back loop between V1 and high layers that may describe how the brain chooses between the various explicit solutions the method provides. It is expected that the addition of extra biological processes will lead to a study of minimal submanifolds of higher codimension in more complicated subRiemannian spaces.

(c) Citti and Sarti also provided a link between the minimal surface problem in the rototranslation space and existing approaches to digital image completion in computer science, including the Mumford-Shah variational approach, Euler's elastica method and the Ambrosio-Masnou level curve approach. Using the explicit solutions to the minimal surface equation obtained in [27], it is hoped computational efficient algorithms may be obtained.

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C Cauchy-Riemann manifolds and the $\bar{\partial}_b$-Neumann problem

C.1 Introduction

When considering a real hypersurface M in $\mathbb{C}^{n+1}$, a natural question is "How much of the ambient complex structure is present on the hypersurface?". The manifold M has odd dimension $2n+1$ so cannot be a complex manifold, but some of the structure does survive. In differential geometric terms, the complex structure can be thought of as an bundle map $J\colon T\mathbb{C}^n \to T\mathbb{C}^n$ with $J^2 = -1$. What J does is tell you how to rotate a vector through 90^o just as multiplication by i does. For our hypersurface, we can now ask which tangent directions rotate to other tangent directions and which directions are spun out of the manifold. This leads us to a $2n$ dimensional distribution inside TM given by $H = TM \cap JTM$. The rotation map does not fully descend to TM, but instead descends to the distribution H. The complex structure survives in all directions except one *characteristic* direction.

An abstract Cauchy-Riemann (CR) manifold can be described in complex terms as an m dimensional manifold M together with a distribution L in the complexified tangent bundle $\mathbb{C}TM$ such that

- $L \cap \overline{L} = \{0\}$,
- $[L, L] \subset L$.

When $m = 2n + 1$ and the dimension of L is n, the CR manifold is called hypersurface type. Sections of L are called CR vector fields and should be considered analogous to holomorphic vector fields. The two conditions are then interpreted as no non-zero CR vector field is real and that the Lie bracket of CR vector fields is a CR vector field. Both of these are true for holomorphic vector fields on a complex manifold. Since the distribution L is complex, the second condition does not imply the presence of foliating submanifolds.

This structure can also be defined in terms of real differential geometry: a CR manifold is a m dimensional manifold M together with an even dimensional distribution H and a bundle map $J\colon H \to H$ such that

- $J^2 = -1$,
- $[X, JY] + [JX, Y] = J([X, Y] + [JX, JY])$ for all sections of H.

The equivalence between the definitions is induced by identifying H with $\mathrm{Re}\left(L \oplus \overline{L}\right)$ and L with the $+i$ eigenspace of the complexification of J. Implicit to the second condition is the requirement that the right hand side make sense. It is then directly equivalent to integrability of L. Hypersurface type CR manifolds require $m = 2n + 1$ and that the dimension of H is $2n$.

A fundamental question in differential geometry is whether or not an abstract manifold can always be embedded into Euclidean space. This difficult problem was solved by Nash and required some significant advances in the theory of partial differential equations. The analogous problem in CR geometry is to decide whether an abstract CR manifold can be embedded into some $\mathbb{C}^N$. This question can be asked both locally and globally. Here, we shall restrict our attention to hypersurface type CR manifolds and henceforth always assume $m = 2n + 1$. and $\dim(H) = 2n$.

Answering this question essentially boils down to the problem of finding N CR functions $F = (f_1, \ldots f_N)$ on M (or an open subset) such that the rank of F is $2n+1$ everywhere. A CR function f is a function satisfying $\overline{Z}f = 0$ for all sections Z of L. Thus CR functions are the direct analogue of holomorphic functions for complex manifolds.

The most interesting work has taken place in the category of strictly pseudoconvex CR manifolds. Locally, we can always find a non-vanishing real 1-form η that annihilates the distribution H. Once η has been chosen, we can consider the bilinear Levi form

$$L_\eta(X,Y) = d\eta(X,JY)$$

on H. The CR manifold is pseudoconvex if the Levi form is either positive or negative semi-definite. Strict pseudoconvexity requires positive or negative definite. Pseudoconvexity of the boundary of a domain in $\mathbb{C}^n$ is known to be equivalent to both the domain being biholomorphic to a convex domain and the domain being a domain of holomorphy.

For compact strictly pseudoconvex CR manifolds, the answer to the global embedding problem has long been known to be no in dimension 3 and yes in dimensions 5 and up. Three dimensional CR manifolds are typically pathological as the integrability condition is trivially satisfied. Strangely, the method used by Boutet de Monvel to solve the compact, dimension ≥ 5 case, relies heavily on the global compactness property of the manifold. The local problem for the noncompact case is significantly harder. It is still false in dimension 3. For dimension ≥ 9 the local embedding property was shown to be true by Kuranishi. This was later reduced to ≥ 7 by Akahori. Both of their proofs are very long and exceedingly difficult. The case of dimension 5 is still open.

This connection between the embedding problem and CR functions motivates interest in the Cauchy-Riemann operator $\bar{\partial}_b$ and the tangential Cauchy-Riemann equations.

C.1.1. The $\bar{\partial}_b$ complex

For a hypersurface in $\mathbb{C}^n$, the operator $\bar{\partial}_b$ can be defined as the projection of $\bar{\partial}$ onto tangential directions. In the abstract setting it can be constructed in a way analogous to that for the exterior derivative. For a differentiable function, $\bar{\partial}_b f$ is defined as the section of $\overline{L}^*$ obtained by restricting df to $\overline{L}$. A function is said to be *CR-holomorphic* (or simply CR) if $\bar{\partial}_b f = 0$.

Extending this operator to higher degree forms requires the introduction of a specialised bigrading. Define $\Lambda^{p,0}$ to be the subbundle of $\Lambda^p M \otimes \mathbb{C}$ containing all p-forms ϕ such that $\bar{Z} \lrcorner \phi = 0$ for all $\bar{Z} \in \overline{L}$. The space of forms of type (p,q) is constructed by setting

$$\Lambda^{p,q} M = \Lambda^{p,0} \otimes \Lambda^q(\overline{L})^*.$$

It should be emphasised that this definition is asymmetric in p and q. Since the CR manifold M has odd dimension there is an extra *characteristic* direction not contained in $L \oplus \overline{L}$. This appears in $\phi \in \Lambda^{p,q}$ as an extra freedom allowed for the first p-arguments of ϕ. These may be selected from any complex tangent vectors including the characteristic direction. The final q-arguments may only be taken from $\overline{L}$.

If ϕ is a smooth global section of $\Lambda^{0,q}$ then $\bar{\partial}_b \phi$ is the smooth global section of $\Lambda^{0,q+1}$ defined by

$$\begin{aligned}\bar{\partial}_b \phi(\bar{Z}_1, \dots, \bar{Z}_{q+1}) &= \sum_{1 \leq i \leq q+1} \bar{Y}_i \phi(\bar{Z}_1, \dots, \hat{\bar{Z}}_i, \dots, \bar{Z}_{q+1}) \\ &+ \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \phi([\bar{Z}_i, \bar{Z}_j], \bar{Z}_1, \dots, \hat{\bar{Z}}_i, \dots, \hat{\bar{Z}}_j, \dots, \bar{Z}_{q+1})\end{aligned}$$

for all $\bar{Z}_1, \dots, \bar{Z}_{q+1} \in \overline{L}$. To extend the definitions to the case $p > 0$ requires a more complicated formula. However, most analysis on CR manifolds is conducted for the case $p = 0$ as there are standard arguments that show that any estimates holding for $(0,q)$ forms also hold on (p,q) forms.

As with $\bar{\partial}$, the operator $\bar{\partial}_b$ satisfies $\bar{\partial}_b \circ \bar{\partial}_b = 0$ on (p,q)-forms and so induces a bigraded cohomology on M known as *Kohn Rossi cohomology.* There is then an associated inhomogenous problem

Question 3. Given a $(p,q+1)$-form f such that $\bar{\partial}_b f = 0$, is there a (p,q)-form u with $\bar{\partial}_b u = f$?

As with $\bar{\partial}$, one method for studying this question is via the $\bar{\partial}_b$-Neumann problem.

In the presence of a Riemannian metric, such as the Levi metric on a strictly pseudoconvex pseudohermitian manifold, these bundles can be identified with subbundles of the space of complexified differential forms on M and the associated *Kohn Laplacian* can be defined by setting $\Box_b = \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^*$.

Example C.1 (Heisenberg Group). For the Heisenberg group, the $(0,q)$-forms are just the linear span of the forms $d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{i_q}$ and the operator $\bar{\partial}_b$ is given by

$$\bar{\partial}_b(f d\bar{z}^I) = \sum_k L_{\bar{k}} f dz^{\bar{k}} \wedge d\bar{z}^I.$$

The Kohn Laplacian can be explicitly computed as

$$\Box_b(f d\bar{z}^I) = \left(-\frac{1}{2} \sum_1^n (L_k L_{\bar{k}} f + L_{\bar{k}} L_k f) + 2i(n-2q)\mathrm{Re}\left(\frac{\partial f}{\partial w}\right)\right) d\bar{z}^I.$$

□

Many authors have studied the properties of the Kohn Laplacian on compact, strictly pseudoconvex CR manifolds. The analysis of $\Box_b$ is more delicate than that of the classical complex Laplacian $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ as $\Box_b$ is not elliptic. Kohn [11] proved that it was subelliptic in the strictly pseudoconvex case, thus establishing strong existence and regularity results for compact manifolds. Later, Folland and Stein [4] introduced an analytic framework where sharp estimates and Fredholm theorems for $\Box_b$ and $\bar{\partial}_b$ could be established. Their work has profoundly influenced the subject with applications in a diverse array of geometric problems such as the CR Yamabe problem [15].

The key concept introduced was a family of restricted Sobolev type spaces adapted to the geometric structure of CR manifolds. For non-negative integers k, define the *Folland-Stein* space $S^k(M)$ to be the Hilbert Space of functions $u \in L^2(M)$ such that $X_1 \dots X_j u \in L^2(M)$ whenever $j \le k$ and $X_1, \dots, X_j$ are smooth sections of the real expression H. For higher degree forms (or more generally vector bundles), define similar spaces either by working with component functions in local frames or by using covariant derivatives instead of ordinary ones. It is convenient to refer to vector fields from H as *Folland-Stein vector fields* and their concatenations as *Folland-Stein derivatives*.

These spaces can be used to describe the fundamental sharp regularity result for $\Box_b$ proved by Folland and Stein [5]. Let M be a compact, strictly pseudoconvex CR manifold. Suppose u is a (p,q)-form such that $u \in L^2(M)$ and $\Box_b u \in S^k(M)$ then $u \in S^{k+2}(M)$ with an estimate of the form

$$\|u\|_{k+2} \le C \left\{ \|\Box_b u\|_k + \|u\|_{L^2} \right\}.$$

From this it is easy to see that $\Box_b$ has closed range and finite-dimensional kernel. Solutions to the inhomogeneous equation $\bar{\partial}_b u = f$ can then be constructed when f satisfies the compatibility condition $\bar{\partial}_b f = 0$. While this result is stated globally for compact CR manifolds the arguments also provide local interior regularity results for domains in the noncompact setting.

C.1.2. The $\bar{\partial}_b$-Neumann problem

Suppose M is a strictly pseudoconvex CR manifold of dimension $2n+1$. Let Ω be a smoothly bounded precompact subset with smooth defining function ρ. The $\bar{\partial}_b$-Neumann problem is then stated as follows:

Question 4. Given a (p,q)-form f smooth up to the boundary of Ω, is there a (p,q)-form u such that $\Box_b u = f$ on Ω and u satisfies the boundary conditions $\bar{\partial}_b \rho \vee u = \bar{\partial}_b \rho \vee \bar{\partial}_b u = 0$ on $\partial\Omega$.

Here $\vee$ is the pointwise adjoint of the wedge product, defined by $\langle \alpha \vee \beta , \gamma \rangle = \langle \beta , \bar{\alpha} \wedge \gamma \rangle$. These boundary conditions are exactly those required for the integration-by-parts formula

$$\left(\Box_b u , v \right)_{L^2(\Omega)} = \left(\bar{\partial}_b u , \bar{\partial}_b v \right)_{L^2(\Omega)} + \left(\bar{\partial}_b^* u , \bar{\partial}_b^* v \right)_{L^2(\Omega)} = \left(u , \Box_b v \right)_{L^2(\Omega)}. \tag{5}$$

to hold and additionally ensure that $u \in \mathrm{Dom}(\Box_b)$.

When Question 4 is solvable, it leads directly to solutions to Question 3. For if $\Box_b u = f$ and $\bar{\partial}_b f = 0$, then it can be shown that $v = \bar{\partial}_b^* u$ is a solution to $\bar{\partial}_b v = f$ with $v \perp \mathrm{Ker}(\bar{\partial}_b)$. If a general solution to Question 3 and Question 4 is attainable, it will have numerous applications to the study of the geometry and analysis of CR manifolds, including embedding problem.

Both of these authors made significant contributions to the $\bar{\partial}_b$-Neumann problem during their attacks on the embedding problem. Kuranishi established the existence of a solution in a singularly weighted L^2 for a small domain Ω provided that Ω admits a defining function that is the real part of a CR-holomorphic function. Part of Akahori's argument in reducing the dimension to 7 was to weaken this condition to one equivalent to

(C1) Ω admits a defining function depending smoothly of the real and imaginary parts of a single CR-holomorphic function w.

This condition (C1) has been present in virtually all subsequent work on the problem. It is however decidedly non-generic. Contrast this with condition $Z(q)$ for the $\bar{\partial}$-Neumann problem which is stable under small C^2 perturbations of the defining function.

Various authors (e.g. M-C. Shaw [16]) have made progress on the problem from the perspective of integral kernel methods. After making some additional geometric assumptions and using the techniques of commutation and integration by parts, R. Diaz [3] was able to establish *a priori* L^2 estimates and prove exact Sobolev regularity at the boundary (if $\Box_b u = f$ then u lies in the same Sobolev space as f) for a closely related problem. Diaz's solutions were only guaranteed to satisfy one of the $\bar{\partial}_b$-Neumann boundary conditions. There are also severe geometric restrictions on allowable domains. He required a pointwise constraint on curvature terms and that $\partial\Omega$ could contain no *characteristic points*; points at which the distribution H is tangent to $\partial\Omega$. While he was able to prove that any point in a CR manifold embedded as a hypersurface is contained in such a domain, these constraints mean the results to do not apply in general. On these domains his results were sufficient to show the existence of solutions to tangential Cauchy-Riemann equations, but these solutions exhibit a loss of 1 degree of Sobolev regularity at the boundary.

The presence of characteristic points complicates the analysis of $\Box_b$ enormously as the dimension of the space of Folland-Stein vector fields tangent to $\partial\Omega$ jumps at these points. The construction of L^2 estimates at the boundary is thus made much more difficult.

All these results have the drawback that they do not provide *sharp* regularity results at the boundary. Ideally, it would desirable to establish a family of function spaces between which $\Box_b$ is an isomorphism. However, since $\Box_b$ is subelliptic (and fully elliptic in the Folland-Stein directions) it seems natural to expect a gain of derivatives at the boundary. Interior estimates follow from the work of Folland and Stein, but these do not extend to the boundary. It is actually possible to show that a full gain of 2 Folland-Stein derivatives does not occur at boundary. I gave a counter-example in [6]. As a consequence some form of weighted regularity will need to be employed.

C.1.3. Previous work

My research focuses on strictly pseudoconvex CR manifolds satisfying condition (C1). The goal is to obtain existence of solutions to the $\bar{\partial}_b$-Neumann problem and sharp estimates at the boundary even in case where the domain possesses characteristic boundary points. The core idea of this approach is to fully exploit the geometric consequences of condition (C1). The existence of such a defining function w implies the presence of a foliation by codimension-2 CR manifolds in a neighbourhood of (almost all) the boundary. The leaves of this foliation are just the level sets of the CR-holomorphic function w. Typically, these leaves will be compact strictly pseudoconvex CR manifolds of dimension $2n-1$.

Using the pseudoHamiltonian vector field of w, defined by

$$\eta(Y) = 0, \quad Y \lrcorner d\eta = \bar{\partial}_b \bar{w}$$

we can orthogonally split $(0,q)$-forms into pieces tangential and transverse to the foliation.

$$\varphi = \varphi^{\top} + \bar{\partial}_b \bar{w} \wedge \varphi^{\perp}, \qquad \overline{Y} \lrcorner \varphi^{\top} = 0 \tag{6}$$

The operator $\Box_b$ can also be decomposed using this split. More importantly, the boundary conditions separate. A complicated subelliptic boundary value problem is reduced (modulo some lower order error terms) to two simpler subelliptic boundary value problems, one with Dirichlet boundary data, the other with more traditional Neumann boundary data.

Under some further mild conditions on w, this split can be explicitly computed and the boundary value problem formally becomes

$$\begin{aligned} \left(\widehat{\Box}_b^w + \overline{Y}^*\overline{Y}\right) u^{\top} + \mathcal{E}_1(u^{\perp}) = f^{\top} \qquad & \overline{Y} u^{\top}_{|\partial\Omega} = 0 \\ \left(\widehat{\Box}_b^w + \overline{Y}\overline{Y}^*\right) u^{\perp} + \mathcal{E}_1^*(u^{\top}) = f^{\perp} \qquad & u^{\perp}_{|\partial\Omega} = 0 \end{aligned} \tag{7}$$

where $\widehat{\Box}_b^w$ denotes the Kohn Laplacian associated to each leaf of the foliation. Since these operators act on compact CR manifolds, the existing theory already provides *a priori* sharp estimates and regularity.

The simplest case is when the leaves of the foliation are all the same normal CR manifold. In this case, the error terms vanish and using Tanaka's eigenspace decomposition for $\Box_b$ on a compact, normal manifold ([18]), we can then reduce both boundary value problems to infinite families of elliptic equations on a domain in the w-complex plane. The problem thus reduces to proving regularity and uniform estimates over these infinite families of elliptic equations.

Establishing estimates for these families can be done using integration-by-parts and commutation arguments on well chosen Dirichlet forms. In [6], I show that $\Box_b$ acts as an isomorphism between the spaces $S^{k;2}(\Omega)$ and $S^k(\Omega)$ on $(0,q)$-forms with $1 \le q \le n-2$. Here $S^k(\Omega)$ is the usual Folland-Stein space and $S^{k;2}(\Omega)$ is the subspace of $S^k(\Omega)$ such that

- u satisfies the $\bar{\partial}_b$-Neumann boundary conditions;
- $u \in S^k(\Omega)$, $\rho u \in S^{k+1}(\Omega)$, $\rho^2 u \in S^{k+2}(\Omega)$, where ρ is a defining function for Ω;
- If Z_1 and Z_2 are Folland-Stein vector fields tangent to the leaves of the foliation then $Z_1 u$ and $Z_2 Z_1 u$ are in $S^k(\Omega)$;
- $\bar{Y} u \in S^{k+1}(\Omega)$;
- The tranverse component $u^{\perp} \in S^{k+2}(\Omega)$.

While the definition of the $S^{k;2}$ spaces are somewhat awkward they contain enough control over the derivatives of u to obtain exact (un-weighted) Folland-Stein regularity at the boundary for the canonical solution of the inhomogeneous equation $\bar{\partial}_b u = f$.

Using these techniques I was able to show the following additional results [6]

- The operator $(1 + \Box_b)^{-1}$ is not compact.
- $\Box_b$ is not globally subelliptic up to the boundary.
- If the leaf N has non-vanishing Kohn-Rossi cohomology, then $\Box_b$ is not hypoelliptic up to the boundary.

When the leaves are not uniform, we cannot do this form of harmonic analysis. However any non-characteristic boundary point does still have a small neighbourhood that is foliated by compact codimension 2 submanifolds. On each leaf, $\Box_b$ will be Fredholm and so will have only finite-dimensional eigenspaces with eigenvalues diverging to $+\infty$. In the case where the leaves do not admit $\Box_b$-harmonic forms, there will be a uniform positive lower bound on these eigenvalues on any sufficient small open set. More generally the eigenvalues can locally be confined within small bands corresponding to finite-dimensional spaces of smooth tangential forms.

In [8], I approached the problem by establishing *a priori* estimates and using the techniques of elliptic regularization. The difficulty with establishing *a priori* estimates for subelliptic operators is in controlling the characteristic direction. However, global estimates on the compact foliating manifolds can be used to control directions tangent to the leaves and the characteristic direction can (modulo some easily controlled terms) be projected onto directions tangent to the leaves. Adapting interpolation techniques used by [17], M-C. Shaw and L. Wang used interpolation techniques for compact CR manifolds, I was able to produce a priori estimates and regularity whenever $\Box_b$ has closed range and Ω is noncharacteristic.

The question of whether $\Box_b$ has closed range is still open in general, even in the noncharacteristic case. However in [8], I was able to prove that if none of the boundary leaves of the foliation admit $\Box_b$-harmonic forms, then $\Box_b$ has closed range on Ω. As an application, I was able to prove the following theorems, which answer Question 3 for a large class of interesting examples:

Theorem A. *If M is a strictly pseudoconvex hypersurface in $\mathbb{C}^{n+1}$ with $n \geq 3$ and Ω a smooth compact domain with noncharacteristic boundary satisfying (C1) then $\Box_b$ is a Fredholm operator on $L^2_{(p,q)}(\Omega)$ for $1 \leq q \leq n-2$.*

Theorem B. *Let M be a strictly pseudoconvex pseudohermitian manifold embedded as a hypersurface in $\mathbb{C}^{n+1}$ with defining function r. Let $\Omega = M \cap \{\varrho < 0\}$ be a bounded domain in M with smooth, strictly convex defining function $\varrho = \varrho(z^1, \bar{z}^1)$. Suppose also that $1 \leq q \leq n-2$ and $dr \wedge dz^1 \wedge d\bar{z}^1 \neq 0$ on $\partial\Omega$ (i.e. that Ω has non-characteristic boundary). Then for any (p,q)-form $\varphi \in C^\infty(\overline{\Omega})$ such that $\bar{\partial}_b \varphi = 0$ there exists $(p,q-1)$-form $u \in C^\infty(\overline{\Omega})$ such that $\bar{\partial}_b u = \varphi$.*

C.1.4. Domains with characteristic boundary points

Consider the example of $\Omega \subset \mathbb{C}^{n+1}$ given by

$$\operatorname{Im}\left(z^{n+1}\right) = f(z^1, \ldots, z^n, \bar{z}^1, \ldots, \bar{z}^n)$$

where $f = A_{ij} z^i z^j + B_{ij} z^i \bar{z}^j + \overline{A}_{ij} \bar{z}^i \bar{z}^j + O(|z|^3)$. The characteristic points of these model examples split into two distinct categories, elliptic and non-elliptic, depending on the asymptotic behavior of f as $z \to 0$. If $B > \operatorname{Re}(A)$ as bilinear forms, then the level sets of f will asymptotically approach a fixed ellipsoid as $z \to 0$. This is the elliptic case.

The elliptic case will be the easiest to study. The axis $\{z = 0\} \subset M$ is foliated by degenerate single point leaves of w and the characteristic boundary points of Ω occur exactly where $\partial\Omega$ intersects $\{z = 0\}$. This prevents the possibility of non-isolated characteristic points and non-characteristic boundary points being contained in degenerate leaves. The asymptotic behavior of eigenvalues of the tangential Kohn Laplacians on the leaves should also be easy to analyze.

In the non-elliptic case, the boundary analysis will be more pathological as the behavior of the degenerate boundary leaves is potentially far more complicated

For more general CR manifolds, the first question is whether there is an equivalent, preferably intrinsic, classification of characteristic points. The notion of ellipticity of a characteristic point should imply the existence of a well-defined asymptotic limit of the foliating leaves.

The most natural example of the elliptic case is the Heisenberg group where $f(z, \bar{z}) = z\bar{z}$. The asymptotic leaf is then just the sphere. I studied this in detail in [7].

The proposed approach to the $\bar{\partial}_b$-Neumann problem in the case of purely elliptic characteristic points largely follows the techniques of [7]. The idea is to use establish uniform local estimates on a locally finite family of small neighborhoods covering every point except the characteristic locus. The arguments of the noncharacteristic case should largely go through here except some care will have to be taken to ensure the uniformity of constants. As in the case of the Heisenberg group [7], I expect that this approach will yield estimate in weighted Folland-Stein spaces where the weights are degenerate at the characteristic locus.

Establishing a closed range estimate will require careful analysis in neighborhoods of the characteristic points themselves. For the elliptic case, the existence of an asymptotic leaf should actually lead to a stronger singular L^2 estimate. This phenomena was seen in the case of the Heisenberg ball [7] and in the earlier work of both Kuranishi and Akahori.

I am confident that the approach outlined here will work in the elliptic case for an appropriate definition of elliptic. The ideas presented here should partially extend to the non-elliptic case, but the degeneracy of the weights and the analysis near the characteristic points may prove much harder to work with.

C.1.5. Further goals and applications

My earlier results on the Heisenberg ball [7] and the results expected from this approach both suggest the possibility of canonical solutions to the equation $\Box_b u = f$ where f is smooth up to the boundary, but u is singular at the characteristic points and smooth elsewhere. The next question is where this is a genuine property or an artifact of the proof. I expect the former. Work done by D. Jerison [9], [10], shows that this indeed occurs for the Dirichlet problem for the Kohn Laplacian. Jerison produced an asymptotic expansion for solution. I plan to first construct specific examples illustrating this behavior and then to produce an asymptotic expansion for the Neumann problem. This will initially focus on the Heisenberg ball but if this is successful for the case of elliptic characteristic points.

The estimates and regularity in the bulk of this proposal would all be in the L^2 category. A natural question is whether they can be extended to other L^p categories or appropriate Hölder type spaces.

As a more long term goal is to determine whether condition (C1) is necessary for the $\bar{\partial}_b$-Neumann problem to be solvable. There are other related conditions that are all reasonable candidates for necessary and sufficient conditions. In successively weaker order, these are:

(C1) Condition (C1).

(C2) A neighbourhood of $\partial\Omega$ in Ω is foliated by codimension-2 CR submanifolds (away from characteristic points).

(C3) $\partial\Omega$ is foliated by codimension-2 CR submanifolds (away from characteristic points).

The techniques of this proposal seem ideally suited for addressing the question of sufficiency and may well shed some light on necessity also.

Another useful technique introduced by Hörmander to study the $\bar{\partial}$ operator and associated $\Box$ Laplacian is to work in weighted L^2 spaces. In particular, when the weights are chosen to be plurisubharmonic exhaustion functions, his work has had many applications in the study of complex variables. A natural question is whether the $\bar{\partial}_b$-Neumann problem and the inhomogenous $\bar{\partial}_b$-equation have stronger solutions in similarly weighted L^2 spaces. Also, is there a CR equivalent to the idea of a plurisubharmonic function that may have an equally powerful effect on the theory of CR manifolds? A detailed understanding of Question 3, both with and without weights, is the first step in constructing a CR equivalent of the powerful $\bar{\partial}$-method in complex analysis.

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