A HYBRID EULER-HADAMARD PRODUCT FOR THE RIEMANN ZETA FUNCTION

S. M. GONEK, C. P. HUGHES, and J. P. KEATING

Abstract

We use a smoothed version of the explicit formula to find an accurate pointwise approximation to the Riemann zeta function as a product over its nontrivial zeros multiplied by a product over the primes. We model the first product by characteristic polynomials of random matrices. This provides a statistical model of the zeta function which involves the primes in a natural way. We then employ the model in a heuristic calculation of the moments of the modulus of the zeta function on the critical line. For the second and fourth moments, we establish all of the steps in our approach rigorously. This calculation illuminates recent conjectures for these moments based on connections with random matrix theory.

Contents

1. Introduction

An important theme in the study of the Riemann zeta function $\zeta(s)$ has been the estimation of the mean values (or moments)

$$I_k(T) = \frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k} \mathrm{d}t.$$

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These have applications to bounding the order of $\zeta(s)$ in the critical strip as well as to estimating the possible number of zeros of the zeta function off the critical line. Moreover, the techniques developed in these problems, in addition to being interesting in their own right, have been used to estimate mean values of other important functions in analytic number theory, such as Dirichlet polynomials.

In 1917 Hardy and Littlewood [9] proved that

$$I_1(T) \sim \log T$$

as $T \to \infty$. Nine years later, in 1926, Ingham [11] showed that

$$I_2(T) \sim \frac{1}{2\pi^2}(\log T)^4.$$

There are no proven asymptotic results for I_k when $k > 2$, although it has long been conjectured that

$$I_k(T) \sim c_k(\log T)^{k^2}$$

for some positive constant c_k. In several lectures starting in the late 1980s, Conrey and Ghosh cast this in a more precise form, namely,

$$I_k(T) \sim \frac{a(k)g(k)}{\Gamma(k^2+1)}(\log T)^{k^2},$$

where

$$a(k) = \prod_p \left(\left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\,\Gamma(k)}\right)^2 p^{-m}\right), \tag{1}$$

the product being taken over all prime numbers, and $g(k)$ is an integer when k is an integer. The results of Hardy and Littlewood and of Ingham give $g(1) = 1$ and $g(2) = 2$, respectively. However, until recently, no one had formed a plausible conjecture for $g(k)$ when $k > 2$. Then, in the early 1990s, Conrey and Ghosh [4] conjectured that $g(3) = 42$. Later, Conrey and Gonek [5] conjectured that $g(4) = 24024$. The method employed by the last two authors reproduced the previous values of $g(k)$ as well, but it did not produce a value for $g(k)$ when $k > 4$.

It was recently suggested by Keating and Snaith [14] that the characteristic polynomial of a large random unitary matrix can be used to model the value distribution of the Riemann zeta function near a large height T. Their idea was that because the zeta function is analytic away from the point $s = 1$, it can be approximated at $s = 1/2 + it$ by polynomials whose zeros are the same as the zeros of $\zeta(s)$ close to t. These zeros (suitably renormalized) are believed to be distributed like the eigenangles of unitary matrices chosen with Haar measure, so they used the characteristic polynomial

$$Z_N(U,\theta) = \prod_{n=1}^{N}(1 - e^{\mathrm{i}(\theta_n - \theta)}), \tag{2}$$

where the θ_n are the eigenangles of a random $N \times N$ unitary matrix U, to model $\zeta(s)$. For scaling reasons, they used matrices of size $N = \log T$ to model $\zeta(1/2 + \mathrm{i}t)$ when t is near T. They then calculated the moments of $|Z_N(U,\theta)|$ and found that

$$\mathbb{E}_N\left[|Z_N(U,\theta)|^{2k}\right] \sim \frac{G^2(k+1)}{G(2k+1)} N^{k^2}, \tag{3}$$

where $\mathbb{E}_N$ denotes expectation with respect to Haar measure, and $G(z)$ is the Barnes G-function. When $k = 1, 2, 3, 4$, they observed that

$$\frac{G^2(k+1)}{G(2k+1)} = \frac{g(k)}{\Gamma(k^2+1)},$$

where $g(k)$ is the same as in the results of Hardy and Littlewood and of Ingham and in the conjectures of Conrey and Ghosh and Conrey and Gonek given above. They then conjectured that this holds in general. That is, they made the following conjecture.

CONJECTURE 1 (see Keating and Snaith [14])
For k fixed with $\mathrm{Re}\, k > -1/2$,

$$\frac{1}{T}\int_T^{2T} \left|\zeta\left(\frac{1}{2} + \mathrm{i}t\right)\right|^{2k} \mathrm{d}t \sim a(k)\frac{G^2(k+1)}{G(2k+1)}(\log T)^{k^2},$$

as $T \to \infty$, *where* $a(k)$ *is given by (1) and* G *is the Barnes* G*-function.*

The characteristic polynomial approach has been successful in providing insight into other important and previously intractable problems in number theory as well (see, e.g., [16] for a survey of recent results). However, the model has the drawback that it contains no arithmetical information—the prime numbers never appear. Indeed, they must be inserted in an ad hoc manner. This is reflected, for example, by the absence of the arithmetical factor $a(k)$ in equation (3). Fortunately, in the moment problem it was only the factor $g(k)$, and not $a(k)$, that proved elusive. A realistic model for the zeta function (and other L-functions) clearly should include the primes.

In this article we present a new model for the zeta function which overcomes this difficulty in a natural way. Our starting point is an explicit formula connecting the zeros and the primes, from which we deduce a representation of the zeta function as a partial Euler product times a partial Hadamard product. Making certain assumptions about how these products behave, we then reproduce Conjecture 1. The representation that we use is the following one.

THEOREM 1

Let $s = \sigma + \mathrm{i}t$ with $\sigma \geqslant 0$ and $|t| \geqslant 2$, let $X \geqslant 2$ be a real parameter, and let K be any fixed positive integer. Let $f(x)$ be a nonnegative C^∞-function of mass one supported on $[0, 1]$*, and set $u(x) = Xf(X\log(x/e) + 1)/x$. Thus, $u(x)$ is a function of mass one supported on* $[e^{1-1/X}, e]$*. Set*

$$U(z) = \int_0^\infty u(x) E_1(z \log x)\,\mathrm{d}x, \tag{4}$$

where $E_1(z)$ is the exponential integral $\int_z^\infty e^{-w}/w\,\mathrm{d}w$. Then

$$\zeta(s) = P_X(s) Z_X(s)\Big(1 + O\Big(\frac{X^{K+2}}{(|s|\log X)^K}\Big) + O(X^{-\sigma}\log X)\Big), \tag{5}$$

where

$$P_X(s) = \exp\Big(\sum_{n\leqslant X} \frac{\Lambda(n)}{n^s \log n}\Big), \tag{6}$$

$\Lambda(n)$ *is the von Mangoldt function, and*

$$Z_X(s) = \exp\Big(-\sum_{\rho_n} U\big((s-\rho_n)\log X\big)\Big). \tag{7}$$

The constants implied by the O-terms depend only on f and K.

We remark that Theorem 1 is unconditional—it does not depend on the assumption of any unproved hypothesis. Moreover, it can easily be modified to accommodate weight functions u supported on the larger interval $[1, e]$. Finally, as is apparent from the proof, the second error term can be deleted if we replace $P_X(s)$ by

$$\widetilde{P}_X(s) = \exp\Big(\sum_{n\leqslant X} \frac{\Lambda(n)}{n^s \log n} v(e^{\log n/\log X})\Big),$$

where $v(t) = \int_t^\infty u(x)\,\mathrm{d}x$.

To clarify (5), we temporarily assume the Riemann hypothesis and take $s = 1/2 + \mathrm{i}t$. We denote the nontrivial zeros of $\zeta(s)$ by $\rho_n = 1/2 + \mathrm{i}\gamma_n$, ordered by their height above the real axis, with $\gamma_{-n} = -\gamma_n$. Since the support of u is concentrated near e, $U(z)$ is roughly $E_1(z)$, which is asymptotic to $-\gamma - \log z$ as $z \to 0$. Here, $\gamma = 0.5772\ldots$ is Euler's constant. Thus, for those ordinates γ_n close enough to t, we see that $\exp\big(-U(\mathrm{i}(t-\gamma_n)\log X)\big)$ looks roughly like $\mathrm{i}(t-\gamma_n)\,e^\gamma \log X$. We expect the ordinates farther away not to contribute substantially to the exponential defining $Z_X(s)$. Now, $P_X(s)$ looks roughly like $\prod_{p\leqslant X}(1-p^{-s})^{-1}$, and hence, our formula for

$\zeta(1/2+\mathrm{i}t)$ looks roughly like

$$\prod_{p\leqslant X}(1-p^{-1/2-\mathrm{i}t})^{-1}\prod_{\substack{\gamma_n\\|t-\gamma_n|<1/\log X}}\big(\mathrm{i}(t-\gamma_n)e^{\gamma}\log X\big). \tag{8}$$

This formula is a hybrid consisting of a truncated Euler product and (essentially) a truncated Hadamard product, with the parameter X mediating between them. Near height T we are approximating part of the zeta function by a polynomial of degree about $\log T/\log X$. The rest of the zeta function, which comes from the zeros we have neglected, is approximated by the finite Euler product. Formally, when we take X large, we reduce the number of zeros used to approximate zeta but make up for it with more primes, and when we take X small, we approach the previous model (2). Note, however, that in order for the error terms in (5) to be smaller than the main term, it is necessary to work in an intermediate regime, where both the zeros and the primes contribute.

To see how to use our formula to model the zeta function and as a test case, we heuristically calculate $I_k(T)$. The new model is more elaborate than the original one, so more work is required. Nevertheless, the idea is straightforward. The $2k$th moment of $|\zeta(1/2+\mathrm{i}t)|$ is asymptotic to the $2k$th moment of $|P_X(1/2+\mathrm{i}t)\,Z_X(1/2+\mathrm{i}t)|$. We argue that when X is not too large relative to T, the $2k$th moment of this product splits as the product of the moments. We call this the *splitting conjecture*.

CONJECTURE 2 (Splitting conjecture)
Let X and $T\to\infty$ with $X=O((\log T)^{2-\epsilon})$. Then for $k>-1/2$, we have

$$\frac{1}{T}\int_T^{2T}\left|\zeta\Big(\frac12+\mathrm{i}t\Big)\right|^{2k}\mathrm{d}t\sim\left(\frac{1}{T}\int_T^{2T}\left|P_X\Big(\frac12+\mathrm{i}t\Big)\right|^{2k}\mathrm{d}t\right)\times\left(\frac{1}{T}\int_T^{2T}\left|Z_X\Big(\frac12+\mathrm{i}t\Big)\right|^{2k}\mathrm{d}t\right).$$

Our motivation for making the splitting conjecture is based on the following observation relating to (8). $P_X(1/2+\mathrm{i}t)$ is approximately given by a product of terms associated with primes $p\le X$. Each of these terms is a periodic function of t with period at least $2\pi/\log X$. On the other hand, $Z_X(1/2+\mathrm{i}t)$ vanishes at the nontrivial zeros and so oscillates on the scale $2\pi/\log|t|$. If $X=o(T)$, then $Z_X(1/2+\mathrm{i}t)$ can thus be thought of as oscillating much faster than $P_X(1/2+\mathrm{i}t)$ (cf., e.g., Figures 2 and 3 in the appendix), and this separation of scales then suggests that they contribute independently to the moments in the limit as $T\to\infty$. When $k=1$ and $k=2$, we can prove this if $X=O((\log T)^{2-\epsilon})$ (see Corollary 1).

In Section 3 we calculate the moments of P rigorously and establish the following theorem.

THEOREM 2
Let $1/2 \leqslant c < 1$, *let* $\epsilon > 0$, *and let* $k > 0$ *be any positive real number. Suppose that* X *and* $T \to \infty$ *and* $X = O\left((\log T)^{1/(1-c+\epsilon)}\right)$. *Then we have*

$$\frac{1}{T}\int_T^{2T} |P_X(\sigma + \mathrm{i}t)|^{2k}\,\mathrm{d}t = a(k,\sigma)F_X(k,\sigma)\Big(1 + O_k\Big(\frac{1}{\log X}\Big)\Big)$$

uniformly for $c \leqslant \sigma \leqslant 1$, *where*

$$a(k,\sigma) = \prod_p \Big\{\Big(1 - \frac{1}{p^{2\sigma}}\Big)^{k^2} \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^{2m\sigma}}\Big\} \tag{9}$$

and

$$F_X(k,\sigma) = \begin{cases} \zeta(2\sigma)^{k^2} e^{-k^2 E_1((2\sigma-1)\log X)} & \text{if } \sigma > 1/2, \\ (e^{\gamma}\log X)^{k^2} & \text{if } \sigma = 1/2. \end{cases}$$

Here, E_1 *is the exponential integral, and* $\gamma = 0.5772\ldots$ *is Euler's constant.*

Note that $a(k, 1/2)$ is the same as $a(k)$ in (1).

In Section 4 we conjecture an asymptotic estimate for $\int_T^{2T} |Z_X(1/2 + \mathrm{i}t)|^{2k}\,\mathrm{d}t$ using random matrix theory. We introduce random matrix theory in the following way. The statistical distribution of the ordinates γ_n is conjectured to coincide with that of the eigenangles θ_n of $N \times N$ random unitary matrices chosen with Haar measure for some N (see, e.g., [17], [18], and [15]). The choice of N requires consideration. The numbers γ_n are spaced $2\pi/\log T$ apart on average, whereas the average spacing of the θ_n is $2\pi/N$, and so we take N to be the greatest integer less than or equal to $\log T$. We therefore conjecture that the $2k$th moment of $|Z_X(1/2+\mathrm{i}t)|$, when averaged over t around T, is asymptotically the same as $|Z_X(1/2+\mathrm{i}t)|^{2k}$ when the γ_n are replaced by θ_n and averaged over all unitary matrices with N as specified above. We perform this random matrix calculation in Section 4 (see Theorem 4, which is stated there) and so obtain the following conjecture.

CONJECTURE 3
Suppose that $X, T \to \infty$ *with* $X = O((\log T)^{2-\epsilon})$. *Then for any fixed* $k > -1/2$, *we have*

$$\frac{1}{T}\int_T^{2T} \Big|Z_X\Big(\frac{1}{2} + \mathrm{i}t\Big)\Big|^{2k}\,\mathrm{d}t \sim \frac{G^2(k+1)}{G(2k+1)}\Big(\frac{\log T}{e^{\gamma}\log X}\Big)^{k^2}.$$

We actually expect Conjecture 3 to hold for a much larger range of X, but the correct bound on the size of X with respect to T is unclear.

We note that this asymptotic formula coincides with that in (3), where N is taken to be on the order of $\log T/e^{\gamma}\log X$. This is consistent with the fact that the polynomial in (8) is of about this degree. Alternatively, the mean density of eigenvalues is N divided by 2π, and this is comparable to the mean density of the ordinates of the zeros when multiplied by $e^{\gamma}\log X$, as they are in (8).

Combining the result of Theorem 2 with the formula in Conjecture 3 and using the splitting conjecture, we recover precisely the conjecture put forward by Keating and Snaith [14]. Note that, as must be the case, all X-dependent terms cancel out.

In Section 5, we prove the following theorem.

THEOREM 3
Let $\epsilon > 0$, and let X and $T \to \infty$ with $X = O((\log T)^{2-\epsilon})$. Then for $k = 1$ and $k = 2$, we have

$$\frac{1}{T}\int_T^{2T}\left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)P_X\Big(\frac{1}{2}+\mathrm{i}t\Big)^{-1}\right|^{2k}\mathrm{d}t \sim \frac{G^2(k+1)}{G(2k+1)}\Big(\frac{\log T}{e^{\gamma}\log X}\Big)^{k^2}.$$

Since $\zeta(1/2+\mathrm{i}t)P_X(1/2+\mathrm{i}t)^{-1} = Z_X(1/2+\mathrm{i}t)(1+o(1))$ for $t \in [T, 2T]$, it follows from this that Conjecture 3 holds when $k = 1$ and $k = 2$. Moreover, combining Theorem 3 with our estimate for

$$\frac{1}{T}\int_T^{2T}\left|P_X\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^{2k}\mathrm{d}t$$

from Theorem 2, we also see that Conjecture 2 holds for $k = 1$ and $k = 2$. Thus, we obtain the following corollary.

COROLLARY 1
Conjectures 2 and 3 are true for $k = 1$ and $k = 2$.

Clearly, our model can be adapted straightforwardly to other L-functions (see [13]). It can also be used to reproduce other moment results and conjectures, such as those given by Gonek [7] and by Hughes, Keating, and O'Connell [10] concerning derivatives of the Riemann zeta function at the zeros of the zeta function. We also expect it to provide further insight into the connection between prime numbers and the zeros of the zeta function. It would be particularly interesting to determine whether the model can be extended to capture lower-order terms in the asymptotic expansions of the moments of $\zeta(1/2+\mathrm{i}t)$ and other L-functions (cf. [3]).

2. Proof of Theorem 1

We begin the proof by stating a smoothed form of the explicit formula due to Bombieri and Hejhal [2, page 837].

LEMMA 1

Let $u(x)$ be a real, nonnegative C^∞-function with compact support in $[1, e]$, and let u be normalized so that if

$$v(t) = \int_t^\infty u(x)\,\mathrm{d}x,$$

then $v(0) = 1$. Let

$$\widetilde{u}(z) = \int_0^\infty u(x) x^{z-1}\,\mathrm{d}x$$

be the Mellin transform of u. Then for s not a zero or pole of the zeta function, we have

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=2}^\infty \frac{\Lambda(n)}{n^s} v(e^{\log n/\log X}) - \sum_\rho \frac{\widetilde{u}(1-(s-\rho)\log X)}{s-\rho} + \frac{\widetilde{u}(1-(s-1)\log X)}{s-1}$$
$$- \sum_{m=1}^\infty \frac{\widetilde{u}(1-(s+2m)\log X)}{s+2m}, \tag{10}$$

where the sum over ρ runs over all the nontrivial zeros of the zeta function.

This lemma is proved in a familiar way, beginning with

$$\frac{1}{2\pi \mathrm{i}} \int_{(c)} \frac{\zeta'}{\zeta}(z+s) \widetilde{u}(1+z\log X)\,\frac{\mathrm{d}z}{z},$$

where the integral is over the vertical line $\operatorname{Re} z = c = \max\{2, 2 - \operatorname{Re} s\}$.

The support condition on u implies that $v(e^{\log n/\log X}) = 0$ when $n > X$, so the sum over n is finite. Furthermore, if $|\operatorname{Im} z| > 2$, say, then by integrating $\widetilde{u}$ by parts K times, we see that

$$|\widetilde{u}(z)| \leqslant \max_x |u^{(K)}(x)| \left|\frac{\Gamma(z)}{\Gamma(z+K)}\right| (e^{\operatorname{Re} z+K} + 1)$$
$$\leqslant \max_x |u^{(K)}(x)| \frac{e^{\max\{\operatorname{Re} z+K, 0\}}}{(1+|z|)^K} \tag{11}$$

for any positive integer K. Thus, the sums over ρ and m on the right-hand side of (10) converge absolutely, so long as $s \neq \rho$ and $s \neq -2m$. This, in fact, is the reason we require smoothing.

Next, we integrate (10) along the horizontal line from $s_0 = \sigma_0 + \mathrm{i}t_0$ to $+\infty$, where $\sigma_0 \geqslant 0$ and $|t_0| \geqslant 2$. If the line does not pass through a zero, then on the left-hand side we obtain $-\log\zeta(s_0)$. We choose the branch of the logarithm here so that $\lim_{\sigma\to\infty}\log\zeta(s) = 0$. If the line of integration does pass through a zero, then we define $\log\zeta(\sigma+\mathrm{i}t) = \lim_{\epsilon\to 0^+}(1/2)\big(\log\zeta(\sigma+\mathrm{i}(t+\epsilon)) + \log\zeta(\sigma+\mathrm{i}(t-\epsilon))\big)$. Recalling the definition of $U(z)$ in (4), we see that

$$\int_{s_0}^{\infty}\frac{\widetilde{u}(1-(s-z)\log X)}{s-z}\,\mathrm{d}s = \int_0^{\infty} u(x)E_1\big((s_0-z)\log X\log x\big)\,\mathrm{d}x$$
$$= U\big((s_0-z)\log X\big), \tag{12}$$

provided that $s_0 - z$ is not real and negative (so as to avoid the branch cut of E_1). If it is, then we use the convention that $U((s_0-z)\log X) = \lim_{\epsilon\to 0^+}(1/2)\big(U((s_0-z)\log X + \mathrm{i}\epsilon) + U((s_0-z)\log X - \mathrm{i}\epsilon)\big)$. Note that the logarithms in (12) are both positive since the support of u is in $[1,e]$ and $X \geqslant 2$. It therefore follows from (10) that

$$\log\zeta(s_0) = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{s_0}\log n}v(e^{\log n/\log X}) - \sum_{\rho}U\big((s_0-\rho)\log X\big)$$
$$+ U\big((s_0-1)\log X\big) - \sum_{m=1}^{\infty}U\big((s_0+2m)\log X\big). \tag{13}$$

The interchange of summation and integration in the sums is justified by absolute convergence. This representation holds for all points in $\operatorname{Re}s \geqslant 0$ not equal to the pole or one of the zeros of the zeta function.

Now suppose that $u(x) = Xf(X\log(x/e)+1)/x$, where f is C^∞, real, nonnegative, has total mass one, and is supported on $[0,1]$. Since $\max_x |f^{(K)}(x)|$ is bounded and independent of X, we see that $\max_x |u^{(K)}(x)| \ll_K X^{K+1}$. It therefore follows from (11) that

$$\widetilde{u}(s) \ll_K \frac{e^{\max\{\sigma,0\}}X^{K+1}}{(1+|s|)^K}.$$

From this, (12), and since $|t_0| \geqslant 2$, we find that if r is real, then

$$\begin{aligned}U\big((s_0-r)\log X\big) &= \int_{s_0}^{\infty}\frac{\widetilde{u}(1-(s-r)\log X)}{s-r}\,\mathrm{d}s\\ &\ll_K \frac{X^{K+1}}{(\log X)^K}\int_{\sigma_0}^{\infty}\frac{X^{\max\{r-\sigma,\,0\}}}{|(\sigma-r)+\mathrm{i}t_0|^{K+1}}\,\mathrm{d}\sigma\\ &\ll_K \frac{X^{K+1+\max\{r-\sigma_0,\,0\}}}{(\log X)^K}\int_{\sigma_0}^{\infty}\frac{1}{|(\sigma-r)+\mathrm{i}t_0|^{K+1}}\,\mathrm{d}\sigma\\ &\ll_K \frac{X^{K+1+\max\{r-\sigma_0,\,0\}}}{(|s_0-r|\log X)^K}.\end{aligned}$$

In particular, for any fixed positive integer K, we have that

$$U\big((s_0-1)\log X\big) \ll_K \frac{X^{K+1+\max\{1-\sigma_0,\,0\}}}{(|s_0|\log X)^K}$$

and, since $\sigma_0 \geqslant 0$, that

$$\sum_{m=1}^{\infty} U\big((s_0+2m)\log X\big) \ll_K \frac{X^{K+1}}{(\log X)^K}\sum_{m=1}^{\infty}\frac{1}{|s_0+2m|^{K+1}}$$
$$\ll_K \frac{X^{K+1}}{(|s_0|\log X)^K}.$$

Inserting these estimates into (13) and replacing s_0 by s, we find that

$$\log\zeta(s) = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s\log n}v(e^{\log n/\log X}) - \sum_{\rho} U\big((s-\rho)\log X\big) + O\Big(\frac{X^{K+2}}{(|s|\log X)^K}\Big)$$

for $\sigma \geqslant 0$, $|t| \geqslant 2$, and K any fixed positive integer. Exponentiating both sides, we obtain

$$\zeta(s) = \widetilde{P}_X(s)Z_X(s)\bigg(1 + O\Big(\frac{X^{K+2}}{(|s|\log X)^K}\Big)\bigg), \tag{14}$$

where

$$\widetilde{P}_X(s) = \exp\Big(\sum_{n\leqslant X}\frac{\Lambda(n)}{n^s\log n}v(e^{\log n/\log X})\Big)$$

and

$$Z_X(s) = \exp\Big(-\sum_{\rho} U\big((s-\rho)\log X\big)\Big).$$

We now show that replacing $\widetilde{P}_X(s)$ by

$$P_X(s) = \exp\Big(\sum_{n\leqslant X}\frac{\Lambda(n)}{n^s\log n}\Big)$$

only introduces a small error term into (14). To see this, note that $v((e^{\log n/\log X})) = 1$ for $n \leqslant X^{1-1/X}$ because the support of $u(x)$ is in $[e^{1-1/X},\ e]$. Therefore,

$$\frac{\widetilde{P}_X(s)}{P_X(s)} = \exp\Big(\sum_{X^{1-1/X}\leqslant n\leqslant X}\frac{\Lambda(n)}{n^s\log n}\big(v(e^{\log X/\log n}) - 1\big)\Big)$$
$$\ll \exp\Big(\sum_{X^{1-1/X}\leqslant n\leqslant X}\frac{1}{n^\sigma}\Big)$$
$$\ll \exp(X^{-\sigma}\log X).$$

This completes the proof of Theorem 1, provided that s is not a nontrivial zero of the zeta function. To remove this restriction, we recall the formula

$$E_1(z) = -\log z - \gamma - \sum_{m=1}^{\infty}\frac{(-1)^m z^m}{m!\,m},$$

where $|\arg z| < \pi$, $\log z$ denotes the principal branch of the logarithm, and γ is Euler's constant. From this and (4), we observe that we may interpret $\exp(-U(z))$ to be asymptotic to Cz for some constant C as $z \to 0$. Thus, both sides of (5) vanish at the zeros. $\square$

3. Proof of Theorem 2

We prove Theorem 2 by first proving a couple of lemmas.

LEMMA 2

Let $k \geqslant 0$, let $1/2 \leqslant c < 1$ be arbitrary but fixed, and suppose that $2 \leqslant X \ll (\log T)^{1/(1-c+\epsilon)}$, where $\epsilon > 0$ is also fixed. Then

$$\frac{1}{T}\int_T^{2T} |P_X(\sigma+\mathrm{i}t)|^{2k}\,\mathrm{d}t = a(k,\sigma)\prod_{p\leqslant X}\Big(1-\frac{1}{p^{2\sigma}}\Big)^{-k^2}\big(1+O_k(X^{-1/2+\epsilon})\big)$$

uniformly for $c \leqslant \sigma \leqslant 1$, where $a(k,\sigma)$ is given by (9).

Proof

Raising $P_X(s)$ to the kth power, where $P_X(s)$ is defined in (6), we have

$$P_X(s)^k = \exp\Big(k\sum_{n\leqslant X}\frac{\Lambda(n)}{n^s\log n}\Big) = \prod_{p\leqslant X}\exp\Big(\log(1-p^{-s})^{-k} - k\sum_{\substack{m\\ p^m>X}}\frac{1}{mp^{ms}}\Big).$$

Thus, if we write

$$P_X(s)^k = \sum_{n=1}^{\infty}\frac{\beta_k(n)}{n^s}, \tag{15}$$

then we immediately see that $\beta_k(n)$ is a multiplicative function of n, $0 \leqslant \beta_k(n) \leqslant d_k(n)$ for all n, where $d_k(n)$ is the kth divisor function; $\beta_k(p^m) = d_k(p^m)$ if $p^m \leqslant X$; and $\beta_k(n) = 0$ if $p \mid n$ for any prime $p > X$.

Let $\mathscr{S}(X)$ denote the set of X-smooth numbers; that is, $\mathscr{S}(X) = \{n : p \mid n \implies p \leqslant X\}$. We truncate the sum in (15) at T^θ, where θ is a small positive number to be chosen later, and we obtain

$$P_X(s)^k = \sum_{\substack{n\in\mathscr{S}(X)\\ n\leqslant T^\theta}}\frac{\beta_k(n)}{n^s} + O\Big(\sum_{\substack{n\in\mathscr{S}(X)\\ n>T^\theta}}\frac{\beta_k(n)}{n^\sigma}\Big).$$

For $\epsilon > 0$ fixed and $\sigma \geqslant c$, the sum in the O-term is

$$\begin{aligned}
&\ll \sum_{\substack{n>T^\theta \\ n\in\mathscr{S}(X)}} \Big(\frac{n}{T^\theta}\Big)^\epsilon \frac{d_k(n)}{n^\sigma} \leqslant T^{-\epsilon\theta} \sum_{n\in\mathscr{S}(X)} \frac{d_k(n)}{n^{c-\epsilon}} \\
&= T^{-\epsilon\theta} \prod_{p\leqslant X} (1-p^{\epsilon-c})^{-k} = T^{-\epsilon\theta} \exp\Bigg(O\Big(k\sum_{p\leqslant X} p^{\epsilon-c}\Big)\Bigg) \\
&\ll T^{-\epsilon\theta} \exp\Bigg(O\Big(\frac{k\,X^{1-c+\epsilon}}{(1-c+\epsilon)\log X}\Big)\Bigg).
\end{aligned}$$

Now suppose that $2 \leqslant X \ll (\log T)^{1/(1-c+\epsilon)}$ with the same ϵ. Then this is

$$\ll T^{-\epsilon\theta} \exp\Bigg(O\Big(\frac{k\,\log T}{\log\log T}\Big)\Bigg) \ll_k T^{-\epsilon\theta/2}.$$

Thus, we find that

$$P_X(s)^k = \sum_{\substack{n\in\mathscr{S}(X) \\ n\leqslant T^\theta}} \frac{\beta_k(n)}{n^s} + O_k(T^{-\epsilon\theta/2}). \tag{16}$$

By the classical mean value theorem for Dirichlet polynomials, we see that

$$\int_T^{2T} \Big| \sum_{\substack{n\leqslant T^\theta \\ n\in\mathscr{S}(X)}} \frac{\beta_k(n)}{n^{\sigma+\mathrm{i}t}} \Big|^2 \,\mathrm{d}t = \big(T + O(T^\theta \log T)\big) \sum_{\substack{n\leqslant T^\theta \\ n\in\mathscr{S}(X)}} \frac{\beta_k(n)^2}{n^{2\sigma}}.$$

Using the method above, we may extend the sum on the right-hand side to infinity with an error again no larger than $O_k(T^{-\epsilon\theta/2})$. Thus, taking $\theta = 1/2$, say, we find that

$$\frac{1}{T}\int_T^{2T} \Big| \sum_{\substack{n\leqslant T^{1/2} \\ n\in\mathscr{S}(X)}} \frac{\beta_k(n)}{n^{\sigma+\mathrm{i}t}} \Big|^2 \,\mathrm{d}t = \sum_{n\in\mathscr{S}(X)} \frac{\beta_k(n)^2}{n^{2\sigma}} \big(1 + O_k(T^{-\epsilon/4})\big).$$

Therefore, from (16) and the Cauchy-Schwarz inequality, it follows that

$$\frac{1}{T}\int_T^{2T} |P_X(\sigma+\mathrm{i}t)|^{2k} \,\mathrm{d}t = \big(1 + O_k(T^{-\epsilon/4})\big) \sum_{n\in\mathscr{S}(X)} \frac{\beta_k(n)^2}{n^{2\sigma}}\,.$$

Since the $\beta_k(n)$ are multiplicative and satisfy $0 \leqslant \beta_k(n) \leqslant d_k(n)$ and $\beta_k(p^m) = d_k(p^m)$ if $p^m \leqslant X$, we have

$$\prod_{p\leqslant X} \Big(\sum_{0\leqslant m\leqslant N_p} \frac{d_k(p^m)^2}{p^{2\sigma m}} \Big) \leqslant \sum_{n\in\mathscr{S}(X)} \frac{\beta_k(n)^2}{n^{2\sigma}} \leqslant \prod_{p\leqslant X} \Big(\sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^{2\sigma m}} \Big),$$

where $N_p = [\log X/\log p]$, the integer part of $\log X/\log p$. Observe that the ratio of the left-hand side to the right-hand side is

$$\prod_{p\leqslant X}\Big(1-\frac{\sum_{m\geqslant N_p+1} d_k(p^m)^2 p^{-2\sigma m}}{\sum_{m\geqslant 0} d_k(p^m)^2 p^{-2\sigma m}}\Big) = \prod_{p\leqslant X}\bigg(1+O\Big(\sum_{m\geqslant N_p+1}\frac{d_k(p^m)^2}{p^{2\sigma m}}\Big)\bigg)$$
$$= \prod_{p\leqslant X}\big(1+O(p^{(N_p+1)(\epsilon-2\sigma)})\big), \tag{17}$$

where we have used the bound $d_k(n) \ll n^{\epsilon/2}$ and summed the geometric series. We split the product over primes into two parts, depending on whether $2 \leqslant p \leqslant \sqrt{X}$ or $\sqrt{X} < p \leqslant X$. In the first, we use the estimate $N_p + 1 > \log X/(\log p)$, and in the second, we use $N_p = 1$. We conclude that (17) equals

$$\prod_{p\leqslant\sqrt{X}}\big(1+O(X^{\epsilon-2\sigma})\big)\times\prod_{\sqrt{X}<p\leqslant X}\big(1+O(p^{2(\epsilon-2\sigma)})\big) = 1+O(X^{-1/2+\epsilon})$$

for $\sigma \geqslant 1/2$. Hence,

$$\sum_{n\in\mathcal{S}(X)}\frac{\beta_k(n)^2}{n^{2\sigma}} = \prod_{p\leqslant X}\Big(\sum_{m=0}^{\infty}\frac{d_k(p^m)^2}{p^{2m\sigma}}\Big)\big(1+O_k(X^{-1/2+\epsilon})\big).$$

Writing the product here as

$$\prod_{p\leqslant X}\bigg(\Big(1-\frac{1}{p^{2\sigma}}\Big)^{k^2}\sum_{m=0}^{\infty}\frac{d_k(p^m)^2}{p^{2m\sigma}}\bigg)\prod_{p\leqslant X}\Big(1-\frac{1}{p^{2\sigma}}\Big)^{-k^2},$$

we note that the first of the two factors may be extended over all the primes because for $\sigma \geqslant 1/2$,

$$\prod_{p>X}\bigg(\Big(1-\frac{1}{p^{2\sigma}}\Big)^{k^2}\sum_{m=0}^{\infty}\frac{d_k(p^m)^2}{p^{2m\sigma}}\bigg) = \prod_{p>X}\bigg(1+O_k\Big(\frac{1}{p^{4\sigma}}\Big)\bigg)$$
$$= 1+O_k\Big(\frac{X^{-1}}{\log X}\Big).$$

Thus, by the definition of $a(k,\sigma)$ in (9), we find that

$$\sum_{n\in\mathcal{S}(X)}\frac{\beta_k(n)^2}{n^{2\sigma}} = a(k,\sigma)\prod_{p\leqslant X}\Big(1-\frac{1}{p^{2\sigma}}\Big)^{-k^2}\big(1+O_k(X^{-1/2+\epsilon})\big),$$

and this completes the proof of Lemma 2. $\square$

LEMMA 3

If k is a real number, then

$$\prod_{p\leqslant X}\left(1-\frac{1}{p^{2\sigma}}\right)^{-k^2}=F_X(k,\sigma)\left(1+O_k\left(\frac{1}{\log X}\right)\right)$$

uniformly for $\sigma\geqslant 1/2$, *where*

$$F_X(k,\sigma)=\begin{cases}\zeta(2\sigma)^{k^2}e^{-k^2E_1((2\sigma-1)\log X)} & \text{if } \sigma>1/2,\\ (e^{\gamma}\log X)^{k^2} & \text{if } \sigma=1/2,\end{cases}$$

and E_1 *is the exponential integral.*

Proof

Mertens's theorem asserts that

$$\prod_{p\leqslant X}\left(1-\frac{1}{p}\right)^{-1}=e^{\gamma}\log X\left(1+O\left(\frac{1}{\log X}\right)\right).$$

Raising both sides to the k^2th power establishes the result when $\sigma=1/2$. When $\sigma>1/2$, we see that

$$\prod_{p\leqslant X}\left(1-\frac{1}{p^{2\sigma}}\right)^{-1}=\zeta(2\sigma)\exp\left(\sum_{p>X}\log\left(1-\frac{1}{p^{2\sigma}}\right)\right).$$

By the prime number theorem in the form $\pi(x)=x/\log x+O(x/(\log x)^A)$, we find that

$$\begin{aligned}\sum_{p>X}\log\left(1-\frac{1}{p^{2\sigma}}\right)&=-\sum_{p>X}\left(\frac{1}{p^{2\sigma}}+O\left(\frac{1}{p^{4\sigma}}\right)\right)\\&=-\int_X^{\infty}\left(\frac{1}{u^{2\sigma}}+O\left(\frac{1}{u^{4\sigma}}\right)\right)\frac{\mathrm{d}u}{\log u}+O\left(\frac{1}{(\log X)^A}\right)\\&=-E_1\big((2\sigma-1)\log X\big)+O\left(\frac{1}{(\log X)^A}\right).\end{aligned}$$

Hence,

$$\prod_{p\leqslant X}\left(1-\frac{1}{p^{2\sigma}}\right)^{-k^2}=\zeta(2\sigma)^{k^2}\exp\big(-k^2E_1((2\sigma-1)\log X)\big)\left(1+O_k\left(\frac{1}{\log X}\right)\right),$$

as asserted. □

The proof of Theorem 2 now follows immediately from Lemmas 2 and 3. □

4. Support for Conjecture 3

In this section we give heuristic arguments supporting Conjecture 3, which we restate as

$$\frac{1}{T}\int_T^{2T}\left|Z_X\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k}\mathrm{d}t \sim \frac{G^2(k+1)}{G(2k+1)}\left(\frac{\log T}{e^{\gamma}\log X}\right)^{k^2}$$

as $T \to \infty$, where $Z_X(s)$ is given by (7).

We assume the Riemann hypothesis. Since $\operatorname{Re} E_1(\mathrm{i}x) = -\operatorname{Ci}(|x|)$ for $x \in \mathbb{R}$, where

$$\operatorname{Ci}(z) = -\int_z^{\infty}\frac{\cos w}{w}\,\mathrm{d}w,$$

we find that

$$\frac{1}{T}\int_T^{2T}\left|Z_X\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k}\mathrm{d}t = \frac{1}{T}\int_T^{2T}\prod_{\gamma_n}\exp\left(2k\int_1^e u(y)\operatorname{Ci}(|t-\gamma_n|\log y\log X)\,\mathrm{d}y\right)\mathrm{d}t, \tag{18}$$

where $u(y)$ is a smooth nonnegative function supported on $[e^{1-1/X}, e]$ and of total mass one. Since the terms in the exponent decay as $|\gamma_n - t|$ increases, this product is effectively a local statistic. That is, the integrand depends only on those zeros close to t. In recent years, considerable evidence has been amassed suggesting that the zeros of the Riemann zeta function around height T are distributed like the eigenangles of unitary matrices of size $\log T$ chosen with Haar measure (see, e.g., the survey article [15]). We therefore model the right-hand side of (18) by replacing the ordinates γ_n by the eigenangles of an $N \times N$ unitary matrix and averaging over all such matrices with Haar measure, where $N = [\log T]$. Thus, the right-hand side of (18) should be asymptotic to

$$\mathbb{E}_N\left[\prod_{n=1}^{N}\exp\left(2k\int_1^e u(y)\operatorname{Ci}(|\theta_n|\log y\log X)\,\mathrm{d}y\right)\right],$$

where the θ_n are the eigenangles of the random matrix and $\mathbb{E}_N[\cdot]$ denotes the expectation with respect to Haar measure. However, since the eigenangles of a unitary matrix are naturally 2π-periodic objects, it is convenient to periodicize our function, which we do by defining

$$\phi(\theta) = \exp\left(2k\int_1^e u(y)\Big(\sum_{j=-\infty}^{\infty}\operatorname{Ci}(|\theta+2\pi j|\log y\log X)\Big)\,\mathrm{d}y\right). \tag{19}$$

It follows from our proof of Lemma 5 that the terms with $j \neq 0$, which make the random matrix calculation much easier, only contribute $\ll_k 1/\log X$ to $\phi(\theta)$ when

$-\pi < \theta \leqslant \pi$. Hence, they do not affect the accuracy of the model. Thus, we argue that

$$\frac{1}{T}\int_T^{2T} \Big|Z_X\Big(\frac{1}{2}+\mathrm{i}t\Big)\Big|^{2k}\,\mathrm{d}t \sim \mathbb{E}_N\Big[\prod_{n=1}^{N}\phi(\theta_n)\Big]. \tag{20}$$

The remainder of this section is devoted to the proof of the following theorem.

THEOREM 4
Let $\phi(\theta)$ be defined as in (19). Then for fixed $k > -1/2$ and $X \geqslant 2$, we have, as $N \to \infty$,

$$\mathbb{E}_N\Big[\prod_{n=1}^{N}\phi(\theta_n)\Big] \sim \frac{(G(k+1))^2}{G(2k+1)}\Big(\frac{N}{e^{\gamma}\log X}\Big)^{k^2}\Big(1+O_k\Big(\frac{1}{\log X}\Big)\Big).$$

Remark 1
The random matrix model of Keating and Snaith [14] for the moments of the Riemann zeta function involves the characteristic polynomial (2). Note that if we set $M = N/(e^{\gamma}\log X)$, then by (3) we have

$$\mathbb{E}_M\big[|Z_M(U,\theta)|^{2k}\big] \sim \frac{(G(k+1))^2}{G(2k+1)}\Big(\frac{N}{e^{\gamma}\log X}\Big)^{k^2},$$

which is the same answer we find in Theorem 4. This is easily explained by the fact that in our model the eigenangles are multiplied by $e^{\gamma}\log X$, and so their mean density is $M/2\pi$. Given that for random matrices the mean density is the only parameter in the asymptotics of local eigenvalue statistics, it is natural that the result should be the same as for unitary matrices of dimension M since their eigenangles have precisely this mean density.

Proof
Heine's identity [20] evaluates the expected value in (20) as a Toeplitz determinant

$$\mathbb{E}_N\Big[\prod_{n=1}^{N}\phi(\theta_n)\Big] = \det[\phi_{i-j}]_{1\leqslant i,j\leqslant N}, \tag{21}$$

where

$$\phi_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(\theta)e^{-\mathrm{i}n\theta}\,\mathrm{d}\theta$$

is the nth Fourier coefficient of $\phi(\theta)$. The Toeplitz symbol $\phi(\theta)$ is singular since it is zero when $\theta = 0$. Thus, the asymptotic evaluation of this determinant requires knowledge of the Fisher-Hartwig conjecture in a form proved by Basor [1].

We factor out the singularity in $\phi(\theta)$ by writing

$$\phi(\theta) = b(\theta)(2 - 2\cos\theta)^k,$$

where

$$b(\theta) = \exp\left(-k\log(2-2\cos\theta) + 2k\int_1^e u(y)\Big(\sum_{j=-\infty}^{\infty} \mathrm{Ci}(|\theta + 2\pi j| \log y \log X)\Big)\,\mathrm{d}y\right). \tag{22}$$

As we see in the proof of Lemma 5, the logarithmic singularities in the exponent on the right-hand side cancel. Thus, $b(\theta)$ never equals zero. The asymptotic behavior of the Toeplitz determinant with these symbols has been determined by Basor [1]. She showed that if $k > -1/2$, then

$$\det[\phi_{i-j}]_{1\leqslant i,j\leqslant N} \sim E\exp\left(\frac{N}{2\pi}\int_{-\pi}^{\pi}\log b(\theta)\,\mathrm{d}\theta\right)N^{k^2} \tag{23}$$

as $N \to \infty$, where the constant E is given by

$$E = \exp\left(\sum_{n=1}^{\infty} n\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log b(\theta)e^{-in\theta}\,\mathrm{d}\theta\Big)^2\right) b(0)^{-k}\frac{G^2(k+1)}{G(2k+1)}.$$

To evaluate E, we need to know $b(0)$ and the Fourier coefficients of $\log b(\theta)$. These are given by the next two lemmas.

LEMMA 4
Let $b(\theta)$ be given by (22). Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log b(\theta)e^{-in\theta}\,\mathrm{d}\theta = \begin{cases} 0 & \text{if } n = 0, \\ \dfrac{k}{n}v(e^{n/\log X}) & \text{if } n \geqslant 1, \end{cases}$$

where

$$v(t) = \int_t^{\infty} u(y)\,\mathrm{d}y.$$

LEMMA 5
Let $b(\theta)$ be given by (22), and let $u(x)$ have total mass one with support in $[e^{1-1/X}, e]$*. Then*

$$b(0) = \exp\big(2k(\log\log X + \gamma)\big)\left(1 + O_k\Big(\frac{1}{\log X}\Big)\right).$$

Before proving the lemmas, we complete the proof of Theorem 4. Since u is a nonnegative function supported in $[e^{1-1/X}, e]$ of total mass one, we see that

$$v(t) = \begin{cases} 1 & \text{if } t \leqslant e^{1-1/X}, \\ 0 & \text{if } t \geqslant e, \end{cases}$$

and $0 \leqslant v(t) \leqslant 1$ if $t \in [e, e^{1-1/X}]$. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n}\Big(v\Big(\exp\Big(\frac{n}{\log X}\Big)\Big)\Big)^2 = \sum_{n \leqslant (1-1/X)\log X} \frac{1}{n} + O\Big(\sum_{(1-1/X)\log X < n \leqslant \log X} \frac{1}{n}\Big).$$

The first sum on the right-hand side equals $\log\log X + \gamma + O(1/\log X)$, and the second is $O(X^{-1})$. Hence, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n} v\Big(\exp\Big(\frac{n}{\log X}\Big)\Big)^2 = \log\log X + \gamma + O\Big(\frac{1}{\log X}\Big).$$

Using this and the value of $b(0)$ given by Lemma 5, we obtain

$$E = \exp\big(-k^2(\log\log X + \gamma)\big)\frac{(G(k+1))^2}{G(2k+1)}\Big(1 + O_k\Big(\frac{1}{\log X}\Big)\Big).$$

The proof of Theorem 4 is completed by combining this, the case $n = 0$ of Lemma 4, (21), and (23). □

Proof of Lemma 4
We evaluate

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log b(\theta) e^{-\mathrm{i}n\theta}\, \mathrm{d}\theta,$$

where $b(\theta)$ is given by (22). After some straightforward algebra, we see that this equals

$$\frac{-k}{\pi}\int_0^{\pi} \log(2 - 2\cos\theta)\cos n\theta\, \mathrm{d}\theta + \frac{2k}{\pi}\int_1^e u(y)\Big(\int_0^{\infty} \mathrm{Ci}(\theta \log y \log X)\cos n\theta\, \mathrm{d}\theta\Big)\, \mathrm{d}y. \tag{24}$$

When $n = 0$, the first integral vanishes by symmetry, and the second vanishes because

$$\int_0^{\infty} \mathrm{Ci}(\theta)\, \mathrm{d}\theta = 0.$$

This is a special case of the formula (see Gradshteyn and Ryzhik [8, page 645])

$$\int_0^\infty \mathrm{Ci}(A\theta)\cos n\theta \,\mathrm{d}\theta = \begin{cases} -\dfrac{\pi}{2n} & \text{if } A < n, \\ -\dfrac{\pi}{4n} & \text{if } A = n, \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

for $A > 0$, which we require in (27) as well. Thus, both terms in (24) vanish, and Lemma 4 holds in this case.

When n is a positive integer, the first term in (24) equals

$$\begin{aligned} -\frac{k}{\pi}\int_0^\pi \log(2-2\cos\theta)\cos n\theta \,\mathrm{d}\theta &= -\frac{k}{\pi}\int_0^\pi \Big(\log 4 + 2\log\Big(\sin\frac{\theta}{2}\Big)\Big)\cos n\theta \,\mathrm{d}\theta \\ &= -\frac{4k}{\pi}\int_0^{\pi/2} \log(\sin\theta)\cos 2n\theta \,\mathrm{d}\theta \\ &= \frac{k}{n} \end{aligned} \tag{26}$$

(see Gradshteyn and Ryzhik [8, page 584]). The second term in (24) is, by (25),

$$\begin{aligned} \frac{2k}{\pi}\int_1^e u(y)\Big(\int_0^\infty \mathrm{Ci}(\theta\log y\log X)\cos n\theta \,\mathrm{d}\theta\Big)\,\mathrm{d}y &= -\frac{k}{n}\int_1^{e^{n/\log X}} u(y)\,\mathrm{d}y \\ &= \frac{k}{n}\big(v(e^{n/\log X})-1\big). \end{aligned} \tag{27}$$

Inserting (27) and (26) into (24), we find that for $n > 0$, an integer

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log b(\theta) e^{-\mathrm{i}n\theta}\,\mathrm{d}\theta = \frac{k}{n} v(e^{n/\log X}).$$

This completes the proof of Lemma 4. □

Proof of Lemma 5
We calculate $b(0)$, where

$$b(\theta) = \exp\Big(-k\log(2-2\cos\theta) + 2k\int_1^e u(y)\Big(\sum_{j=-\infty}^{\infty}\mathrm{Ci}(|\theta+2\pi j|\log y\log X)\Big)\,\mathrm{d}y\Big). \tag{28}$$

Using the expansion

$$\mathrm{Ci}(x) = \gamma + \log x + O(x^2)$$

for $x > 0$, we find that the first term in the exponent and the $j = 0$ term combined contribute

$$\begin{aligned}
&-k\log(2-2\cos\theta) + 2k\int_1^e u(y)\,\mathrm{Ci}(|\theta|\log y\log X)\,\mathrm{d}y \\
&\quad = 2k\Big\{-\log(|\theta|) + O(\theta^2) + \int_1^e u(y)\big(\log(|\theta|\log y\log X) + \gamma + O_X(\theta^2)\big)\,\mathrm{d}y\Big\} \\
&\quad = 2k\Big\{\gamma + \log\log X + \int_1^e u(y)\log\log y\ \mathrm{d}y + O_X(\theta^2)\Big\}
\end{aligned}$$

since $u(x)$ has total mass one. Moreover, $u(x)$ is supported in $[e^{1-1/X}, e]$, so we have

$$\int_1^e u(y)\log\log y\,\mathrm{d}y \ll \frac{1}{X}.$$

Therefore, we find that

$$\begin{aligned}
&\lim_{\theta\to 0}\Big\{-k\log(2-2\cos\theta) + 2k\int_1^e u(y)\,\mathrm{Ci}(|\theta|\log y\log X)\,\mathrm{d}y\Big\} \\
&\quad = 2k(\log\log X + \gamma) + O_k\Big(\frac{1}{X}\Big).
\end{aligned} \tag{29}$$

Now consider the contribution of the terms with $j \neq 0$ in (28). An integration by parts shows that

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,\mathrm{d}t = \frac{\sin x}{x} + O\Big(\frac{1}{x^2}\Big)$$

for x positive and $\gg 1$. Thus, since $(1-1/X)\log X \leqslant \log y\ \log X \leqslant \log X$, $X > 2$, and $\theta \in (-\pi, \pi]$, we see that

$$\begin{aligned}
&\sum_{\substack{j=-\infty\\ j\neq 0}}^{\infty} \mathrm{Ci}(|\theta + 2\pi j|\log y\log X) \\
&\quad = \frac{1}{\log y\log X}\sum_{\substack{j=-\infty\\ j\neq 0}}^{\infty}\frac{\sin(|\theta+2\pi j|\log y\log X)}{|\theta+2\pi j|} + O\Big(\frac{1}{(\log X)^2}\Big).
\end{aligned}$$

In a standard way (via Abel partial summation), one can show that the series on the right-hand side is uniformly convergent for $y \in [e^{1-1/X}, e]$, except possibly in the neighborhood of a finite number of points, and boundedly convergent over the whole interval. Moreover, the series may be bounded independently of $\theta \in (-\pi, \pi]$. We may therefore multiply by the continuous function $u(y)$ and integrate to find that

$$\int_1^e u(y)\Big(\sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \mathrm{Ci}(|\theta+2\pi j|\log y\log X)\Big)\,\mathrm{d}y \ll \frac{1}{\log X}\int_1^e \frac{u(y)}{\log y}\,\mathrm{d}y + O\Big(\frac{1}{(\log X)^2}\Big)$$
$$\ll \frac{1}{\log X}$$

uniformly for $\theta \in (-\pi,\pi]$. Combining this and (29) with (28), we obtain

$$\begin{aligned} b(0) &= \exp\left(2k(\log\log X+\gamma)+O_k\Big(\frac{1}{\log X}\Big)\right) \\ &= \exp\big(2k(\log\log X+\gamma)\big)\left(1+O_k\Big(\frac{1}{\log X}\Big)\right). \end{aligned}$$

This completes the proof of Lemma 5. □

5. Proof of Theorem 3

We begin with a lemma.

LEMMA 6
Define

$$Q_X(s) = \prod_{p\leqslant\sqrt{X}}\Big(1-\frac{1}{p^s}\Big)\prod_{\sqrt{X}<p\leqslant X}\Big(1-\frac{1}{p^s}+\frac{1}{2p^{2s}}\Big)$$

and

$$R_X(s) = \prod_{p\leqslant\sqrt{X}}\Big(1-\frac{2}{p^s}+\frac{1}{p^{2s}}\Big)\prod_{\sqrt{X}<p\leqslant X}\Big(1-\frac{2}{p^s}+\frac{2}{p^{2s}}\Big).$$

Then for X sufficiently large, we have

$$P_X(s)^{-1} = Q_X(s)\left(1+O\Big(\frac{1}{\log X}\Big)\right) \tag{30}$$

and

$$P_X(s)^{-2} = R_X(s)\left(1+O\Big(\frac{1}{\log X}\Big)\right) \tag{31}$$

uniformly for $\sigma \geqslant 1/2$.

Proof
By (6), we have

$$P_X(s) = \prod_{p \leqslant X} \exp\Big(\sum_{1 \leqslant j \leqslant N_p} \frac{1}{j\, p^{js}} \Big),$$

where $N_p = [\log X / \log p]$, the integer part of $\log X / \log p$. The primes $\sqrt{X} < p \leqslant X$ have $N_p = 1$, so a straightforward calculation shows that

$$P_X(s) Q_X(s) = \exp\bigg(-\sum_{p \leqslant \sqrt{X}} \sum_{j > N_p} \frac{1}{j p^{js}} + \sum_{\sqrt{X} < p \leqslant X} \Big(\frac{1}{p^s} - \sum_{j=1}^{\infty} \frac{1}{j} \Big(\frac{1}{p^s} - \frac{1}{2p^{2s}} \Big)^j \Big) \bigg).$$

The argument of the exponential is clearly

$$\ll \sum_{p \leqslant \sqrt{X}} \frac{1}{p^{\sigma(N_p+1)}} + \sum_{\sqrt{X} < p \leqslant X} \frac{1}{p^{3\sigma}}.$$

Since $N_p + 1 > \log X / \log p$, we have $p^{N_p+1} > X$. Thus, for $\sigma \geqslant 1/2$, this is

$$\ll X^{-1/2} \sum_{p \leqslant \sqrt{X}} 1 + \sum_{\sqrt{X} < p \leqslant X} \frac{1}{p^{3/2}} \ll \frac{1}{\log X}.$$

It follows that

$$P_X(s) Q_X(s) = 1 + O\Big(\frac{1}{\log X} \Big),$$

which is equivalent to (30).

To prove (31), we square $Q_X(s)$ and note that for X large enough,

$$\begin{aligned}
\prod_{\sqrt{X} < p \leqslant X} \Big(1 - \frac{1}{p^s} + \frac{1}{2p^{2s}} \Big)^2 &= \prod_{\sqrt{X} < p \leqslant X} \Big(1 - \frac{2}{p^s} + \frac{2}{p^{2s}} + O\Big(\frac{1}{p^{3\sigma}} \Big) \Big) \\
&= \prod_{\sqrt{X} < p \leqslant X} \Big(1 - \frac{2}{p^s} + \frac{2}{p^{2s}} \Big) \Big(1 + O\Big(\frac{1}{p^{3\sigma}} \Big) \Big) \\
&= \prod_{\sqrt{X} < p \leqslant X} \Big(1 - \frac{2}{p^s} + \frac{2}{p^{2s}} \Big) \Big(1 + O\Big(\frac{1}{\log X} \Big) \Big).
\end{aligned}$$

The approximation in (31) now follows.

We now prove Theorem 3 when $k = 1$. In this case, $G^2(k+1)/G(2k+1) = G^2(2)/G(3) = 1$, and by Lemma 6 we may replace $P_X(1/2+\mathrm{i}t)^{-1}$ by $Q_X(1/2+\mathrm{i}t)$. Thus, it suffices to show that for $X \to \infty$ with $X \ll (\log T)^{2-\epsilon}$,

$$\frac{1}{T}\int_T^{2T} \left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)Q_X\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2 \mathrm{d}t = \frac{\log T}{e^{\gamma}\log X}\left(1+O\Big(\frac{1}{\log X}\Big)\right).$$

Let $\mathscr{S}(X) = \{n : p \mid n \implies p \leqslant X\}$ be the set of X-smooth numbers, and write

$$Q_X\Big(\frac{1}{2}+\mathrm{i}t\Big) = \sum_{n\in\mathscr{S}(X)} \frac{\alpha_{-1}(n)}{n^{1/2+\mathrm{i}t}},$$

where $\alpha_{-1}(n) = \mu(n)$, the Möbius function, if $n \in \mathscr{S}(\sqrt{X})$, $\alpha_{-1}(p) = \mu(p)$ for all $p \leqslant X$, and $\alpha_{-1}(n) \ll d(n)$ for all $n \in \mathscr{S}(X)$. As in (16), if the ϵ above is sufficiently small, then we find that

$$Q_X\Big(\frac{1}{2}+\mathrm{i}t\Big) = \sum_{\substack{n\leqslant T^{\theta}\\ n\in\mathscr{S}(X)}} \frac{\alpha_{-1}(n)}{n^{1/2+\mathrm{i}t}} + O(T^{-\theta\epsilon/10}). \tag{32}$$

(The exponent $1/10$ in place of $1/2$ is accounted for by the slight difference between the conditions $X \ll (\log T)^{2-\epsilon}$ and $X \ll (\log T)^{1/(1/2+\epsilon)}$.) Now for m and n coprime positive integers, we have the formula

$$\int_T^{2T}\left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2\Big(\frac{m}{n}\Big)^{\mathrm{i}t}\mathrm{d}t = \frac{T}{\sqrt{mn}}\left(\log\Big(\frac{T}{2\pi mn}\Big)+2\gamma-1\right)+O\big(mnT^{8/9}(\log T)^6\big)$$

(see, e.g., [12, Corollary 24.5]). Using this and the main term in (32) with $\theta = 1/20$, we find that

$$\begin{aligned}\frac{1}{T}\int_T^{2T} &\left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2\bigg|\sum_{\substack{n\leqslant T^{1/20}\\ n\in\mathscr{S}(X)}}\frac{\alpha_{-1}(n)}{n^{1/2+\mathrm{i}t}}\bigg|^2\mathrm{d}t\\ &= \sum_{\substack{m,n\leqslant T^{1/20}\\ m,n\in\mathscr{S}(X)}}\frac{\alpha_{-1}(m)}{m}\frac{\alpha_{-1}(n)}{n}(m,n)\bigg\{\log\Big(\frac{T(m,n)^2}{2\pi mn}\Big)+2\gamma-1\\ &\qquad\qquad +O\Big(\frac{mn}{(m,n)^2}T^{-1/9}(\log T)^6\Big)\bigg\},\end{aligned} \tag{33}$$

where (m,n) denotes the greatest common divisor of m and n. The O-term contributes

$$\ll T^{-1/9}(\log T)^6\Big(\sum_{n\leqslant T^{1/20}} d(n)\Big)^2 \ll T^{-1/90}(\log T)^8.$$

Grouping together those m and n for which $(m,n)=g$, replacing m by gm and n by gn, and then using the inequality $d(ab)\leqslant d(a)d(b)$, we find that

$$\sum_{\substack{m,n\leqslant T^{\theta}\\ m,n\in\mathscr{S}(X)}}\frac{\alpha_{-1}(m)}{m}\frac{\alpha_{-1}(n)}{n}(m,n)\bigg(\log\Big(\frac{(m,n)^2}{2\pi mn}\Big)+2\gamma-1\bigg)$$
$$\ll\sum_{g\in\mathscr{S}(X)}\frac{1}{g}\sum_{\substack{m,n\in\mathscr{S}(X)\\ (m,n)=1}}\frac{d(gm)d(gn)\log mn}{mn}\ll\sum_{g\in\mathscr{S}(X)}\frac{d(g)^2}{g}\Big(\sum_{n\in\mathscr{S}(X)}\frac{d(n)\log n}{n}\Big)^2.$$

If we write $f(\sigma)=\sum_{n\in\mathscr{S}(X)}d(n)n^{-\sigma}=\prod_{p\leqslant X}(1-p^{-\sigma})^{-2}$, then the sum over n is $-f^{'}(1)$, which, by logarithmic differentiation, is $2f(1)\sum_{p\leqslant X}\log p/(p-1)\ll f(1)(\log X)\ll(\log X)^3$. We also have $\sum_{g\in\mathscr{S}(X)}d(g)^2g^{-1}\ll\prod_{p\leqslant X}(1-p^{-1})^{-4}\ll(\log X)^4$, and so the expression above is $\ll(\log X)^{10}$.

Thus far, then, we have

$$\frac{1}{T}\int_T^{2T}\Big|\zeta\Big(\frac12+\mathrm{i}t\Big)\Big|^2\Big|\sum_{\substack{n\leqslant T^{1/20}\\ n\in\mathscr{S}(X)}}\frac{\alpha_{-1}(n)}{n^{1/2+\mathrm{i}t}}\Big|^2\,\mathrm{d}t$$
$$=\log T\sum_{\substack{m,n\leqslant T^{1/20}\\ m,n\in\mathscr{S}(X)}}\frac{\alpha_{-1}(m)}{m}\frac{\alpha_{-1}(n)}{n}(m,n)+O\big((\log X)^{10}\big). \tag{34}$$

Since $\sum_{g|n}\phi(g)=n$, the remaining sum here is

$$\sum_{\substack{m,n\leqslant T^{1/20}\\ m,n\in\mathscr{S}(X)}}\frac{\alpha_{-1}(m)}{m}\frac{\alpha_{-1}(n)}{n}\Big(\sum_{\substack{g|m\\ g|n}}\phi(g)\Big)=\sum_{\substack{g\leqslant T^{1/20}\\ g\in\mathscr{S}(X)}}\frac{\phi(g)}{g^2}\Big(\sum_{\substack{n\leqslant T^{1/20}g^{-1}\\ n\in\mathscr{S}(X)}}\frac{\alpha_{-1}(gn)}{n}\Big)^2. \tag{35}$$

We extend the sums on the right-hand side to all of $\mathscr{S}(X)$. For this, we use several estimates. First,

$$\sum_{n\in\mathscr{S}(X)}\frac{|\alpha_{-1}(gn)|}{n}\leqslant d(g)\sum_{n\in\mathscr{S}(X)}\frac{d(n)}{n}=d(g)\prod_{p\leqslant X}\Big(1-\frac1p\Big)^{-2}\ll d(g)(\log X)^2.$$

Second,

$$\sum_{\substack{n>T^{1/20}g^{-1}\\ n\in\mathscr{S}(X)}}\frac{|\alpha_{-1}(gn)|}{n}\leqslant d(g)\sum_{\substack{n>T^{1/20}g^{-1}\\ n\in\mathscr{S}(X)}}\frac{d(n)}{n}\leqslant d(g)\Big(\frac{T^{1/20}}{g}\Big)^{-1/4}\sum_{n\in\mathscr{S}(X)}\frac{d(n)}{n^{3/4}}$$
$$\ll d(g)g^{1/4}T^{-1/80}\prod_{p\leqslant X}\Big(1-\frac{1}{p^{3/4}}\Big)^{-2}$$
$$\ll d(g)g^{1/4}T^{-1/80}e^{10X^{1/4}/\log X}\ll d(g)g^{1/4}T^{-1/100},$$

say. From these it follows that the square of the sum over n in (35) is

$$\Big(\sum_{n\in\mathscr{S}(X)}\frac{\alpha_{-1}(gn)}{n}\Big)^2+O\big(d(g)^2g^{1/2}T^{-1/200}\big). \tag{36}$$

Similarly, we find that

$$\sum_{g\in\mathscr{S}(X)}\frac{\phi(g)d(g)^2}{g^{3/2}}\ll T^{1/400}\qquad\text{and}\qquad\sum_{\substack{g>T^{1/20}\\ g\in\mathscr{S}(X)}}\frac{\phi(g)d(g)^2}{g^2}\ll T^{-1/100}.$$

Using these and (36), we find that the right-hand side of (35) equals

$$\begin{aligned}&\Big(\sum_{g\in\mathscr{S}(X)}-\sum_{\substack{g>T^{1/20}\\ g\in\mathscr{S}(X)}}\Big)\frac{\phi(g)}{g^2}\Big(\sum_{n\in\mathscr{S}(X)}\frac{\alpha_{-1}(gn)}{n}\Big)^2+O\Big(T^{-1/200}\sum_{\substack{g\leqslant T^{1/20}\\ g\in\mathscr{S}(X)}}\frac{\phi(g)d(g)^2}{g^{3/2}}\Big)\\ &=\sum_{g\in\mathscr{S}(X)}\frac{\phi(g)}{g^2}\Big(\sum_{n\in\mathscr{S}(X)}\frac{\alpha_{-1}(gn)}{n}\Big)^2+O(T^{-1/400}).\end{aligned}$$

Combining this with (34), we now have

$$\begin{aligned}&\frac{1}{T}\int_T^{2T}\Big|\zeta\Big(\frac12+\mathrm{i}t\Big)\Big|^2\Big|\sum_{\substack{n\leqslant T^{1/20}\\ n\in\mathscr{S}(X)}}\frac{\alpha_{-1}(n)}{n^{1/2+\mathrm{i}t}}\Big|^2\,\mathrm{d}t\\ &=\log T\sum_{g\in\mathscr{S}(X)}\frac{\phi(g)}{g^2}\Big(\sum_{n\in\mathscr{S}(X)}\frac{\alpha_{-1}(gn)}{n}\Big)^2+O\big((\log X)^{10}\big).\end{aligned} \tag{37}$$

Since α_{-1} and ϕ are multiplicative functions, we may expand the entire sum into the Euler product

$$\prod_{p\leqslant X}\Big(\sum_r\sum_j\sum_k\frac{\varphi(p^r)\alpha_{-1}(p^{j+r})\alpha_{-1}(p^{k+r})}{p^{2r+j+k}}\Big).$$

Recall that $\alpha_{-1}(n)=\mu(n)$, the Möbius function, if $n\in\mathscr{S}(\sqrt{X})$, $\alpha_{-1}(p)=\mu(p)$ for all $p\leqslant X$, and $\alpha_{-1}(n)\ll d(n)$ for all $n\in\mathscr{S}(X)$. Thus, the product equals

$$\begin{aligned}\prod_{p\leqslant\sqrt{X}}\Big(1-\frac1p\Big)\prod_{\sqrt{X}<p\leqslant X}\Big(1-\frac1p+O\Big(\frac{1}{p^2}\Big)\Big)&=\prod_{p\leqslant X}\Big(1-\frac1p\Big)\prod_{\sqrt{X}<p\leqslant X}\Big(1+O\Big(\frac{1}{p^2}\Big)\Big)\\ &=\frac{1}{e^{\gamma}\log X}\Big(1+O\Big(\frac{1}{\log X}\Big)\Big).\end{aligned}$$

Since $\log X \ll \log\log T$, it now follows from (37) that

$$\frac{1}{T}\int_T^{2T} \left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2 \Big| \sum_{\substack{n\leqslant T^{1/20}\\ n\in\mathscr{S}(X)}} \frac{\alpha_{-1}(n)}{n^{1/2+\mathrm{i}t}}\Big|^2 \,\mathrm{d}t = \frac{\log T}{e^{\gamma}\log X}\Big(1+O\Big(\frac{1}{\log X}\Big)\Big). \tag{38}$$

Rewriting (32) (with $\theta = 1/20$) as $Q_X(1/2+\mathrm{i}t) = \sum + O(T^{-\epsilon/200})$, we see that

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T} \left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)Q_X\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2 \mathrm{d}t\\
&\quad= \frac{1}{T}\int_T^{2T} \left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2 \Big|\sum + O(T^{-\epsilon/200})\Big|^2 \,\mathrm{d}t\\
&\quad= \frac{1}{T}\int_T^{2T} \left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2 \Big|\sum\Big|^2 \,\mathrm{d}t + O\Big(\frac{1}{T^{1+\epsilon/200}}\int_T^{2T}\left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2\Big|\sum\Big|\,\mathrm{d}t\Big)\\
&\qquad+ O\Big(\frac{1}{T^{1+\epsilon/100}}\int_T^{2T}\left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2\mathrm{d}t\Big).
\end{aligned}$$

The final term is $O(T^{-\epsilon/200})$ since the second moment of the zeta function is $O(T\log T)$. Also, by the Cauchy-Schwarz inequality and (38), the second term is

$$\begin{aligned}
&\ll \frac{1}{T^{1+\epsilon/200}}\Big(\int_T^{2T}\left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\sum\right|^2\mathrm{d}t \int_T^{2T}\left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2\mathrm{d}t\Big)^{1/2}\\
&\ll \frac{1}{T^{1+\epsilon/200}}\Big(\frac{T^2\log^2 T}{\log X}\Big)^{1/2} \ll T^{-\epsilon/400}.
\end{aligned}$$

From these estimates and (38), we may now conclude that

$$\frac{1}{T}\int_T^{2T} \left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)Q_X\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2 \mathrm{d}t = \frac{\log T}{e^{\gamma}\log X}\Big(1+O\Big(\frac{1}{\log X}\Big)\Big)$$

for $X = O((\log T)^{2-\epsilon})$. This completes the proof of Theorem 3 in the case $k=1$.

We now prove Theorem 3 for $k=2$. By Lemma 6, we may replace $P_X(1/2+\mathrm{i}t)^{-2}$ by $R_X(1/2+\mathrm{i}t)$. Furthermore, $G^2(3)/G(5) = 1/12$, so it suffices to show that

$$\frac{1}{T}\int_T^{2T} \left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)^2 R_X\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2 \mathrm{d}t = \frac{1}{12}\big(1+o(1)\big)\Big(\frac{\log T}{e^{\gamma}\log X}\Big)^4 \tag{39}$$

for $X \ll (\log T)^{2-\epsilon}$. By (16) (see (32) also and the comment following it), we have

$$R_X\Big(\frac{1}{2}+\mathrm{i}t\Big) = \sum_{\substack{n\leqslant T^{\theta}\\ n\in\mathscr{S}(X)}} \frac{\alpha_{-2}(n)}{n^{1/2+\mathrm{i}t}} + O(T^{-\epsilon\theta/10}),$$

say, where $\alpha_{-2}(p) = -2$ for all $p \leqslant X$, $\alpha_{-2}(p^2) = 1$ if $p \leqslant \sqrt{X}$, $\alpha_{-2}(p^2) = 2$ if $\sqrt{X} < p \leqslant X$, and $\alpha_{-2}(p^j) = 0$ otherwise. In particular, we note that $|\alpha_{-2}(n)| \leqslant d(n)$.

In carrying out the proof of splitting for this case, we gloss over some of the less important steps as these are handled analogously to those for the $k = 1$ case. In particular, by an argument similar to the one at the end of the proof of the case $k = 1$, one can show that

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T} \left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big) P_X\Big(\frac{1}{2}+\mathrm{i}t\Big)^{-1}\right|^4 \mathrm{d}t \\
&\quad = \left(1+O\Big(\frac{1}{\log X}\Big)\right)\frac{1}{T}\int_T^{2T} \left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)^2 R_X\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2 \mathrm{d}t \\
&\quad = \left(1+O\Big(\frac{1}{\log X}\Big)\right)\frac{1}{T}\int_T^{2T} \left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)^2 \sum_{\substack{n\leqslant Y\\ n\in\mathcal{S}(X)}} \frac{\alpha_{-2}(n)}{n^{1/2+\mathrm{i}t}}\right|^2 \mathrm{d}t, \qquad (40)
\end{aligned}$$

where $Y = T^{\epsilon_1}$ and $\epsilon_1 = 1/\log X$.

To estimate the right-hand side, we use an analogue of (33) due to Jose Gaggero Jara [6]. Let $A(s) = \sum_{n\leqslant Y} a_n n^{-s}$, where the a_n are complex coefficients and $Y = T^\theta$ with $\theta < 1/150$. Gaggero's formula is

$$\begin{aligned}
&\left(1+O\Big(\frac{1}{(\log T)^B}\Big)\right)\frac{1}{T}\int_T^{2T} \left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big) A\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2 \mathrm{d}t \\
&= \sum_{k=1}^{4}\left\{\sum_{m,n\leqslant Y} \frac{c_k(m,n)a_m\overline{a_n}}{mn}(m,n)\left(\log^k\Big(\frac{YT(m,n)}{2\pi mn}\Big) + \log^k\Big(\frac{T(m,n)}{2\pi Y}\Big)\right)\right\} \\
&\quad - \sum_{m,n\leqslant Y}\frac{a_m\overline{a_n}}{mn}\sum_{0<d<Y/4}\frac{(m,d)(n,d)}{d}\left(\log\Big(\frac{Y}{4d}\Big)+O(1)\right)\sum_{v<V_1}\frac{1}{v}\sum_{\substack{u<U_1\\ (n_d u, m_d v)=1}}\frac{1}{u} \\
&\quad - \sum_{m,n\leqslant Y}\frac{a_m\overline{a_n}}{mn}\sum_{0<d<mn/4Y}\frac{(m,d)(n,d)}{d}\left(\log\Big(\frac{mn}{4dY}\Big)+O(1)\right)\sum_{v<V_1'}\frac{1}{v}\sum_{\substack{u<U_1'\\ (n_d u, m_d v)=1}}\frac{1}{u}.
\end{aligned}$$

Here,

$$\begin{aligned}
U_1 &= \frac{CYT}{dn_d}, & V_1 &= \frac{CYT}{dm_d}, \\
U_1' &= \frac{CmnT}{Ydn_d}, & V_1' &= \frac{CmnT}{Ydm_d},
\end{aligned} \qquad (41)$$

$C = 2/\pi$, B is an arbitrary positive number, and for integers n and d we write $n_d = n/(n,d)$. Also, $c_4(m,n) = (1/4\pi^2)\delta(m_n)\delta(n_m)$, where

$$\delta(n) = \prod_{p^r \| n} \Big(1 + r\frac{(1-1/p)}{(1+1/p)}\Big),$$

and $c_j(m,n) \ll |c_4(m,n)|(\log\log 3mn)^{4-j}$ for $j = 1, 2, 3$.

To estimate the right-hand side of (40), we take $a_n = \alpha_{-2}(n)$ and $Y = T^{\epsilon_1}$ in this and obtain

$$\begin{aligned}
&\Big(1 + O\Big(\frac{1}{\log X}\Big)\Big)\frac{1}{T}\int_T^{2T} \Big|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)^2 R_X\Big(\frac{1}{2}+\mathrm{i}t\Big)\Big|^2 \,\mathrm{d}t \\
&= \Big(\frac{1}{2\pi^2} + O(\epsilon_1)\Big)\log^4 T \sum_{\substack{m,n\leqslant Y\\ m,n\in\mathscr{S}(X)}} \frac{\alpha_{-2}(m)\alpha_{-2}(n)\delta(m/(m,n))\delta(n/(m,n))}{mn}(m,n) \\
&\quad - \sum_{\substack{m,n\leqslant Y\\ m,n\in\mathscr{S}(X)}} \frac{\alpha_{-2}(m)\alpha_{-2}(n)}{mn} \sum_{0<d<Y/4} \frac{(m,d)(n,d)}{d}\Big(\log\Big(\frac{Y}{4d}\Big) + O(1)\Big) \\
&\quad \times \sum_{v<V_1}\frac{1}{v} \sum_{\substack{u<U_1\\ (n_d u, m_d v)=1}} \frac{1}{u} - \sum_{\substack{m,n\leqslant Y\\ m,n\in\mathscr{S}(X)}} \frac{\alpha_{-2}(m)\alpha_{-2}(n)}{mn} \\
&\quad \times \sum_{0<d<mn/4Y} \frac{(m,d)(n,d)}{d}\Big(\log\Big(\frac{mn}{4dY}\Big) + O(1)\Big) \sum_{v<V_1'}\frac{1}{v} \sum_{\substack{u<U_1'\\ (n_d u, m_d v)=1}} \frac{1}{u} \\
&= \mathscr{T}_1 - \mathscr{T}_2 - \mathscr{T}_3, \qquad (42)
\end{aligned}$$

say.

Let us denote the sum in $\mathscr{T}_1$ by S_1. Grouping together those terms for which $(m,n) = g$ and then replacing m by mg and n by ng, we obtain

$$S_1 = \sum_{\substack{g\leqslant Y\\ g\in\mathscr{S}(X)}} \frac{1}{g} \sum_{\substack{n\leqslant Y/g\\ n\in\mathscr{S}(X)}} \frac{\alpha_{-2}(gn)\delta(n)}{n}\Big(\sum_{\substack{m\leqslant Y/g\\ (m,n)=1\\ m\in\mathscr{S}(X)}} \frac{\alpha_{-2}(gm)\delta(m)}{m}\Big). \qquad (43)$$

Let $P = \prod_{p\leqslant X} p$. Since α_{-2} is supported on cube-free integers, the g's we are summing over may be restricted to numbers of the form

$$g = g_1 g_2^2, \quad \text{where } g_1 \mid P,\ g_2 \mid \Big(\frac{P}{g_1}\Big).$$

Note that this representation is unique and that $(g_1, g_2) = 1$. The summation over g in (43) may therefore be replaced by the double sum

$$\sum_{\substack{g_1 \leqslant Y \\ g_1 | P}} \sum_{\substack{g_2 \leqslant (Y/g_1)^{1/2} \\ g_2 | (P/g_1)}} .$$

In the sum over n, we group terms together according to their greatest common divisor with $g = g_1 g_2^2$. Observe that we may assume that $(n, g_2) = 1$; otherwise, a cube divides $g_1 g_2^2 n$, and $\alpha_{-2}(gn)$ vanishes. If we then write $(n, g_1) = r$ and $n = rN$, we may replace the sum over n in (43) by

$$\sum_{r | g_1} \sum_{\substack{N \leqslant (Y/rg_1 g_2^2) \\ N \in \mathscr{S}(X) \\ (N, (g_1/r) g_2) = 1}} .$$

Ignoring the restriction $(m, n) = 1$ for the moment, we may similarly write the sum over m in (43) as

$$\sum_{s | g_1} \sum_{\substack{M \leqslant (Y/sg_1 g_2^2) \\ M \in \mathscr{S}(X) \\ (M, (g_1/s) g_2) = 1}} .$$

Instead of $(m, n) = 1$, we now have $(sM, rN) = 1$ or, equivalently, $(M, N) = (r, s) = (N, s) = (M, r) = 1$. We may impose the condition $(r, s) = 1$ by replacing $s \mid g_1$ in (5) by $s \mid (g_1/r)$ since g_1 is square-free. Furthermore, since $(N, g_1/r) = 1$ and $s \mid (g_1/r)$, we automatically have $(N, s) = 1$. Thus, the coprimality conditions on M are $(M, (g_1/s)g_2) = (M, N) = (M, r) = 1$. The first condition implies the third because $r \mid (g_1/s)$. Thus, we need only require that $(M, N(g_1/s)g_2) = 1$. The sum over m may therefore be written

$$\sum_{s | g_1/r} \sum_{\substack{M \leqslant (Y/sg_1 g_2^2) \\ M \in \mathscr{S}(X) \\ (M, N(g_1/s) g_2) = 1}} .$$

We now have

$$S_1 = \sum_{\substack{g_1 \leqslant Y \\ g_1 | P}} \frac{1}{g_1} \sum_{\substack{g_2 \leqslant (Y/g_1)^{1/2} \\ g_2 | (P/g_1)}} \frac{1}{g_2^2} \sum_{r | g_1} \frac{1}{r} \sum_{s | (g_1/r)} \frac{1}{s} \sum_{\substack{N \leqslant (Y/rg_1 g_2^2) \\ N \in \mathscr{S}(X) \\ (N, (g_1/r) g_2) = 1}} \frac{\alpha_{-2}(r^2 g_2^2 N (g_1/r)) \delta(rN)}{N}$$
$$\times \sum_{\substack{M \leqslant (Y/sg_1 g_2^2) \\ M \in \mathscr{S}(X) \\ (M, N(g_1/s) g_2) = 1}} \frac{\alpha_{-2}(s^2 g_2^2 M (g_1/s)) \delta(sM)}{M}.$$

Note that if N and r have a common factor, then $\alpha_{-2}(r^2 g_2^2 N(g_1/r)) = 0$ and similarly for M and s. We may therefore replace the coprimality conditions in the sums over N and M by $(N, g_1 g_2) = 1$ and $(M, N g_1 g_2) = 1$, respectively. The new conditions then imply that $\alpha_{-2}(r^2 g_2^2 N(g_1/r)) = \alpha_{-2}(r^2)\alpha_{-2}(g_2^2)\alpha_{-2}(N)\alpha_{-2}(g_1/r)$, $\delta(rN) = \delta(r)\delta(N)$, and similarly for $\alpha_{-2}(s^2 g_2^2 M(g_1/s))$ and $\delta(sM)$. Hence,

$$S_1 = \sum_{\substack{g_1 \leqslant Y \\ g_1 | P}} \frac{\alpha_{-2}(g_1)^2}{g_1} \sum_{\substack{g_2 \leqslant (Y/g_1)^{1/2} \\ g_2 | (P/g_1)}} \frac{\alpha_{-2}(g_2^2)^2}{g_2^2} \sum_{r | g_1} \frac{\alpha_{-2}(r^2)\delta(r)}{\alpha_{-2}(r) r} \sum_{s | (g_1/r)} \frac{\alpha_{-2}(s^2)\delta(s)}{\alpha_{-2}(s) s}$$
$$\times \sum_{\substack{N \leqslant (Y/r g_1 g_2^2) \\ N \in \mathscr{S}(X) \\ (N, g_1 g_2) = 1}} \frac{\alpha_{-2}(N)\delta(N)}{N} \sum_{\substack{M \leqslant (Y/s g_1 g_2^2) \\ M \in \mathscr{S}(X) \\ (M, N g_1 g_2) = 1}} \frac{\alpha_{-2}(M)\delta(M)}{M}.$$

We next extend each of the sums here to all of $\mathscr{S}(X)$. The error terms that this introduces are handled as they were in the case $k = 1$, and they contribute at most "little o" of the main term. Observing also that M and N may be restricted to cube-free integers, we obtain

$$S_1 = \big(1 + o(1)\big) \sum_{g_1 | P} \frac{\alpha_{-2}(g_1)^2}{g_1} \sum_{g_2 | (P/g_1)} \frac{\alpha_{-2}(g_2^2)^2}{g_2^2} \sum_{r | g_1} \frac{\alpha_{-2}(r^2)\delta(r)}{\alpha_{-2}(r) r} \sum_{s | (g_1/r)} \frac{\alpha_{-2}(s^2)\delta(s)}{\alpha_{-2}(s) s}$$
$$\times \sum_{N | (P/g_1 g_2)^2} \frac{\alpha_{-2}(N)\delta(N)}{N} \sum_{M | (P/N g_1 g_2)^2} \frac{\alpha_{-2}(M)\delta(M)}{M}. \tag{44}$$

We now define the following multiplicative functions:

$$A(n) = \sum_{d | n} \frac{\alpha_{-2}(d)\delta(d)}{d} = \prod_{p^a || n} \Big(1 + \frac{\alpha_{-2}(p)\delta(p)}{p} + \cdots + \frac{\alpha_{-2}(p^a)\delta(p^a)}{p^a}\Big),$$

$$B(n) = \sum_{d | n} \frac{\alpha_{-2}(d)\delta(d)}{d A(d^2)} = \prod_{p^a || n} \Big(1 + \frac{\alpha_{-2}(p)\delta(p)}{p\, A(p^2)} + \cdots + \frac{\alpha_{-2}(p^a)\delta(p^a)}{p^a\, A(p^{2a})}\Big),$$

$$C(n) = \sum_{d | (n, P)} \frac{\alpha_{-2}(d^2)\delta(d)}{\alpha_{-2}(d) d} = \prod_{p | (n, P)} \Big(1 + \frac{\alpha_{-2}(p^2)\delta(p)}{\alpha_{-2}(p) p}\Big),$$

$$D(n) = \sum_{d | (n, P)} \frac{\alpha_{-2}(d^2)\delta(d)}{\alpha_{-2}(d) C(d) d} = \prod_{p | (n, P)} \Big(1 + \frac{\alpha_{-2}(p^2)\delta(p)}{\alpha_{-2}(p) C(p) p}\Big),$$

$$E(n) = \sum_{d | n} \frac{\alpha_{-2}(d^2)^2}{A(d^2) B(d^2)\, d^2} = \prod_{p | l} \Big(1 + \frac{\alpha_{-2}(p^2)^2}{A(p^2) B(p^2)\, p^2}\Big),$$

and

$$F(n) = \sum_{d | n} \frac{\alpha_{-2}(d)^2 C(d) D(d)}{A(d^2) B(d^2) E(d)\, d} = \prod_{p | l} \Big(1 + \frac{\alpha_{-2}(p)^2 C(p) D(p)}{A(p^2) B(p^2) E(p)\, d}\Big).$$

Using these definitions and working from the inside out in (44), we find first that the sum over M is $A((P/Ng_1g_2)^2) = A(P^2)/A(N^2)A(g_1^2)A(g_2^2)$. The contribution of the sums over M and N together is then $(A(P^2)/A(g_1^2)A(g_2^2))(B(P^2)/B(g_1^2)B(g_2^2))$. Thus, so far, we have

$$S_1 = \big(1+o(1)\big)A(P^2)B(P^2)\sum_{g_1|P}\frac{\alpha_{-2}(g_1)^2}{g_1A(g_1^2)B(g_1^2)}\sum_{g_2|(P/g_1)}\frac{\alpha_{-2}(g_2^2)^2}{g_2^2A(g_2^2)B(g_2^2)}$$
$$\times\sum_{r|g_1}\frac{\alpha_{-2}(r^2)\delta(r)}{\alpha_{-2}(r)r}\sum_{s|(g_1/r)}\frac{\alpha_{-2}(s^2)\delta(s)}{\alpha_{-2}(s)s}.$$

The sums over r and s contribute $C(g_1)D(g_1)$, and the sum over g_2 is then $E(P)/E(g_1)$. Thus, we see that

$$S_1 = \big(1+o(1)\big)A(P^2)B(P^2)E(P)\sum_{g_1|P}\frac{\alpha_{-2}(g_1)^2C(g_1)D(g_1)}{g_1A(g_1^2)B(g_1^2)E(g_1)}$$
$$= \big(1+o(1)\big)A(P^2)B(P^2)E(P)F(P).$$

Using the expression for $F(P)$ as a product, we see that this is the same as

$$S_1 = \big(1+o(1)\big)\prod_{p|P}\Big(A(p^2)B(p^2)E(p)+\frac{\alpha_{-2}(p)^2C(p)D(p)}{p}\Big). \tag{45}$$

By the definitions of C and D, we see that

$$C(p)D(p) = \Big(1+\frac{\alpha_{-2}(p^2)\delta(p)}{\alpha_{-2}(p)p}\Big)+\frac{\alpha_{-2}(p^2)\delta(p)}{\alpha_{-2}(p)p}$$
$$= 1-\frac{\alpha_{-2}(p^2)\delta(p)}{p} \tag{46}$$

since $\alpha_{-2}(p) = -2$ for p dividing P. Similarly,

$$A(p^2)\,B(p^2)\,E(p) = A(p^2)\,B(p^2)+\frac{\alpha_{-2}(p^2)^2}{p^2}.$$

It is clear that $A(p^2) = A(p^3) = \cdots$. Therefore,

$$B(p^2) = 1+\frac{\alpha_{-2}(p)\delta(p)}{p\,A(p^2)}+\frac{\alpha_{-2}(p^2)\delta(p^2)}{p^2\,A(p^2)}$$
$$= 1+\frac{1}{A(p^2)}\big(A(p^2)-1\big)$$
$$= 2-\frac{1}{A(p^2)}$$

and

$$A(p^2)\,B(p^2)\,E(p) = 2A(p^2) - 1 + \frac{\alpha_{-2}(p^2)^2}{p^2}.$$

We use this, (46), and $\alpha_{-2}(p) = -2$, and we obtain

$$\begin{aligned}
&A(p^2)B(p^2)E(p) + \frac{\alpha_{-2}(p)^2 C(p)D(p)}{p}\\
&= 2A(p^2) - 1 + \frac{\alpha_{-2}(p^2)^2}{p^2} + \frac{4}{p} - \frac{4\alpha_{-2}(p^2)^2\delta(p)}{p^2}\\
&= 2\Big(1 - \frac{2\delta(p)}{p} + \frac{\alpha_{-2}(p^2)\delta(p^2)}{p^2}\Big) - 1 + \frac{\alpha_{-2}(p^2)^2}{p^2} + \frac{4}{p} - \frac{4\alpha_{-2}(p^2)^2\delta(p)}{p^2}\\
&= 1 + \frac{4 - 4\delta(p)}{p} + \frac{\alpha_{-2}(p^2)(\alpha_{-2}(p^2) - 4\alpha_{-2}(p^2)\delta(p) + 2\delta(p^2))}{p^2}.
\end{aligned}$$

Recall that $\delta(p^r) = 1 + r((1-1/p)/(1+1/p))$, so that $\delta(p) = 2/(1+1/p)$ and $\delta(p^2) = 2\delta(p) - 1$. Also recall that $\alpha_{-2}(p^2) = 1$ if $p \leqslant \sqrt{X}$. Thus, for $p \leqslant \sqrt{X}$, the last line is

$$\begin{aligned}
&= 1 + \frac{4 - 4\delta(p)}{p} + \frac{1 - 4\delta(p) + 2(2\delta(p) - 1)}{p^2} = 1 + \frac{4 - 4\delta(p)}{p} - \frac{1}{p^2}\\
&= 1 + \frac{4}{p} - \frac{8}{p+1} - \frac{1}{p^2} = \frac{(1-1/p)^3}{1+1/p} = \frac{(1-1/p)^4}{1-1/p^2}.
\end{aligned}$$

On the other hand, if $\sqrt{X} < p \leqslant X$, then $\alpha_{-2}(p^2) = 2$, and the last line is

$$= \frac{(1-1/p)^4}{1-1/p^2} + O\Big(\frac{1}{p^2}\Big).$$

Combining these results in (45), we find that

$$\begin{aligned}
S_1 &= \big(1 + o(1)\big) \prod_{p \leqslant \sqrt{X}} \Big(\frac{(1-1/p)^4}{1-1/p^2}\Big) \prod_{\sqrt{X} < p \leqslant X} \Big(\frac{(1-1/p)^4}{1-1/p^2} + O\Big(\frac{1}{p^2}\Big)\Big)\\
&= \big(1 + o(1)\big) \prod_{p \leqslant X} \Big(\frac{(1-1/p)^4}{1-1/p^2}\Big) \prod_{\sqrt{X} < p \leqslant X} \Big(1 + O\Big(\frac{1}{p^2}\Big)\Big)\\
&= \big(1 + o(1)\big) \prod_{p \leqslant X} \Big(1 - \frac{1}{p}\Big)^4 \prod_{p} \Big(1 - \frac{1}{p^2}\Big)^{-1}\\
&= \big(1 + o(1)\big) \frac{\pi^2}{6} (e^\gamma \log X)^{-4}.
\end{aligned}$$

Since

$$\mathscr{T}_1 = S_1\left(\frac{1}{2\pi^2} + O(\epsilon_1)\right)\log^4 T$$

and $\epsilon_1 = 1/\log X$, we now see that

$$\mathscr{T}_1 = \left(\frac{1}{12} + o(1)\right)\left(\frac{\log T}{e^\gamma \log X}\right)^4. \tag{47}$$

□

To treat the second term on the right-hand side of (42), $\mathscr{T}_2$, we require two lemmas.

LEMMA 7

Suppose that a and b are positive integers with $(a, b) = 1$. Then for $b \leqslant x$, we have

$$\sum_{\substack{n\leqslant x\\(an,b)=1}} \frac{1}{n} = \frac{\phi(b)}{b}\log x + O(\log\log 2b).$$

Proof

Since $(a, b) = 1$, the condition $(an, b) = 1$ is equivalent to $(n, b) = 1$. Thus, the sum is

$$\sum_{\substack{n\leqslant x\\(n,b)=1}} \frac{1}{n} = \sum_{n\leqslant x}\frac{1}{n}\sum_{\substack{d|n\\d|b}}\mu(d) = \sum_{d|b}\frac{\mu(d)}{d}\sum_{m\leqslant x/d}\frac{1}{m} = \sum_{d|b}\frac{\mu(d)}{d}\left(\log\left(\frac{x}{d}\right) + O(1)\right).$$

Now, $\sum_{d|b}\mu(d)/d = \phi(b)/b$ and

$$\sum_{d|b}\frac{\mu(d)\log d}{d} = \frac{\phi(b)}{b}\sum_{p|b}\frac{\log p}{p-1} \ll \frac{\phi(b)}{b}\log\log 2b \ll \log\log 2b.$$

Furthermore,

$$\sum_{d|b}\frac{|\mu(d)|}{d} = \prod_{p|b}\left(1+\frac{1}{p}\right) \leqslant \frac{b}{\phi(b)} \ll \log\log 2b.$$

Thus, we find that

$$\sum_{\substack{n\leqslant x\\(an,b)=1}} \frac{1}{n} = \frac{\phi(b)}{b}\log x + O(\log\log 2b).$$

□

LEMMA 8

Let $\kappa(n)=\prod_{p|n}(1+1/p)^{-1}$, and let U,V be either U_1,V_1 or U_1',V_1' as defined in (41). If $m,n,d\ll Y\leqslant T^{1/150}$ and $(n_d,m_d)=1$, then

$$\sum_{v<V}\frac{1}{v}\sum_{\substack{u<U\\(n_du,m_dv)=1}}\frac{1}{u}=\frac{6}{\pi^2}\kappa(m_d)\kappa(n_d)\log U\ \log V+O(\log T\log\log T).$$

Proof

The conditions $(n_du,m_dv)=1$ and $(n_d,m_d)=1$ are equivalent to $(v,n_d)=1$ and $(u,m_dv)=1$. Hence, by Lemma 7, the double sum equals

$$\begin{aligned}\sum_{\substack{v<V\\(v,n_d)=1}}\frac{1}{v}\sum_{\substack{u<U\\(u,m_dv)=1}}\frac{1}{u}&=\log U\sum_{\substack{v<V\\(v,n_d)=1}}\frac{1}{v}\Big(\frac{\phi(m_dv)}{m_dv}+O\big(\log\log(m_dV)\big)\Big)\\&=\log U\sum_{\substack{v<V\\(v,n_d)=1}}\frac{1}{v}\Big(\frac{\phi(m_dv)}{m_dv}\Big)+O\left(\log T\log\log T\right).\end{aligned}\tag{48}$$

By denoting the sum on the right-hand side by $\sum$, we have

$$\sum=\sum_{\substack{v<V\\(v,n_d)=1}}\frac{1}{v}\sum_{r|m_dv}\frac{\mu(r)}{r}=\sum_{r<m_dV}\frac{\mu(r)}{r}\sum_{\substack{v<V\\(v,n_d)=1\\r|m_dv}}\frac{1}{v}.$$

Now set $(m_d,r)=g$, and write $r=gR$. Then $(m_d/g,R)=1$, and we find that

$$\sum=\sum_{g|m_d}\frac{1}{g}\sum_{\substack{R<m_dV/g\\(m_d/g,R)=1}}\frac{\mu(gR)}{R}\sum_{\substack{v<V\\(v,n_d)=1\\R|v}}\frac{1}{v}.$$

If we set $v=Rw$, then $w<V/R$, and $(Rw,n_d)=1$ is the same as the two conditions $(R,n_d)=1$ and $(w,n_d)=1$. Thus, using Lemma 7 and the observation that the inner sum vanishes unless $R<V$, we obtain

$$\begin{aligned}\sum&=\sum_{g|m_d}\frac{1}{g}\sum_{\substack{R<m_dV/g\\(m_d/g,R)=1\\(n_d,R)=1}}\frac{\mu(gR)}{R^2}\sum_{\substack{w<V/R\\(w,n_d)=1}}\frac{1}{w}\\&=\sum_{g|m_d}\frac{1}{g}\sum_{\substack{R<V\\(m_d/g,R)=1\\(n_d,R)=1}}\frac{\mu(gR)}{R^2}\Big(\frac{\phi(n_d)}{n_d}\log\frac{V}{R}+O(\log\log 2n_d)\Big).\end{aligned}$$

We may assume that $(R, g) = 1$; otherwise, $\mu(gR) = 0$. The coprimality conditions on the sum may then be written $(m_d n_d, R) = 1$, and we find that

$$\begin{aligned}\sum &= \sum_{g|m_d} \frac{\mu(g)}{g} \sum_{\substack{R<V \\ (R,m_d n_d)=1}} \frac{\mu(R)}{R^2}\Big(\frac{\phi(n_d)}{n_d}\log\frac{V}{R} + O(\log\log 2n_d)\Big) \\ &= \frac{\phi(n_d)}{n_d}\log V \sum_{g|m_d}\frac{\mu(g)}{g} \sum_{\substack{R<V \\ (R,m_d n_d)=1}} \frac{\mu(R)}{R^2} \\ &\quad + O\Big(\log\log 2n_d \sum_{g|m_d}\frac{|\mu(g)|}{g}\sum_{R<V}\frac{\log R}{R^2}\Big).\end{aligned}$$

Since $\sum_{g|m_d} |\mu(g)|/g = \prod_{p|m_d}(1+1/p) \ll \log\log 2m_d$, the error term is $\ll (\log\log 2m_d \, \log\log 2n_d)$. The main term is

$$\begin{aligned}&= \frac{\phi(n_d)}{n_d}\log V \sum_{g|m_d}\frac{\mu(g)}{g}\Big(\sum_{\substack{R=1 \\ (R,m_d n_d)=1}}^{\infty} \frac{\mu(R)}{R^2} + O(V^{-1})\Big) \\ &= \zeta(2)^{-1}\prod_{p|m_d n_d}\Big(1-\frac{1}{p^2}\Big)^{-1}\frac{\phi(n_d)}{n_d}\log V\sum_{g|m_d}\frac{\mu(g)}{g} + O\Big(\frac{\log V}{V}\sum_{g|m_d}\frac{|\mu(g)|}{g}\Big) \\ &= \frac{6}{\pi^2}\prod_{p|m_d n_d}\Big(1-\frac{1}{p^2}\Big)^{-1}\frac{\phi(m_d)}{m_d}\frac{\phi(n_d)}{n_d}\log V + O\Big(\frac{\log V \log\log 2m_d}{V}\Big).\end{aligned}$$

By hypothesis, $(m_d, n_d) = 1$. Furthermore, $\prod_{p|l}(1-1/p^2)^{-1}(\phi(l)/l) = \prod_{p|l}(1+1/p)^{-1} = \kappa(l)$. Thus, combining our estimates, we obtain

$$\sum = \frac{6}{\pi^2}\kappa(m_d)\kappa(n_d)\log V + O(\log\log 2m_d \log\log 2n_d).$$

Since $m, n \ll T^{1/150}$, we obtain from this and (48) that

$$\sum_{v<V}\frac{1}{v}\sum_{\substack{u<U \\ (n_d u, m_d v)=1}}\frac{1}{u} = \frac{6}{\pi^2}\kappa(m_d)\kappa(n_d)\log U \log V + O(\log T \log\log T). \qquad \square$$

Returning to $\mathcal{T}_2$ in (42) and using Lemma 8, we have

$$\begin{aligned}\mathcal{T}_2 = \frac{6}{\pi^2} &\sum_{\substack{m,n\leqslant Y \\ m,n\in\mathcal{S}(X)}} \frac{\alpha_{-2}(m)\alpha_{-2}(n)}{mn} \sum_{\substack{0<d<Y/4 \\ d\in\mathcal{S}(X)}} \frac{(m,d)(n,d)}{d}\Big(\log\Big(\frac{Y}{4d}\Big) + O(1)\Big) \\ &\times \big(\kappa(m_d)\,\kappa(n_d)\log U_1 \, \log V_1 + O(\log T \log\log T)\big),\end{aligned}$$

where $U_1 = CYT/dn_d$, $V_1 = CYT/dm_d$, and $Y = T^{\epsilon_1}$. By interchanging the order of summation, we find that

$$\mathscr{T}_2 = \frac{6}{\pi^2} \sum_{\substack{0<d<Y/4 \\ d\in\mathscr{S}(X)}} \frac{1}{d}\left(\log\left(\frac{Y}{4d}\right) + O(1)\right) \sum_{\substack{m,n\leqslant Y \\ m,n\in\mathscr{S}(X)}} \frac{\alpha_{-2}(m)\alpha_{-2}(n)(m,d)(n,d)}{mn}$$
$$\times \big(\kappa(m_d)\,\kappa(n_d)\log U_1\,\log V_1 + O(\log T\,\log\log T)\big).$$

Since $\kappa(n) \ll \log\log 3n$, the expression in the last parentheses is

$$=\kappa(m_d)\,\kappa(n_d)\big(\log U_1\,\log V_1 + O(\log T\log\log^3 T)\big)=\big(1+O(\epsilon_1)\big)\kappa(m_d)\,\kappa(n_d)\,\log^2 T.$$

Thus,

$$\mathscr{T}_2 = \frac{6}{\pi^2}\big(1+O(\epsilon_1)\big)\log^2 T$$
$$\times \sum_{\substack{0<d<Y/4 \\ d\in\mathscr{S}(X)}} \frac{(\log(Y/4d)+O(1))}{d}\Bigg(\sum_{\substack{n\leqslant Y \\ n\in\mathscr{S}(X)}} \frac{\alpha_{-2}(n)\,(n,d)\kappa(n/(n,d))}{n}\Bigg)^2. \tag{49}$$

Denote the inner sum by $S(d)$. By extending the sum to all of $\mathscr{S}(X)$, as we did, for example, after (35), we introduce an error term that is $o(1)$ times the main term. Thus, grouping together terms in $S(d)$ for which $(n,d)=e$, say, we obtain

$$S(d) = \big(1+o(1)\big)\sum_{e|d} e \sum_{\substack{n\in\mathscr{S}(X) \\ (n,d)=e}} \frac{\alpha_{-2}(n)\,\kappa(n/e)}{n}$$
$$= \big(1+o(1)\big)\sum_{e|d} \sum_{\substack{N\in\mathscr{S}(X) \\ (N,d/e)=1}} \frac{\alpha_{-2}(eN)\,\kappa(N)}{N}.$$

Since α_{-2} is supported on only cube-free numbers in $\mathscr{S}(X)$, we may assume that $e \mid P^2$. Therefore, $e \mid (d, P^2) = D$, say. Now D may be written uniquely as $D = D_1 D_2^2$, where $D_1 \mid P$ and $D_2 \mid (P/D_1)$, so that, in particular, $(D_1, D_2) = 1$. Furthermore, we may write any divisor e of D as $e = e_1 e_2 e_3^2$, where $e_1 \mid D_1$, $e_2 \mid D_2$, and $e_3 \mid (D_2/e_2)$. Note that this means that the e_i are pairwise coprime. The condition $(N, d/e) = 1$ is now $(N, (D_1 D_2^2/e_1 e_2 e_3^2)) = 1$. Also, $\alpha_{-2}(eN) = \alpha_{-2}(e_1 e_2 e_3^2 N)$, so we may assume that $(N, e_3) = 1$ and therefore that $(N, (D_1 D_2^2/e_1 e_2)) = 1$. Observe, moreover, that $e_2 \mid D_2$ implies $e_2 \mid (D_2^2/e_2)$. Thus, $(N, (D_1 D_2^2/e_1 e_2)) = 1$ is the same as $(N, (D_1 D_2/e_1)) = 1$. It follows that N and e_1 can have a common factor, but N and e_2 or e_3 cannot. We may therefore write $\alpha_{-2}(e_1e_2e_3^2N) = \alpha_{-2}(e_1N)\alpha_{-2}(e_2)\alpha_{-2}(e_3^2)$ and

$$\begin{aligned}
S(d) &= S(D) \\
&= \big(1+o(1)\big) \sum_{e_1|D_1} \sum_{e_2|D_2} \alpha_{-2}(e_2) \sum_{e_3|(D_2/e_2)} \alpha_{-2}(e_3^2) \sum_{\substack{N\in\mathscr{S}(X)\\(N,(D_1D_2/e_1))=1}} \frac{\alpha_{-2}(e_1N)\,\kappa(N)}{N} \\
&= \big(1+o(1)\big) \sum_{e_1|D_1} \sum_{\substack{N\in\mathscr{S}(X)\\(N,(D_1D_2/e_1))=1}} \frac{\alpha_{-2}(e_1N)\,\kappa(N)}{N} \sum_{e_3|D_2} \alpha_{-2}(e_3^2) \sum_{e_2|(D_2/e_3)} \alpha_{-2}(e_2).
\end{aligned}$$

The innermost sum is

$$\begin{aligned}
\sum_{e_2|(D_2/e_3)} \alpha_{-2}(e_2) = \prod_{p|(D_2/e_3)} \big(1+\alpha_{-2}(p)\big) &= \prod_{p|(D_2/e_3)} (1-2) \\
= \mu\Big(\frac{D_2}{e_3}\Big) &= \mu(D_2)\mu(e_3).
\end{aligned}$$

We also have

$$\sum_{e_3|D_2} \mu(e_3)\alpha_{-2}(e_3^2) = \prod_{p|D_2} \big(1-\alpha_{-2}(p^2)\big)\,.$$

At this point, it is convenient to define numbers

$$P_1 = \prod_{p\leqslant\sqrt{X}} p \qquad \text{and} \qquad P_2 = \prod_{\sqrt{X}<p\leqslant X} p.$$

Notice that $P = P_1P_2$. Since $\alpha_{-2}(p^2) = 1$ if $p \mid P_1$ and $\alpha_{-2}(p^2) = 2$ if $p \mid P_2$, the sum over e_3 equals zero unless $D_2 \mid P_2$, in which case it equals $\mu(D_2)$. Thus, if D_2 and P_1 have a common factor, $S(D_1D_2^2) = 0$, whereas if $D_2 \mid P_2$, then

$$S(d) = S(D_1D_2^2) = \big(1+o(1)\big) \sum_{e_1|D_1} \sum_{\substack{N\in\mathscr{S}(X)\\(N,(D_1D_2/e_1))=1}} \frac{\alpha_{-2}(e_1N)\,\kappa(N)}{N}.$$

From this point on, we therefore assume that $D_2 \mid P_2$.

Now set $(N, e_1) = r$, and write $N = rM$. Then we have

$$\begin{aligned}
S(D_1D_2^2) &= \big(1+o(1)\big) \sum_{e_1|D_1} \sum_{r|e_1} \sum_{\substack{N\in\mathscr{S}(X)\\(N,e_1)=r\\(N,(D_1D_2/e_1))=1}} \frac{\alpha_{-2}(e_1N)\,\kappa(N)}{N} \\
&= \big(1+o(1)\big) \sum_{e_1|D_1} \sum_{r|e_1} \frac{1}{r} \sum_{\substack{M\in\mathscr{S}(X)\\(M,e_1/r)=1\\(rM,(D_1D_2/e_1))=1}} \frac{\alpha_{-2}(r^2M(e_1/r))\,\kappa(rM)}{M}.
\end{aligned}$$

We may assume that $(M, r) = 1$ and $(r, e_1/r) = 1$ since, otherwise, $\alpha_{-2}(r^2M(e_1/r)) = 0$. Actually, $(r, e_1/r) = 1$ is automatically satisfied because $r \mid e_1$ and e_1 is square-free.

It follows that $\kappa(rM) = \kappa(r)\kappa(M)$ and, since we also have $(M, e_1/r) = 1$, that $\alpha_{-2}(r^2M(e_1/r)) = \alpha_{-2}(r^2)\alpha_{-2}(M)\alpha_{-2}(e_1/r)$. The coprimality conditions in the sum are now seen to be equivalent to the conditions $(M, r) = (r, e_1/r) = (M, e_1/r) = (M, (D_1D_2/e_1)) = (r, (D_1D_2/e_1)) = 1$. As we have already pointed out, the second of these is automatic. Similarly, so is the last. The remaining conditions are equivalent to $(M, D_1D_2) = 1$, so we find that

$$S(D_1D_2^2) = \big(1 + o(1)\big) \sum_{e_1|D_1} \alpha_{-2}(e_1) \sum_{r|e_1} \frac{\alpha_{-2}(r^2)\kappa(r)}{r\alpha_{-2}(r)} \sum_{\substack{M\in\mathscr{S}(X)\\(M,D_1D_2)=1}} \frac{\alpha_{-2}(M)\,\kappa(M)}{M}.$$

The sum over M equals

$$\prod_{p|(P/D_1D_2)} \Big(1 + \frac{\alpha_{-2}(p)\,\kappa(p)}{p} + \frac{\alpha_{-2}(p^2)\,\kappa(p^2)}{p^2}\Big) = G\Big(\frac{P}{D_1D_2}\Big), \tag{50}$$

say. Hence,

$$S(D_1D_2^2) = \big(1 + o(1)\big)G\Big(\frac{P}{D_1D_2}\Big) \sum_{e_1|D_1} \alpha_{-2}(e_1) \sum_{r|e_1} \frac{\alpha_{-2}(r^2)\kappa(r)}{r\alpha_{-2}(r)}.$$

The double sum equals

$$\begin{aligned}\sum_{r|D_1} \frac{\alpha_{-2}(r^2)\kappa(r)}{r\alpha_{-2}(r)} \sum_{f_1|(D_1/r)} \alpha_{-2}(f_1) &= \mu(D_1) \sum_{r|D_1} \frac{\mu(r)\alpha_{-2}(r^2)\kappa(r)}{r\alpha_{-2}(r)}\\ &= \mu(D_1) \prod_{p|D_1} \big(1 + \alpha_{-2}(p^2)\kappa(p)2p\big)\\ &= \mu(D_1)H(D_1),\end{aligned} \tag{51}$$

say. Thus,

$$S(d) = S(D_1D_2^2) = \big(1 + o(1)\big)G(P)\frac{\mu(D_1)H(D_1)}{G(D_1)G(D_2)}, \tag{52}$$

provided $D_2 \mid P_2$; otherwise, $S(d) = 0$.

We use this in (49). Recall that for each $d < Y/4$, we had set $(d, P^2) = D_1D_2^2$ with $D_1 \mid P$ and $D_2 \mid (P/D_1)$. Recall also that $P = P_1P_2$ and $Y = T^{\epsilon_1}$. We therefore see that

$$\begin{aligned}\mathcal{T}_2 &= \frac{6}{\pi^2}\big(1+O(\epsilon_1)\big)\log^2 T \sum_{\substack{0<d<Y/4\\ d\in\mathcal{S}(X)}} \frac{(\log(Y/4d)+O(1))}{d} S(d)^2\\ &= \frac{6}{\pi^2}\big(1+O(\epsilon_1)\big)\log^2 T \sum_{D_2|P_2} \frac{1}{D_2^2}\\ &\quad\times \sum_{D_1|(P/D_2)} \frac{S(D_1D_2^2)^2}{D_1} \sum_{\substack{0<\delta<Y/4D_1D_2^2\\ (\delta,(P_1P_2)^2/D_1D_2^2)=1}} \frac{(\log(Y/(4D_1D_2^2\delta))+O(1))}{\delta}.\end{aligned}$$

The coprimality condition in the last sum is equivalent to $(\delta, P_1P_2/D_2)=1$. Thus, using (52), we find that

$$\begin{aligned}\mathcal{T}_2 = \frac{6}{\pi^2}\big(1+O(\epsilon_1)\big)G(P)^2\log^2 T \sum_{D_2|P_2} \frac{1}{D_2^2G(D_2)^2} \sum_{D_1|(P/D_2)} \frac{H(D_1)^2}{D_1G(D_1)^2}\\ \times \sum_{\substack{0<\delta<Y/4D_1D_2^2\\ (\delta,P_1P_2/D_2)=1}} \frac{(\log(Y/(4D_1D_2^2\delta))+O(1))}{\delta}.\end{aligned}$$

By Lemma 7, the sum over δ is

$$\ll \log Y \sum_{\substack{0<\delta<Y/4D_1D_2^2\\ (\delta,P_1P_2/D_2)=1}} \frac{1}{\delta} \ll \frac{\phi(P)}{P}\frac{D_2}{\phi(D_2)}\log^2 Y.$$

Thus,

$$\mathcal{T}_2 \ll G(P)^2\frac{\phi(P)}{P}\log^2 T\log^2 Y \sum_{D_2|P_2} \frac{1}{D_2\phi(D_2)G(D_2)^2} \sum_{D_1|(P/D_2)} \frac{H(D_1)^2}{D_1G(D_1)^2}.$$

If we denote the innermost sum by $I(P/D_2)$, then

$$I\Big(\frac{P}{D_2}\Big) = \prod_{p|(P/D_2)}\Big(1+\frac{H(p)^2}{pG(p)^2}\Big), \tag{53}$$

and we find that

$$\begin{aligned}\mathcal{T}_2 &\ll G(P)^2I(P)\frac{\phi(P)}{P}\log^2 T\log^2 Y \sum_{D_2|P_2} \frac{1}{D_2\phi(D_2)G(D_2)^2I(D_2)}\\ &\ll \epsilon_1^2 G(P)^2I(P)\frac{\phi(P)}{P}\log^4 T \prod_{p|P_2}\Big(1+\frac{1}{p\phi(p)G(p)^2I(p)}\Big).\end{aligned}$$

Now, by the definitions of G, H, and I in (50), (51), and (53), we have $G(p) = 1 - 2/p + O(1/p^2)$, $H(p) = 1 + O(1/p)$, and $I(p) = 1 + 1/p\big((1 + O(1/p))/(1 - 2/p + O(1/p^2))\big)^2 = 1 + 1/p + O(1/p^2)$. From these estimates, it is clear that the product over p dividing P_2 here is $\prod_{\sqrt{X}<p\leqslant X}(1 + O(1/p^2)) \ll 1$. Thus,

$$\begin{aligned}\mathscr{T}_2 &\ll \epsilon_1^2 G(P)^2 I(P)\frac{\phi(P)}{P}\log^4 T\\ &\ll \epsilon_1^2 \log^4 T \prod_{p|P}\Big(\Big(1 - \frac{4}{p} + O\Big(\frac{1}{p^2}\Big)\Big)\Big(1 + O\Big(\frac{1}{p^2}\Big)\Big)\Big)\\ &\ll \epsilon_1^2 \log^4 T \prod_{p|P}\Big(1 - \frac{1}{p}\Big)^4 \ll {\epsilon_1}^2\Big(\frac{\log T}{\log X}\Big)^4.\end{aligned}$$

The treatment of $\mathscr{T}_3$ is almost identical and leads to the same bound. Thus, by combining our estimates for $\mathscr{T}_1$ (see (47)), $\mathscr{T}_2$, and $\mathscr{T}_3$ with (42) and noting that we may take $\epsilon_1 > 0$ as small as we like, we obtain (39). This completes the proof of the case $k = 2$ of Theorem 3 and, thus, the proof of the theorem. □

Appendix. Graphs

To illustrate Theorem 1, in Figures 1–3 we have plotted $|Z_X(1/2+\mathrm{i}t)|$ and $|P_X(1/2+\mathrm{i}t)|$ for t near the 10^{12}th zero for two values of X, and we have compared their product with the Riemann zeta function. The values of X used are $X = 26.31 \approx \log\gamma_{10^{12}}$ and $X = 1000$. Though the functions P_X and Z_X depend on X, when multiplied together the X dependence mostly cancels out, and we have an accurate pointwise

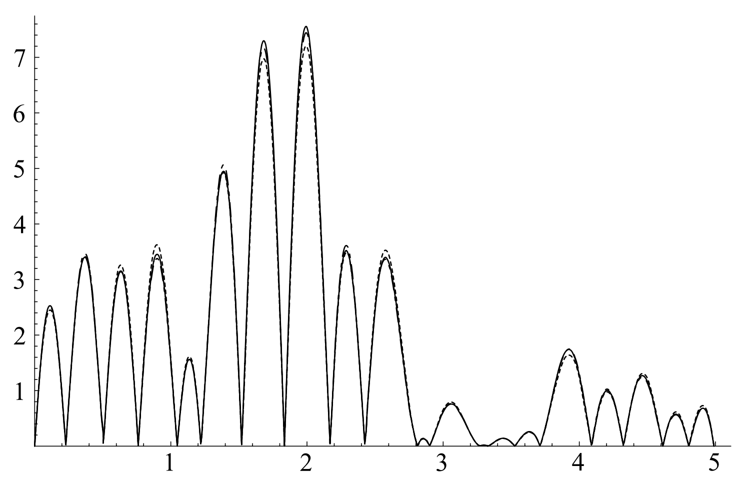

Figure 1. Graph of $|\zeta(1/2 + \mathrm{i}(x + t_0))|$ (solid) and $|P_X(1/2 + \mathrm{i}(x + t_0))Z_X(1/2 + \mathrm{i}(x + t_0))|$ with $t_0 = \gamma_{10^{12}+40}$, $X = \log t_0$ (dots), and $X = 1000$ (dash dots)

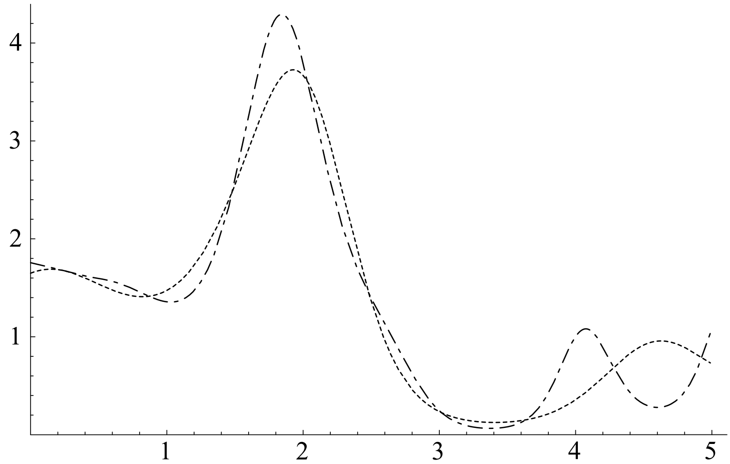

Figure 2. Graph of $|P_X(1/2+\mathrm{i}(x+t_0))|$ with $t_0=\gamma_{10^{12}+40}$, $X=\log t_0$ (dots), and $X=1000$ (dash dots)

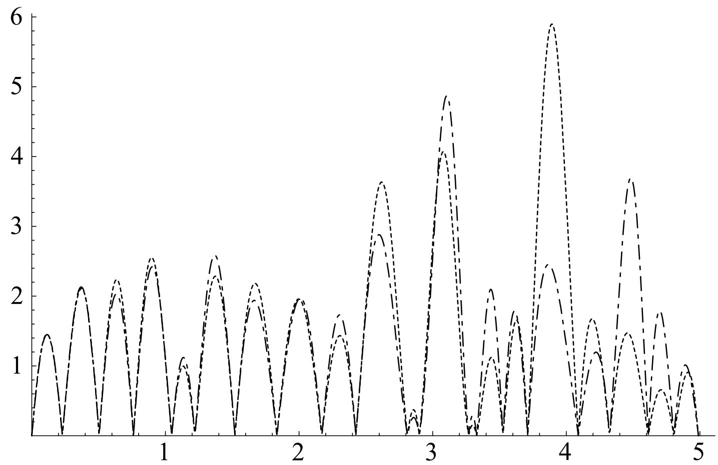

Figure 3. Graph of $|Z_X(1/2+\mathrm{i}(x+t_0))|$ with $t_0=\gamma_{10^{12}+40}$, $X=\log t_0$ (dots), and $X=1000$ (dash dots)

approximation to the zeta function. The actual functions plotted are

$$\left|P_X\Big(\frac{1}{2}+\mathrm{i}(x+t_0)\Big)\right| = \exp\Big(\sum_{n\leqslant X}\frac{\Lambda(n)\cos((x+t_0)\log n)}{\log n\sqrt{n}}\Big)$$

and

$$\left|Z_X\Big(\frac{1}{2}+\mathrm{i}(x+t_0)\Big)\right| = \prod_{n=N+1}^{N+100}\exp\big(\mathrm{Ci}(|x+t_0-\gamma_n|\log X)\big),$$

where $t_0 = \gamma_{10^{12}+40}$. The values of the zeros of the zeta function came from Andrew Odlyzko's table [19]. The functions were plotted for x between 0 and 5, a range covering the zeros between $\gamma_{10^{12}+40}$ and $\gamma_{10^{12}+60}$. Note that the function Z_X that we have plotted is an unsmoothed, truncated form of the function Z_X that appears in Theorem 1. Note also that $Z_X(1/2+\mathrm{i}t)$ oscillates more rapidly than $P_X(1/2+\mathrm{i}t)$, as asserted in our motivation for the splitting conjecture, Conjecture 2.

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Gonek
Department of Mathematics, University of Rochester, Rochester, N.Y. 14627, USA; gonek@math.rochester.edu

Hughes
Department of Mathematics, University of York, Heslington, York, YO10 5DD, United Kingdom; ch540@york.ac.uk

Keating
Department of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, United Kingdom; j.p.keating@bristol.ac.uk