

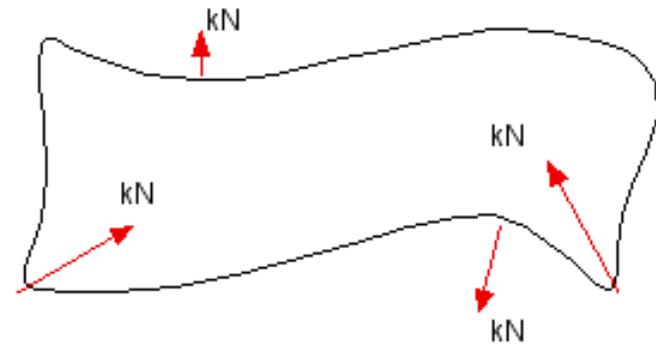
**Minkowski geometry,
curve shortening
and
flow by weighted mean curvature.**

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University of Rochester**

February 21, 2003

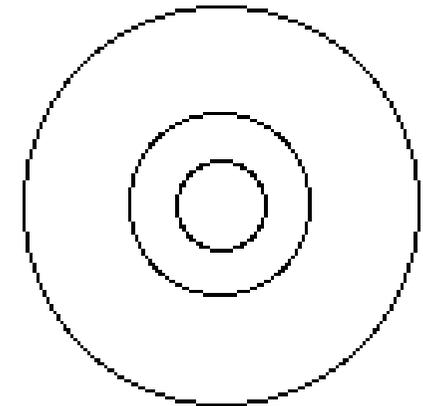
The flow by curvature

What is the “asymptotic shape” of this curve as it evolves along its curvature vector?



Does it depend on initial conditions?

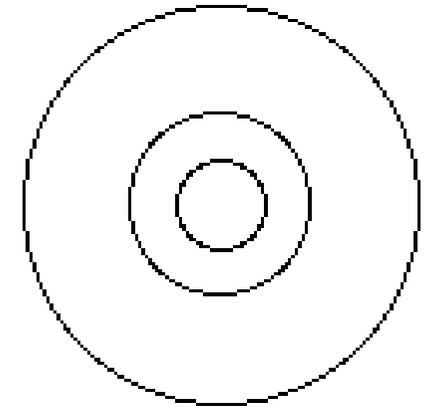
A circle clearly shrinks to a point.



Shrinking circles attract all flows

Prof. J. Sethian's curve shortening applet

A circle shrinks to a point.
This flow attracts the flow
of all other embedded
curves. Every embedded
curve shrinks to a point
and becomes asymptotically
circular.



Flow by curvature is the L^2 gradient flow of the length

First variation formula for length

$$d\mathcal{L}(\alpha) = \left. \frac{\partial \mathcal{L}(\gamma(t) + \epsilon \alpha(t))}{\partial \epsilon} \right|_{\epsilon=0} = \int_{\gamma} \langle \alpha, -kN \rangle ds$$

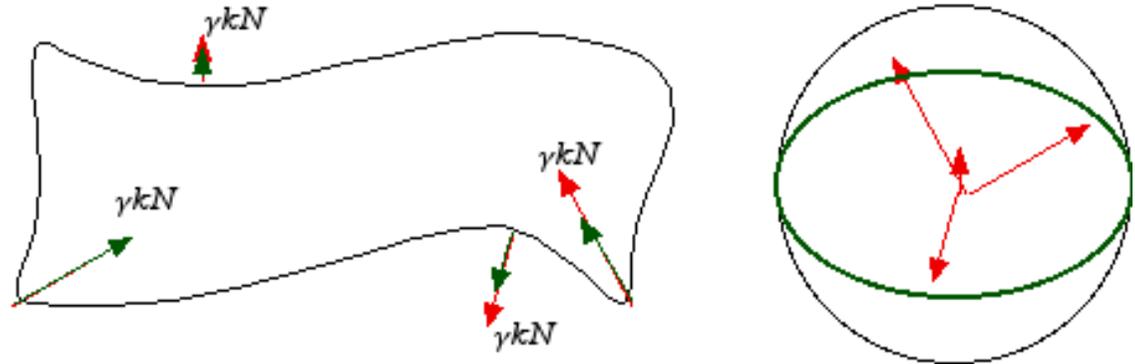
The negative gradient vector field, with respect to the L^2 metric on curves is

$$\nabla \mathcal{L} = kN$$

Flow by curvature is the curve shortening flow. It shortens length as 'quickly as possible'.

Weighted curvature flow

What if the velocity is not uniformly proportional to the curvature?



Suppose that the proportionality constant γ depends on the direction of the normal vector? or the position?

- What is the asymptotic shape?
- Is there are self-similar flow?
- Is it a gradient flow? and of what?

The plan for this talk:

- Early history of the curve shortening flow
- Other flows
- Applications
- Minkowski geometries and weighted curvature flow
- Crystalline geometry flows
- Open questions

Every (reasonable) weighted curvature flow is the curve shortening flow for some unique Minkowski geometry, and vice versa. It possesses a unique self-similar solution which makes the identification.

Early history of curve shortening

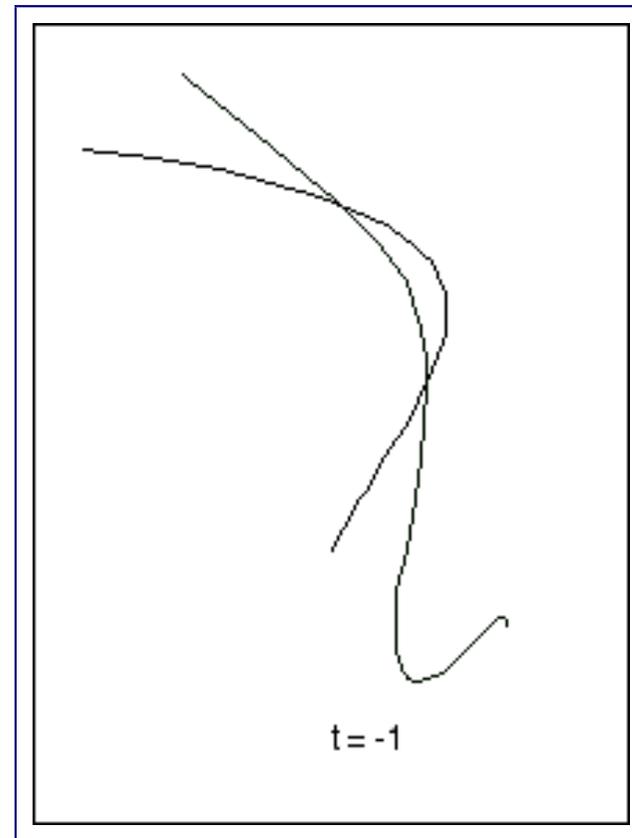
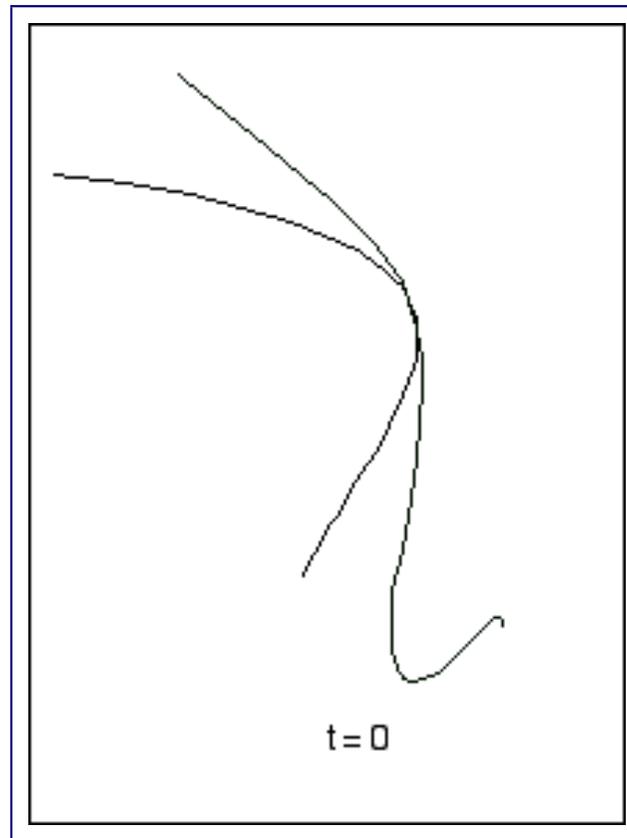
- 1981 Karen Uhlenbeck: Find embedded closed geodesics on surfaces by deforming the curve along its curvature vector.
- 1981 Herman Gluck, and UPenn math dept: Find asymptotic shape for curve shortening flow in the plane.
- 1982 Gage: The isoperimetric ratio $\frac{L^2}{A}$ decreases under the flow
- 1982 Gage: If the curvature doesn't blow up "prematurely", the area decreases to zero, $\frac{L^2}{A}$ decreases to 4π and the shape becomes circular.
- 1983 Gage and Richard Hamilton: For convex curves the curvature can't blow up prematurely. (JDG 1986)

- 1986 Matt Grayson: Every embedded closed curve in the plane flows to a convex curve, and then to a 'round' point.
- Matt Grayson: Every embedded closed curve on a surface flows to a round point or to a geodesic.

Expand on history

Why curves have to stay embedded: **The maximum principle**

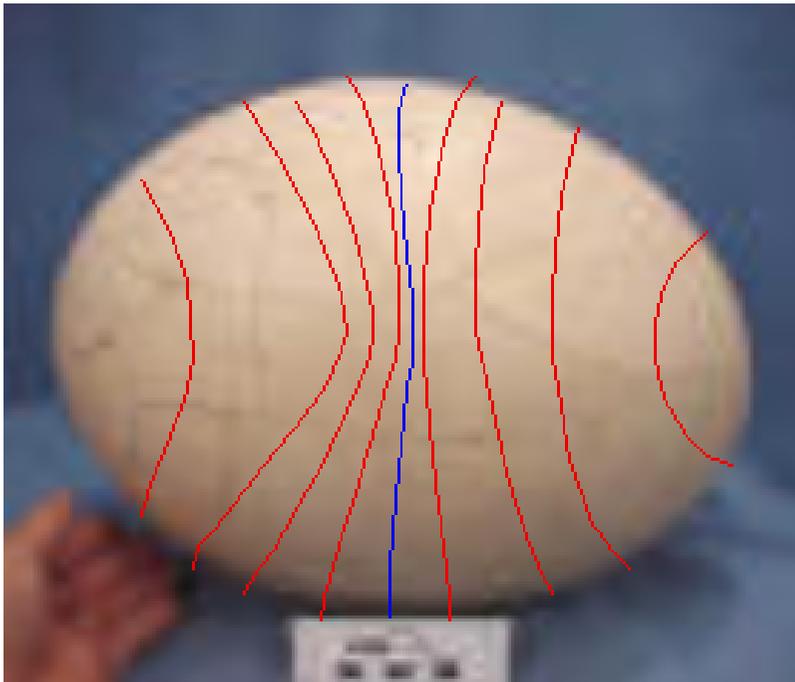
Embedded
curves cannot
cross



Finding embedded geodesics

Matt Grayson: On surfaces embedded curves flow to round points or to geodesics.

Find embedded closed geodesics by using a mini-max procedure due to Lyusternik and Schnierlmann.



Finding geodesics on ovaloids as global minima

Theorem: (Poincare-Croke 1982) On an ovaloid, consider the set of curves which bisect the total Gauss curvature $\int K dA$. The curve in this set which minimizes length is an embedded geodesic.

By the Gauss-Bonnet theorem, curves α which bisect the total Gauss curvature have total geodesic curvature equal to zero.

$$\int_{\alpha} k ds = 0.$$

Designer flows

The weighted curvature flow

$$\frac{\partial X}{\partial t} = \frac{k}{K} N$$

where k is geodesic curvature and K is Gauss curvature of the surface, preserves the quantity $\int_{\alpha} k ds$ and also shortens the length.

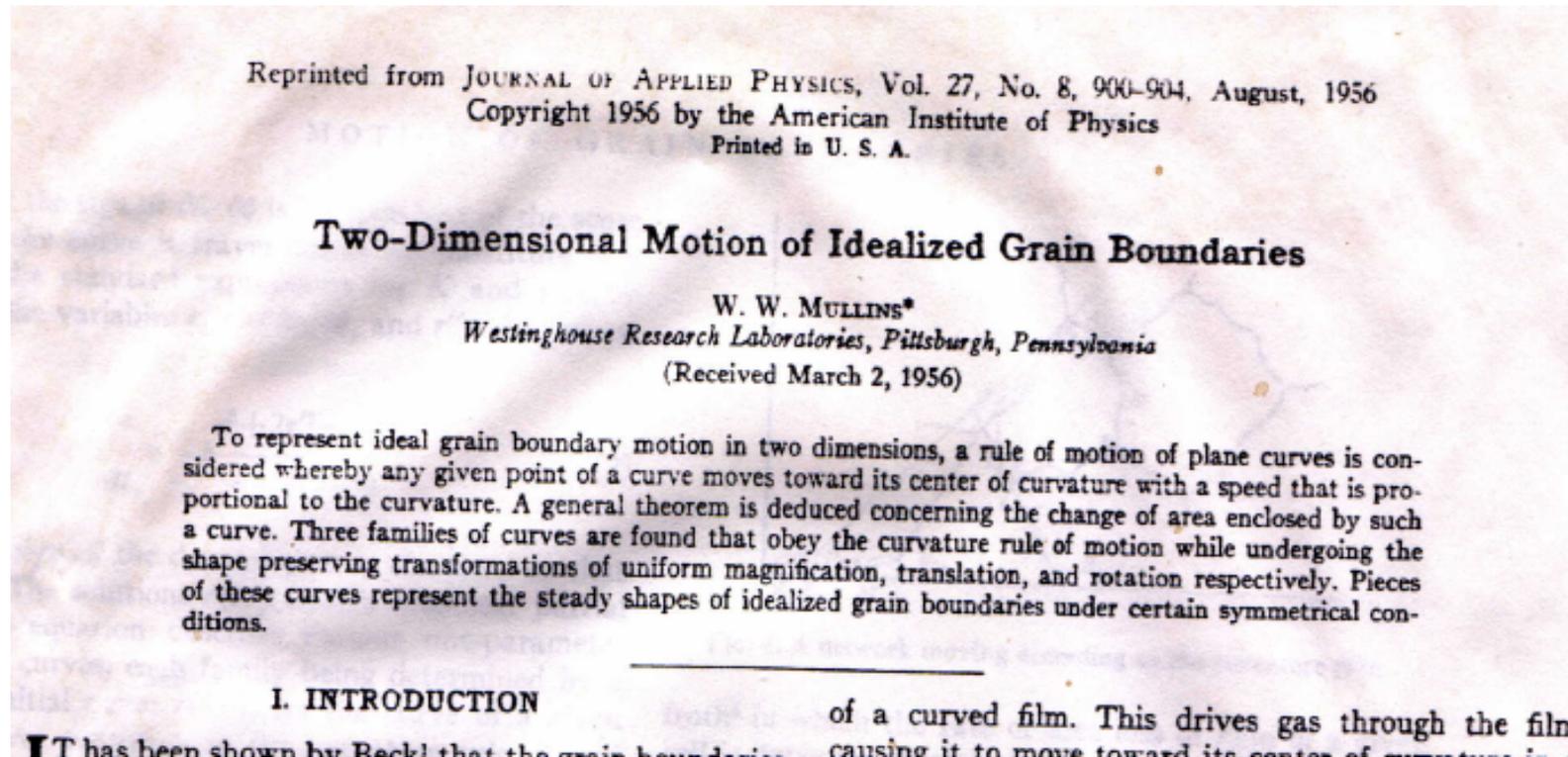
Theorem (Gage - 1985) If cusps don't develop then embedded curves with zero total geodesic curvature flow to embedded geodesics.

Theorem (Oaks 1990) Embedded curves can't develop cusps.

Every embedded curve with total curvature equal to zero flows “downhill” to a geodesic under this weighted curvature flow.

Prehistory 1956 Mullins

Mullins: Described several self-similar flows for $X_t = kN$ and showed that enclosed area decreases at a constant rate.



Context — other flows

1980's were a good decade for curvature flows:

The idea of taking an existing object and deforming it into a better object has a natural appeal.

- Curve shortening flow: $X_t = kN$
- R. Hamilton: Ricci flow for metrics $g_t = -Ric(g)$
- G. Huisken: The mean curvature flow ("area shortening")
 $X_t = HN$
- Gerhardt: Inverse mean curvature flow for star-like objects.
 $X_t = \frac{-1}{H}N$

Prehistory: Firey 1974 Flow by Gauss curvature $X_t = KN -$

Shapes of worn stones – see also B. Andrews

Flows have a similar character.

Curve shortening flow:

a two dimensional system of equations on a line (or S^1):

$$\frac{\partial x(s, t)}{\partial t} = k(x, y)N(x, y) \cdot \vec{e}_1$$

$$\frac{\partial y(s, t)}{\partial t} = k(x, y)N(x, y) \cdot \vec{e}_2$$

Ricci flow on 3 manifolds:

A 6 dimensional system of equations on charts in R^3 .

$$\frac{\partial g_{ij}(\vec{x})}{\partial t} = -Ric_{ij}(g(\vec{x}))$$

Flows have a similar character

Each of the flows is parabolic – it acts like an ODE, but with a laplacian term on the right hand side which smooths things out.

The curve shortening flow is the simplest in the sense that it can be reduced to a single equation for the curvature, rather than a system. It has served as the simplest non-trivial case for illustrating many techniques which apply to the other flows as well.

$$k_t = k_{ss} + k^3$$

Convex curves stay convex.

Applications of the flows

The curve shortening flow indicated a new way to find embedded closed geodesics.

Ricci curvature flow (Hamilton 1983):

- If a compact, simply connected three manifold has positive Ricci curvature, the metric deforms under the Ricci flow to one of constant positive curvature.
- Only 3-spheres have constant positive curvature
- The only simply connected, compact three manifolds carrying positive Ricci curvature are topological three spheres.

Broader significance of the flows –recent developments

Hamilton : In other cases the Ricci flow has singularities.

Understanding these singularities could allow decomposition of the three manifold in an understandable way, thus proving Thurston's geometrization conjecture.

G.

Perelman: The entropy formula for the Ricc flow and its geometric applications

Very recent results indicate a way to understand these singularities.

<http://arxiv.org/abs/math/0211159>

Stay tuned....

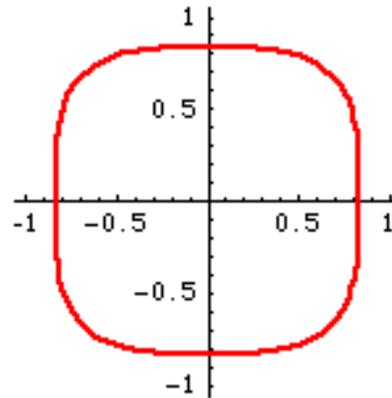
Today's subject: anisotropic curvature flows

Physics equals geometry

A weighted curvature
flow $\gamma(\theta)kN$

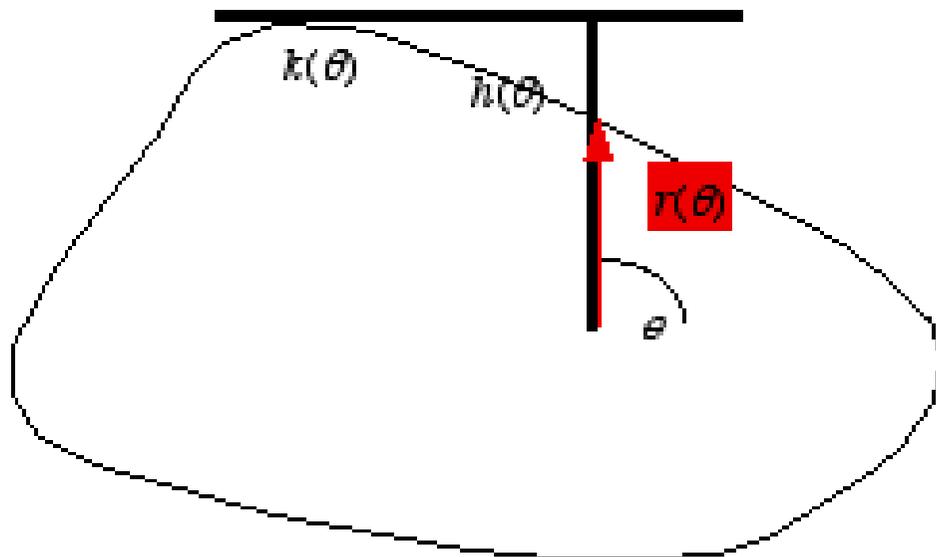


A specific Minkowski geometry



A unique self-similarly shrinking shape identifies the geometry and the weight γ with one important exception and a few technical caveats.

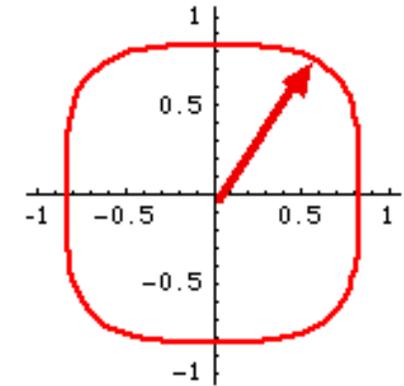
Notation: Support function and radial function



$$\frac{1}{k} = h_{\theta\theta} + h$$

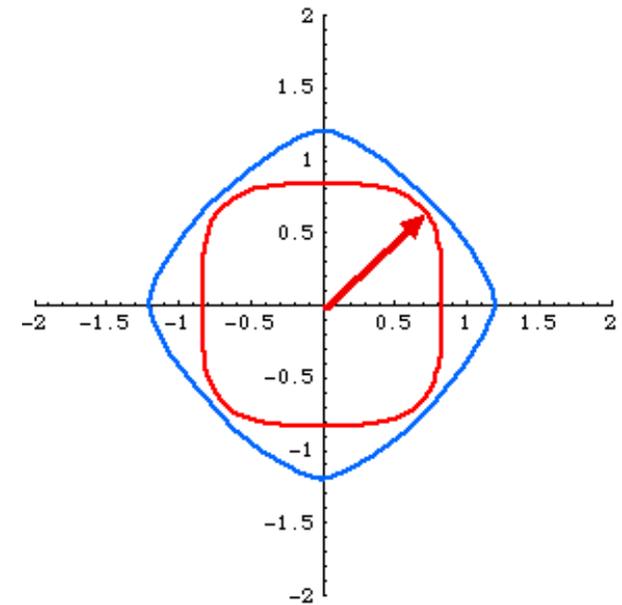
Minkowski geometry

A convex body with a distinguished center point defines the **unit length** in each direction.



- For most of this talk we'll assume the **unit ball** is symmetric through its center point and defines a norm.
- Our restrictions on γ (positive and smooth) will limit us to unit balls which have smooth boundary and are strictly convex.
- Define lengths of curves by approximation by polygonal lines.

The **isoperimetrix** solves the Minkowski isoperimetric problem



- Has the boundary of least length among those bodies with fixed (Euclidean) area.
- In metallurgy terms this shape is called the **Wulff** shape.
- The support function $\tilde{h}(\theta)$ of the isoperimetrix is the inverse of the radial function $r(\theta)$ of the unit ball.
- The isoperimetrix is the dual of the unit ball, rotated 90 degrees.

Frenet frames

- Euclidean: $\frac{d\mathbf{e}_1}{d\theta} = \mathbf{e}_2 \quad \frac{d\mathbf{e}_2}{d\theta} = -\mathbf{e}_1$
- Minkowski: $\frac{d\mathbf{t}}{d\theta} = a(\theta)\mathbf{n} \quad \frac{d\mathbf{n}}{d\theta} = b(\theta)\mathbf{t}$
- Up to a constant a and b are uniquely defined.
- $\mathbf{t} = r(\theta)\mathbf{e}_1 \quad \mathbf{n} = -\tilde{h}_\theta(\theta)\mathbf{e}_1 + \tilde{h}(\theta)\mathbf{e}_2 \quad \tilde{h}(\theta) = \frac{1}{r}$
- $\frac{d\mathbf{t}}{d\theta} = r^2\mathbf{n} \quad \frac{d\mathbf{n}}{d\theta} = -\tilde{h}(\tilde{h}_{\theta\theta} + \tilde{h})\mathbf{t}$
- $\frac{d\mathbf{t}}{d\theta} = r^2\mathbf{n} \quad \frac{d\mathbf{n}}{d\theta} = -\frac{\tilde{h}}{\tilde{k}}\mathbf{t}$

Frenet frames

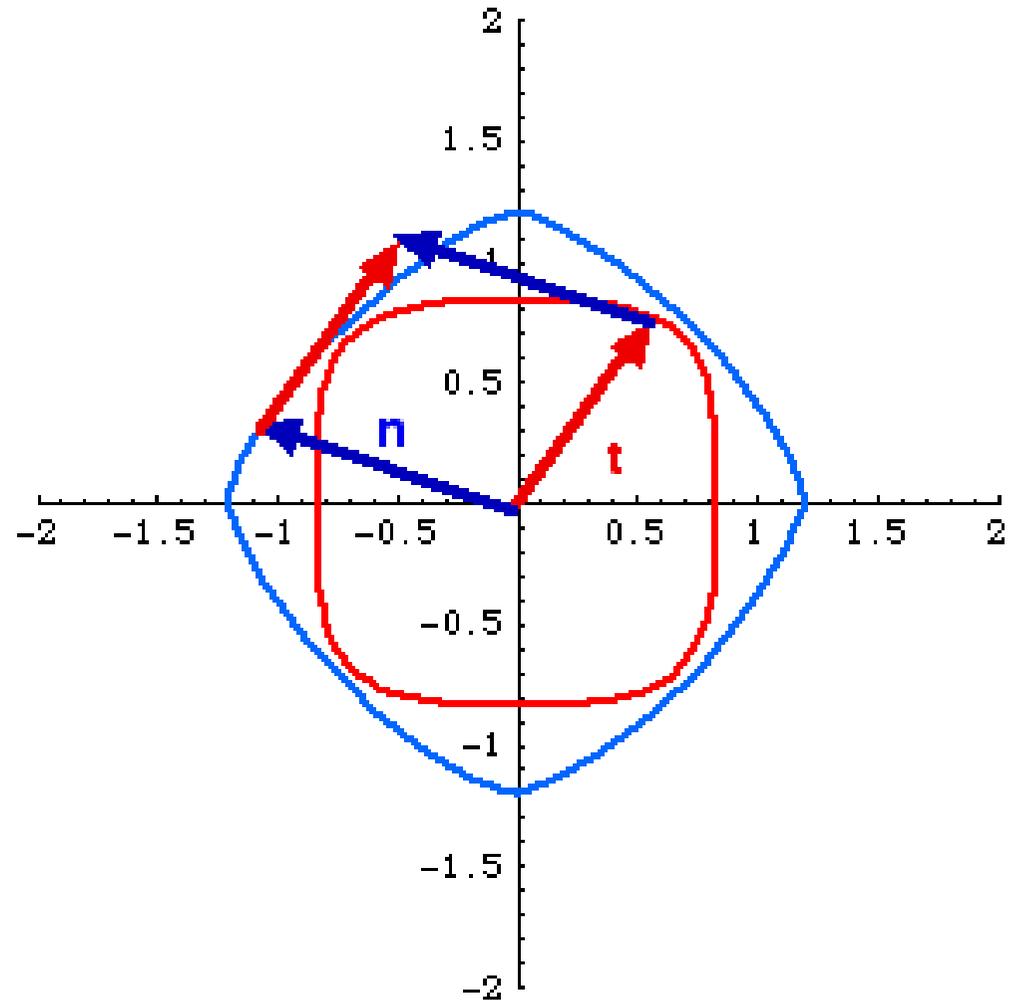
$$\frac{d\mathbf{t}}{d\theta} = r^2 \mathbf{n} \quad \frac{d\mathbf{n}}{d\theta} = -\frac{\tilde{h}}{\tilde{k}} \mathbf{t}$$

$$\mathbf{t} = r(\theta) \mathbf{e}_1$$

$$\mathbf{n} = -\tilde{h}_\theta(\theta) \mathbf{e}_1 + \tilde{h}(\theta) \mathbf{e}_2$$

$$\mathbf{t} \wedge \mathbf{n} = 1$$

The isoperimetrix is the trace of \mathbf{n}

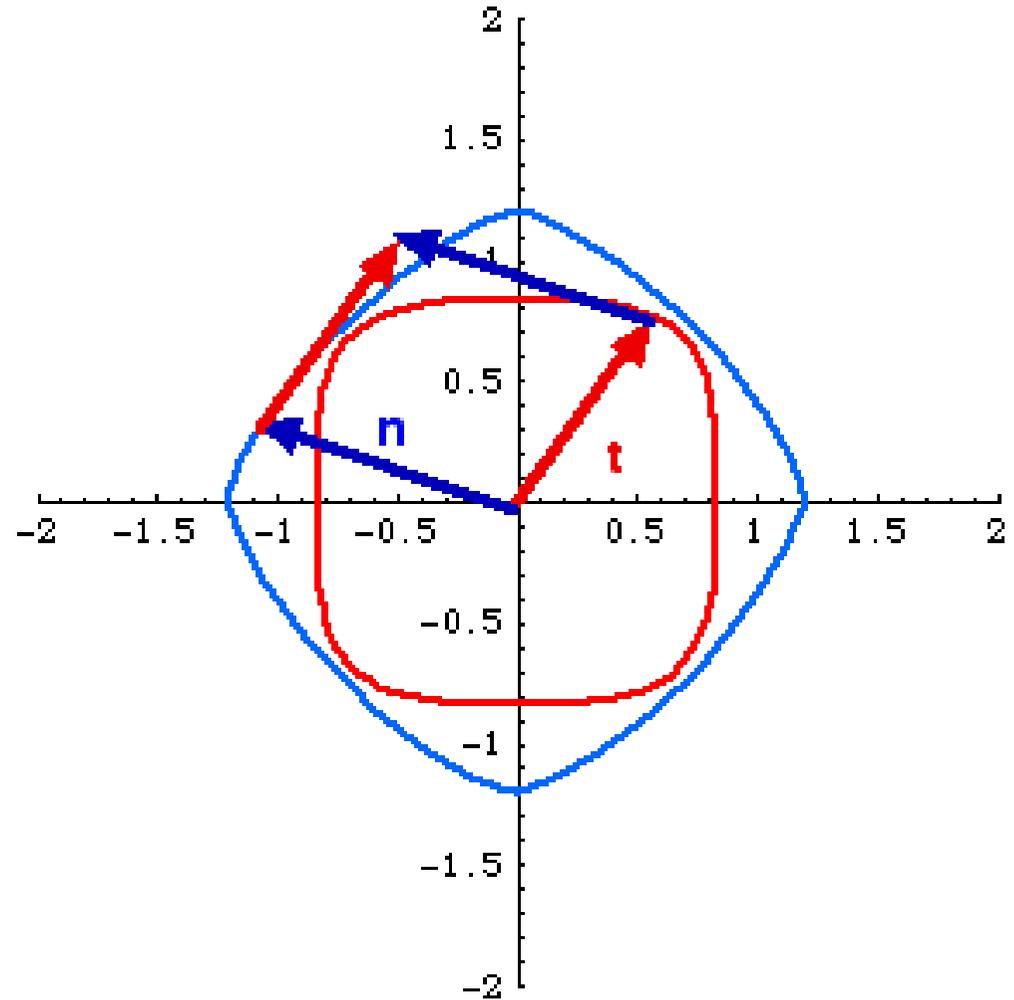


First variation

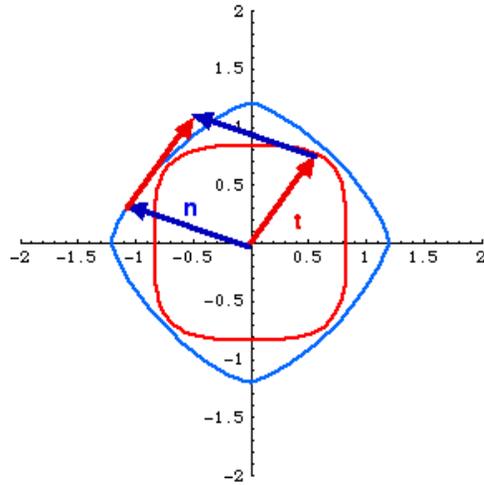
How length of vector changes at \mathbf{t} .

$$dl(\mathbf{n}) = \frac{dl(\mathbf{t} + \epsilon \mathbf{n})}{d\epsilon} = 0$$

$$dl(\mathbf{t}) \frac{dl(\mathbf{t} + \epsilon \mathbf{t})}{d\epsilon} = 1$$



First variation



$X(\sigma, \epsilon)$ is a one parameter family of curves. Write variation field using \mathbf{t} and \mathbf{n} .

$$\frac{\partial X}{\partial \epsilon} = \alpha(\sigma)\mathbf{t} + \beta(\sigma)\mathbf{n}$$

$$\mathcal{L}(X(\cdot, 0)) = \int l \left(\frac{\partial X}{\partial \sigma} \right) d\sigma$$

- $\frac{\partial}{\partial \epsilon} \mathcal{L}|_{\epsilon=0} = \int dl \left(\frac{\partial^2 X}{\partial \epsilon \partial \sigma} \right) d\sigma = \int -\beta \frac{k}{\tilde{k}} d\sigma$
- Among $\beta(\sigma)$ with $\int \beta^2 d\sigma = 1$ $\beta = \frac{k}{\tilde{k}}$ will decrease the fastest.
- Suggests gradient vector field: $\nabla \mathcal{L} = -\frac{k}{\tilde{k}} \mathbf{n} = \frac{k}{\tilde{k}} (\tilde{h} \mathbf{e}_2 - \tilde{h}_\theta \mathbf{e}_1)$

Curve shortening equation for a Minkowski geometry

$$\frac{\partial X}{\partial t} = \left(\frac{1}{\tilde{k}} \right) k \mathbf{n}$$

or

$$\frac{\partial X}{\partial t} = \left(\frac{\tilde{h}}{\tilde{k}} \right) k \mathbf{e}_2$$

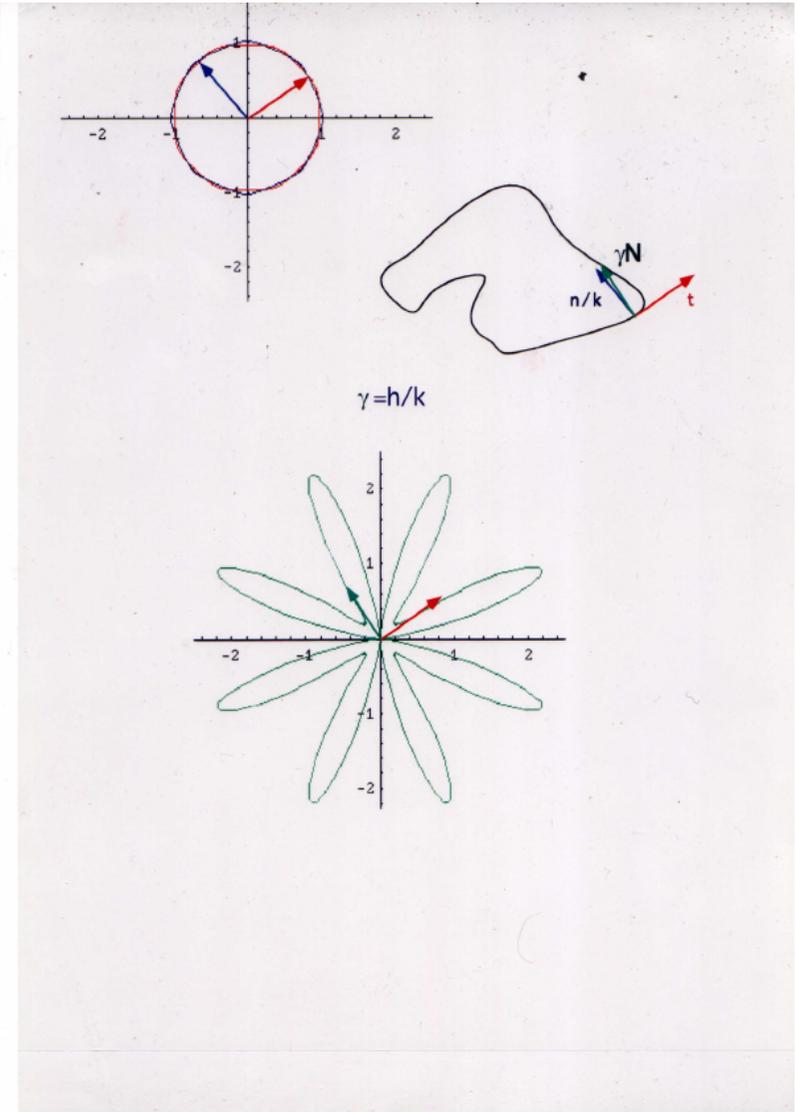
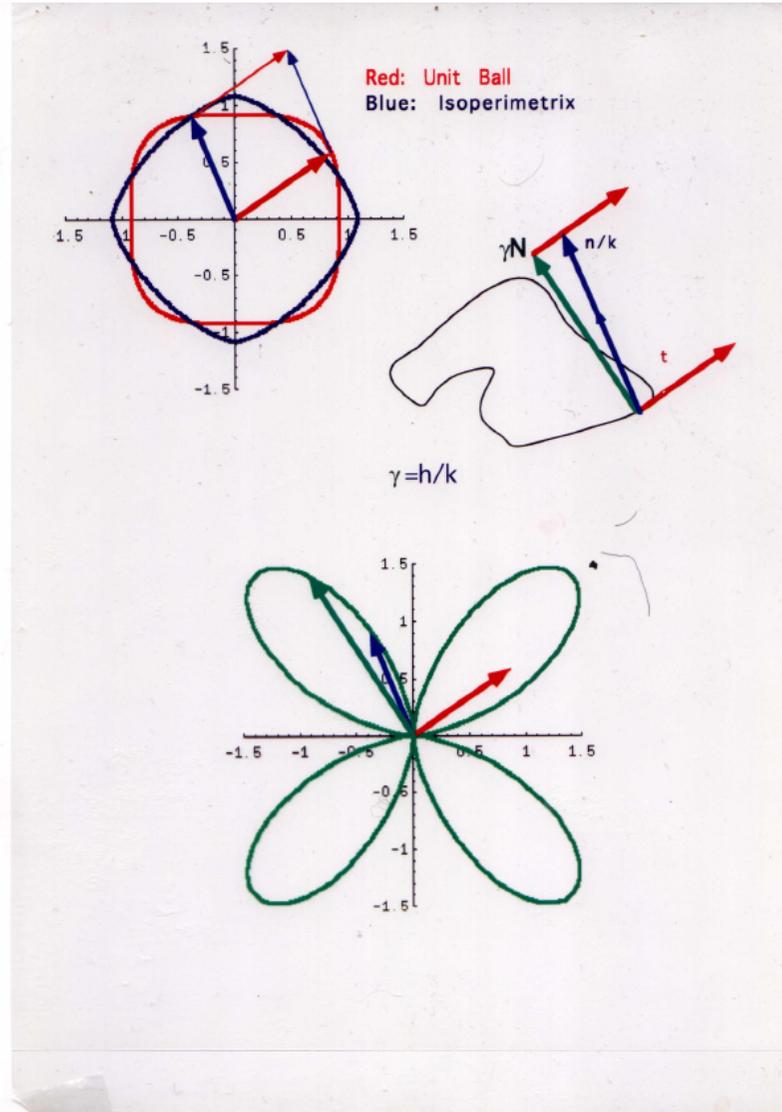
and for convex curves the evolution of the support function:

$$\frac{d}{dt} \left(\frac{h(\theta, t)}{\tilde{h}(\theta)} \right) = - \frac{k(\theta, t)}{\tilde{k}(\theta)}$$

$$\gamma = \frac{\tilde{h}}{\tilde{k}}$$

Lemma: The isoperimetrix shrinks self-similarly for the curve shortening flow of a Minkowski geometry.

Pictures of Minkowski geometry quantities



Unique attractor for this flow is the isoperimetrix

Show that the Minkowski isoperimetric inequality

$$\frac{\mathcal{L}^2}{Area} - 2 \int \gamma d\theta$$

is always decreasing along a flow

- As it decreases to zero the shape approaches the isoperimetrix in the Hausdorff metric.
- The self-similar flow is an attractor among all convex curves
- and therefore unique.
- This proof requires that the Minkowski ball be symmetric through the origin.

Domain of attraction

- Jeff Oaks: Flows of this type take non-convex embedded curves into convex curves.
- All embedded curves are attracted by the shrinking isoperimetrix.
- This theorem also requires that the Minkowski ball be symmetric, otherwise embedded curves may cross each other.

Existence

To find a geometry for a smooth, positive γ

- Find a self-similar flow for γ
- This means finding a convex body satisfying

$$\frac{\tilde{h}}{\tilde{k}} = \tilde{h}(\tilde{h}_{\theta\theta} + \tilde{h}) = \gamma$$

- or alternatively find a positive 2π periodic solution to the elliptic equation
- The shape of the self-similar flow will be the isoperimetrix of the geometry you seek.

Existence

Reduce the periodic ODE to a corresponding to PDE.

Gage-Y.Li: Every curve sub-converges to a flow which is self-similar. (Even if γ is only 2π periodic.)

This is actually a common trick. Choose something with the right topology (periodicity in this case) and then deform it until it satisfies the equation.

One can prove uniqueness only when γ is π periodic.

Elliptic version: Dohmen, Giga, and Mizoguchi

Dohmen, Giga and Mizoguchi solve the equation directly using elliptic methods.

$$\frac{\tilde{h}}{\tilde{k}} = \tilde{h}(\tilde{h}_{\theta\theta} + \tilde{h}) = \gamma$$

Still can prove uniqueness only when γ is π periodic.

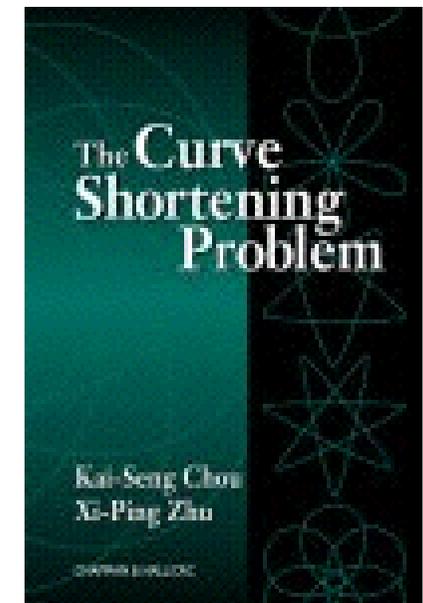
Summary of these results

The Curve Shortening Problem

by Kai-Seng Chou (Tso) and Xi-Ping Zhu
published by Chapman and Hall/CRC 2001

provides an excellent and unified account of many results related
to flowing curves by curvature.

Results of: Altschuler, B. Andrews, S. Angenent, K-S. Chou, B. Chow, Y. Giga, M. Grayson, R. Hamilton, G. Huisken, J. Oaks, G. Sapiro, A. Tannenbaum, J. Taylor, X-P. Zhu. and others.



Crystalline geometry case

Suppose the unit ball is a polygon? Only flat sides and corners.

- Curvature is the Jacobian of the Gauss map: $\frac{d\theta}{\tilde{k}}$ is the measure of the length of curve whose tangent lies in the range $d\theta$.
- Replace $\frac{1}{\tilde{k}}$ by \tilde{l}_i , the length of the i th flat side.
- The ODE equations in polar coordinates

$$\left(\frac{h(\theta_i)}{\tilde{h}(\theta_i)} \right)_t = \frac{\tilde{l}_i}{l_i}$$

- Conjecture:(J. Taylor, Gurtin-Angenent) If there are more than 4 equations symmetric through the origin, there is a unique attracting self-similar flow.

Existence and Uniqueness –A. Stancu

There is always a self-similar flow.

- When the equations are symmetric through the origin, the self-similar flow is attracting and unique, except when the unit ball is a parallelogram.
- For example, if the unit ball is a square, then rectangles with parallel sides flow self-similarly.
- In very recent work Stancu has shown uniqueness for many cases where symmetry through the origin does not hold. It remains to see how (if?) this can be transferred to the smooth case.

Open questions:

- Piecewise smooth boundary case.
- **Do geometries which are close in Hausdorff distance have flows which are close in Hausdorff distance** – this would unify the work done so far in the smooth category.
- What is the relation of 3dim weighted mean curvature flows to area in Minkowski geometry.
- How does weighted curvature flow relate to Finsler metrics on manifolds? Are there analogs of the k/K flow?

