The purpose of this file (always under construction) is to provide corrections, and references to further developments regarding the material in the publication list. Right now the corrections are in better shape than updates, which are incomplete and in the form of informal pointers to myself. The updates covered in later papers are not usually mentioned. Most updates and corrections (before 2004) have been incorporated in my book 'Function Field Arithmetic', and can be found (together with updates for the book!) on my homepage. But I still would add misprints/corrections here.

Any questions, comments, suggestions and corrections are most welcome. Especially, if you know any progress about the open questions mentioned in my papers, which is not mentioned in these updates, please let me know.

This file is unfortunately, quite outdated (on updates) as I have not worked on it systematically for a few years.

The main basic references for the arithmetic of function fields, in the book form are (A) Springer Lecture notes 1231 by Gekeler on Drinfeld modular curves, (B) Proceedings 'Arithmetic of function fields', of conference in 91 at Ohio University, pub. by Walter de Gruyter and edited by David Goss, David Hayes and Michael Rosen, (C) 'Basic structures of function field Arithmetic' by David Goss, Ergibnesse published (1996) by Springer Verlag. (D) 'Function Field Arithmetic' by Dinesh thakur (2004), pub. by World Scientific. These contain extensive bibliographies and I plan to refer to them soon.

The corrections to my old papers are usually given at the end of my more recent papers: we will also collect those below.

Papers have been identified by numerals referring to the publication list, and/or by catch words, journal and year.

1. (1) (Hengelhoef 86) (Updates) See (2).

2. (2) (Thesis 87) (Corrections): Most of the misprints are corrected in Gamma paper, but anyway:

   Pa. 14, 3rd displayed eqn. and also the one on the last line should be $(D_0 \cdots D_n)^{q-1} = D_{n+1}/([1] \cdots [n+1])$.

   Pa. 16, In the second line in the second claim, $g(0)$ should be replaced by $g(0)^{1/2}$.

   Pa. 20, In the second displayed formula, the last exponent should be q^d rather than $1 - q^d$.

   Pa. 26, $M_r = []$ should be $M_r = -\lim[]$. Last line: $(-P_1)^{t_r}$ should be $(-1)^{h-1}(-P_1)^{t_r}$.

   Pa. 27, thm. 3.1: $\Gamma_P(q^r/(1-q^r)) = ((-1)^{h-1}(-P_1)^{t_r})^{1/(q^h-1)}$.

   Pa. 31, In theorem 3.5, $\tilde{P}$ should be replaced by $\tilde{\wp}$. Monicity condition before lemma 3.6 is only for $F_q[t]$.

   Pa. 36, in the definition, need b of finite order. Also better to put sign condition on α in the hypothesis, ord_∞ stands for $\sum n_{\infty_i} v_{\infty_i}$. In the second line in the proof, the exponent is q^{hf-1} rather than $q^{hf} - 1$.

   Pa. 40, in the summation in the third paragraph, 'n monic' is missing, and in (**) on the last line we need $\Lambda_{m-1}^{(k)}$ rather than $\Lambda_m^{(k)}$.

   Pa. 41, 3rd line last term should be $[k]^q/[1]$ rather than $[k]/[1]$.

   (Updates) Gamma chapter has appeared in (9) (Gamma Annals 91), Gauss sum chapter has appeared, with different proofs and better formulation in (3) (Gauss Inven. 88). The Zeta chapter, which contained the first explicit transcendence and irrationality results on Carlitz zeta values, did not appear, as it was quickly improved by joint work with Greg Anderson (5) (Tensor, Annals 90) and by the work of Jing Yu (Annals 90, 96?). The preliminary results of this chapter were announced in (1) Hengelhoef conference paper (86).

   Work of Cherif, de Mathan, Hellagouarch, Dammame, Allouche, Berthe, Denis takes the results further.

3. (3) (Gauss, Inv 88): (Updates) For A, other than $F_q[t]$, the theory is wildly different: See (10) (Gauss JNT 91) and (14) (Shtukas Invent. 93) and (15) (Gauss BLMS 93).

   Greg Anderson, Zhao, Feng and Chapman. Ref.?

4. (4) (Gross-Koblitz, Trento 89) (Updates) See (14) (Shtukas, Invent. 93).

5. (5) (Tensor, Annals 90) (Updates) Applications to transcendence and generalizations to higher genus and class numbers. Jing Yu's papers in Annals 90, 96, Ohio volume, Anderson's papers in Duke, JNT, Dammame's thesis and Ohio paper, Dammame-Hellagouarch paper in JNT. Paper by Chang-Yu determining algebraic relations.

6. (6)(Analogies, BMC 90) (Corrections) On pa. 86, the first part of the last dispalyed equation should be $z/e(z) = 1 - \sum(z/\tilde{\pi}a)/(1-(z/\tilde{\pi}a))$.

   (Updates) Some more refined analogies in (16) (Iwasawa, CM 94).

7. (8) (Zeta measure JNT 90) Meaning and generalization to other A's still open. Carlitz claim: Javier Diaz-Vargas, Poonen, Sheats. Sangtae Jeong does variants.

8. (9) (Gamma, Annals 91) (Corrections): The last exponent $1 - q^d$ in the first formula in 3.4, should be replaced by q^d. In 7.8, K-linearly should read Q-linearly. On pa. 50, one g is genus. On pa. 51, first displayed formula line 3, exponent of q is j-c: it should be mentioned that c is constant comming from Riemann-Rocah, and it is -1 for rational field. The reference [Y1] in 1.6 is wrong and is corrected in (4) (Gamma, Ohio 92) paper end.

   (Updates) Anderson, Sinha papers, (Gamma, Ohio 92), ((21) Gamma, Annals 96), ((14) Shtuka, Invent. 93), ((26) Soliton). Bae-Yin-Yu preprint have some answers to questions at end. Brownawell-Papanikolas preprint generalizes Sinha.

9. (10) (Gauss JNT 91) (Corrections) pa. 246, line -14, χ_{j-1} should be χ_{j+1}.

   (Updates) On pa. 244, it is shown that ρ_I determines ρ, if it is known to be sgn-normalized and it is stated that it is not known what happens if we drop the normalization. Lingsueh Shu provided me the following simple argument which shows that ρ_I for non trivial I does not, in general, determine ρ: Let A be with $\delta = 1$ and with class number more than one and with a place I of degree one. Let $\rho_I = c + F$, with $c \in H$. Let σ be a non-trivial element of the Galois group of H over K. Let μ be a $q-1$-th root of c/c^σ. Then clearly, ρ^σ and $\mu^{-1}\rho\mu$ are non-isomorphic, but both have I-th isogeny $c^\sigma + F$. (Since $(\mu^{-1}\rho\mu)_I = 1/(\text{leadingterm})(\mu^{-1}\rho_I\mu)$).

   On page 247 of this paper and in other papers, book I have quoted then famous result of Leitzel-Madan-Queen (JNT175) that there are only 7 global function fields of positive genus and class number 1. But in Nov. 2013, Claudio Stirpe (see ArXive, to be published in JNT) produced one more of genus 4 over F_2, exactly the one LMQ tried to show did not exist! This exception needs to be added to Thm. 8.3.2 of the book and thm. 3.2 of Inventions 1993 paper., as it does not have prime of degree 1, 2 or 3. On the other hand, it does not have prime of degree one, so there are still only 4 known A's of class no. 1 which are not $F_q[t]$. (January 2014)

   This is the only exception. For details see papers of Mercuri-Stirpe and Shen-Shi in Vol. 154 JNT 2015 (and original Stirpe Vol. 143 JNT 2014).

   Open questions on pa. 250: (1) and (3) are answered in positive in (14) (Shtukas, Invent 93) and different proofs are given together with higher class number results and relation of factorization to theta divisor. For (4), see Anderson and Zhao papers in JNT. For more details on assertions in Cyclotomic theory summary, see Hayes paper in Ohio proceedings.

10. (11) (Continued JNT 92) (Update) See (19)(Continued JNT 96) and (22) (Patterns 97). Allouche Toeplitz connection. Van Hamme student Ann Verdoot, Annales Math. Blaise Pascal, 1, 1994, 71-84 Continued fractions for finite sums.

    (Correction) On pa. 153, there is a sign misprint in displayed formulas for x_{n+1} and $\overline{x}_{n+1}$: The exponents $q^n(q-2)$ should be $-q^n(q-2)$. (Thanks to Diana Mecum for pointing this out).

11. (12) (Zeta, IMRN 92) (Corrections): In the published version, by mistake of the journal, the following abstract was dropped: In this paper, arithmetically interesting quantities such as factorial, gamma functions, binomial coefficients, zeta functions, in the context of function fields, are related to quantities connected with Drinfeld modules. This is then applied to obtain the algebraicity of Drinfeld exponentials of some special zeta values. The analogues, due to Jing yu, of Hermite-Lindemann and Gelfand-Schneider theorems about the transcendence properties of the exponential then imply transcendence of such values and their ratios with the periods.

    pa. 196, line 2, F_4 should be F_q.

    Pa. 188, line -7: $g-1$ should be $2g-1$.

    (Updates) Conjecture E is proved by Anderson, Hypothesis (almost). Refer to Anderson's two (Duke, JNT) papers.

12. (13) (Gamma:Ohio 92) (Corrections) Last paragraph of (4) on pa. 85 is garbled: The interpolation we have defined earlier works fine and agrees with what is said there under the correct hypothesis that denominator of z divides $v-1$ missing there. There is no need of redefinition.

    The second to last statement in (5) pa. 85 should be deleted, as we do not yet know v-adic interpolations.

    (Updates) Anderson, Sinha papers, (Gamma, trans, Annals), (14) (Shtuka, Invent 93) and (26) (Soliton). Mention Local gamma, gamma ideal, shu's paper. In Crelle 1997 paper (and preprints) Manjul Bhargava has given a very nice general recipe for factorials which specializes to usual notion for Z and Carlitz factorial for $F_q[t]$, but is different in general. (Remark is put in soliton paper). There does not seem to be simultaneous P-ordering for higher genus and if you do it for each prime, one ends up with Sinnott formula (Goss gamma ideal): whose local nature is explained in Shu (JNT).

13. (14) (Shtukas, Invent 93) (Corrections) 5.1 to 5.4: The hypothesis $\delta = 1$ is missing.

    Pa. 559, line 3, the first $\overline{X}$ should be X.

    Pa. 560, line 2 and 3 from bottom, minus sign is missing between '$W = V$' and '$(\xi) + (\eta)$', so it should be $W = V - (\xi) + (\eta)$.

    Pa. 561, 0.3.5: when we describe how to get Drinfeld module from f, by convention in 0.1.1 it is supposed to be sgn-normalized, but that works only for $\delta = 1$, since $sgn(f) = 1$ implies $sgn(f^{(\delta)}) = 1$ (as δ-th twist is just q^δ-th power on signs extended as in 0.0), but $sgn(f^{(1)})$ need not be 1 in general. So in this correspondence we should drop the 'sgn-normalized' requirement for $\delta > 1$ (we can drop sign normalization of f also then) and be content with one choice in the isomorphism class. It will be sgn-normalized, if $sgn(ff^{(1)}\cdots f^{(\delta-1)}) = 1$ instead. Thanks to Angus Chung for pointing this out (18 September 2020).

    Pa. 561, Proposition 0.3.8: Thanks a lot to Angus Chung, who pointed out (on 18 September 2020!) that this only works for $\delta = 1$, since otherwise for $n < \delta$, the contribution of residues at the infinite places do not work out as claimed (i.e. contribution zero, for $n > 0$ and 1 for $n = 0$ corresponding to the linear term).

    The correction is that the proposition and the proof works when the following correct normalization of the differential w in 0.3.7 is used: Replace $2\overline{\infty}$ there by $\overline{\infty}^{(-2)} + \overline{\infty}^{(-1)}$ and normalize w by requiring the residue of $w^{(1)}/f$ at $\overline{\infty}^{(-1)}$ to be 1. (I believe that the applications and examples worked out, by me and others so far, were only for $\delta = 1$ and are not affected, even though this proposition with the original (wrong for $\delta > 1$) normalization may have been repeated in other places, including my and David Goss' books (same for the remark above on 0.3.8).

In Theorem 3.2, one more exception needs to be added: See detailed comments in (10) on the correction (discovered in 2013) to LMQ1975 referred there.

14. (15) (Gauss, BLMS 93)

15. (16) (Iwasawa, CM 94) Shu, Anderson and Mazur on class groups and Vandiver. A-modules suggestions achieved by Greg Anderson and Lenny Taelman! In paragraph, after Thm. 7, in discussion of Carlitzian analog of Wieferich prime and its connection with extra zeta divisibility, in the equivalence, q should be greater than 2 so that 1 is 'odd' as in Thm. 7. The equivalence works unless $q = 2$ and $\wp = t$ or $t + 1$, but when $q = 2$ we have degenerate case $\zeta_\wp(1) = 0$, and we also have (by equivalence) $C_{\wp-1}(1) = 0$, when $q = 2$ unless $\wp$ is of degree one. This is directly seen also, as $t^2 + t$ then divides $\wp - 1$ and 1 is $t^2 + t$-torsion when $q = 2$. So we call Wieferich for $q > 2$ only. (Search by my student George Todd found (Sept. 2012) for $q = 5$ only one such 'wieferich' prime $t^5 - t + 1$ of degree at most 5, for q=3, only two such $t^6 + t^4 + t^3 + t^2 - t - 1$ and $t^9 + t^6 + t^4 + t^2 - t - 1$ of degree at most 9, and for $q = 7$, nothing for degree less than 7 and at least one for degree 7, namely $t^7 - t + 3$. Do they exist only in degrees divisible by the characteristic?: I thank Dong Quan Nguyen who pointed out that the general answer is no, as there are some examples by Mauduit of degree 3 in characteristic 2, with q not 2. (August 13)). Alex Lara found (Jan 2013) $t^{12} - t^{10} + t^9 - t^4 - t^3 + t^2 + 1$ works for $q = 3$.

16. (17) (Hypergeometric, FFA 95) (28) (Hyp. II 2000) has updates. Kochubei. On pa. 229, $log(1 - z)$ should be $-z$ times $F(1, 1, 2, z)$. Analog of this corresponds to $\sum z^{q^n}/[n]$ rather.

17. (18) (Zeta, Compositio 95) (Corrections) Pa. 232 First paragraph, last line: r_1 in the exponent should be $r_1 s$.

    Pa. 237: In formula for S_k, minus sign between $\mathcal{J}^{-1}$ and $\{0\}$ is missing. In third paragraph, one closing bracket is missing after $(j_2 + r(i)$.

    Pa. 239: Last but one paragraph: Reference to Theorem 2, should be to Theorem 1.

    Pa. 240: Remarks (ii): l(6infty) should be 2 and not 3, where it is mentioned, but in the next example of voloch, it should be 3.

    Pa. 245: Last paragraph: 'The results of Goss-Sinnott mentioned above': Somehow I forgot to mention these results: For K of class number one, for which there is exceptional vanishing, they imply non-vanishing of class group components of the $\wp$-th cyclotomic extensions of K, for any $\wp$.

    (Updates) A theorem and several examples of extra vanishing for higher class numbers are obtained in my student's thesis: 'On zeros of characteristic p zeta functions', Javier Diaz-Vargas, U. of Arizona, May 96. On the overall subject of the paper, see Goss book, Wan, Taguchi-Wan, Diaz-Vargas, Sheats for latest developments.

    Theorem 1.3 of Math. Z. 2013 paper of Gebhard Boeckle calculates the exact order of vanishing in one class number one example at all points, thus generalizing the proposition after thm. 8 in this case.

18. (19) (Continued JNT 96) (Correction) The formula for w_i on page 253 (also quoted in Thm. I in patterns paper) has to be multiplied by an appropriate element of F_q^* by taking into account the sign of the denominator of the initial convergent.

On pa. 254, there is a sign mistake in definition of r_2: The numerator $-(bg+b'D)$ should be $(bg-b'D)$. (Thanks to Diana Mecum for pointing this out).

(Update) (22) (Patterns, JNT 97) solves the main open question. A version of the Folding Lemma seems to have appeared even before the reference [S1] I gave, in Acta Arith. 23 (1973), 207-215 in an article by Mendes France and another article by him in Colloq. Math. 1974. Thanks to him for pointing this out.

19. (20) (q transc, JNT 96) (Correction) Ref. to Voloch should be Vol. 58, no. 1, 55-59 rather than vol. 57 no. 2.

    (Update) See (23) (Automata) paper and (27). In 2011, in letters to the author, Deligne provided moduli theoretic proof of the identities proved in this paper using Ramanujan's identity.

20. (21) (Gamma trans, Annals) (Corrections) In theorem 2, the hypothesis that 'α_l are not all zero' is missing.

21. (22) (Patterns JNT97) (Correction) See (19). (Updates) $\theta \neq 1$ still open for $q = 2$. (Work in Progress: My student Aaron Ekstrom at Arizona has formulated some conjectures in this case.)

22. (23) (Automata) paper. (Corrections) (i) First page, first theorem 'algebraic power series over k(x)' should be 'power series over k algebraic over k(x)'

    (ii) Second page, displayed equation, '$2\pi i$' is missing. (Thanks to Somnath for pointing out these misprints).

    (iii) On page 3, theorem 3, 'non-periodic' should be 'non eventually periodic'.

    (iv) On pa.7 bottom, 'turns out to be trivial monomial' should be 'turns out to be trivial monomial, after a suitable integral translation of arguments, which changes values by rational functions.'

    (Updates) Allouche, Mendes France, Yao. Now Adamcrezcki and Bugeaud have proved Cobham's conjecture mentioned that automatic irrational real numbers are transcendental.

23. (24) (Computational classification IMRN) (Update) Dale Brownawell pointed out to me that Mahler refined his classification mentioned in the paper after 30 years in the reference Acta Arith. XVIII (1971): On the order function of a transcendental number, to get many classes, but it is hard to show that elements exist in his classes.

24. (25) (Dioph approx) (update) Abhyankar conference paper improves and updates this, provides non-Riccati examples.

    The remark 3 at the end: See updates to (48).

25. (28) (Hyp II) (Update) Kochubei's work.

26. (29-30) In the cyclotomic fields book, pa. 328 paragraph last but one, $p = 2$ is misprint for $a_p = 2$.

    Pa. 166, para.4 line 6, modulo 3, when p=3 should be modulo 9 when p=3.

    Pa. 167, 4th para. from bottom: γ should be δ

    Pa. 169, third para. from bottom, last line: $\mathcal{O}_L^*$ should be $\mathcal{O}_L - \{0\}$.

    Pa. 37, 3ed paragraph, 4th line: 'b^2' should be dropped.

    Pa. 38, line 10 from bottom, $a/q - b$ should be $a/b - q$.

    Pa. 39, 3rd para., 5th line, 'cube of an ideal' should better be 'cube of a principal ideal'.

    Pa. 41, In the factorization claim (and may be elsewhere too) one should assume n is not congruent to 2 modulo 4. (Note that if n is odd, $-\zeta_n$ is ζ_{2n}).

    Pa. 43, line 7, superscript of $Q(\zeta_n)$ should be + rather than $*$.

27. (31) (Exponents and Deformation) Page 597, note added in the proof: γ there is misprint for r.

28. (32) TIFR paper: m should be $m_\wp$. Kazdan-Flicker seems to have done Langlands with special condition at ∞, at least after later work. Lafforgue completes anyway.

29. (34) Elliptic survey paper: (Corrections) HS reference missing (pa. 231) is Hindry-Silvermann, Inventiones 93, 1988, 419-450. Pa. 231, 4th para. reference [Ab, Y] should be [Ab, Ya].

    (updates) Here is a nice argument/email (March 25, 03) from Bjorn Poonen answering the question (pa. 232) of Fpbar gonality of modular curves raised in the survey: THEOREM: There exists a function p(n) such that for any separable extension of fields L/k and any (smooth, projective, geometrically integral) curve X over k with a k-point, the gonalities $G_k(X)$ and $G_L(X)$ of X over k and L respectively satisfy $G_k(X) <= p(G_L(X))$.

    PROOF: It suffices to consider the case L=ksep. I will use the notation L(X) to denote the function field of the base extension X_L. Fix f in L(X) of degree $d := G_L(X)$. By descent theory, the subfield of L(X) spanned by the Galois conjugates of f is L(Y) for some k-curve Y, and we have a dominant morphism $h : X - - > Y$ such that f = f' h, for some $f' : Y_L - - > P^1$. Since $[L(Y) : L(P^1)] <= [L(X) : L(P^1)] = d$, at most d conjugates of f' are needed to generate L(Y) over $L(P^1)$ (in fact this can be improved to $log_2 d$, by looking at how the degree of the generated field grows with the addition of each new conjugate). Let $f'_1, ..., f'_e$ be these conjugates, so $e <= d$. Then $f'_1, ..., f'_e$ map Y_L birationally to its image curve in $(P^1)^e$ of multidegree at most (d,d,...,d), so the genus g of Y_L is bounded in terms of d. Also, $G_k(Y) <= g + 1$, by Riemann-Roch applied to (g+1)P, where P is in Y(k). Finally, the degree of X–¿Y divides d, so $G_k(X)$ also is bounded in terms of d alone.

    As we discussed, this can be used to bound the Fpbar-gonality of modular curves from below, given a lower bound on the Fp-gonality.

    I asked Matt Baker about this and he wrote

    ”The general method of using descent theory together with birational maps to bound gonality over separable field extensions appears in a few places: in my thesis, in a paper of Silverman and Harris, and in an unpublished paper by Nguyen and Saito (paper 11 at http://www.math.s.kobe-u.ac.jp/HOME/mhsaito/).”

    although he did not know an explicit reference mentioning the application to modular curves over Fpbar. Best regards, Bjorn

30. (35) (Diophantine proceedings 2008) pa. 762, in Riccati equation $a\alpha 2$ is misprint for $a\alpha^2$.

31. (40) (App. exponent, Roth 2009) pa. 427, lemma 1, minus sign is missing on the second displayed quantity.

32. (41) (Hypergeometric CR note) See (49).

33. (42) (Power sums article) Page 540, the last paragraph can be a little confusing because of the issue of uniqueness of m_i's as well as y's in the combinatorial argument there, so here are more details: If there is unique y and unique m_i's, we get the first sentence there. Uniqueness is clear for q prime, since then as said on the next page, we fill m_d, m_{d-1}, etc. in order, by $p-1$ of least possible p powers (that we need since $m_i > 0$ are divisible by $q - 1$), so that there is no carry over. Since carry-over starts after digit $q - 1$, this minimal greedy process leads to all $q - 1$ digits (except possibly the top digit, if $k-1$ is not divisible by $q-1$, when the remainder goes in m_0 -part) for $(k-1)+y$, leading to $q^n - 1$. The same greedy algorithm (but with more complicated description of how each m_i gets filled in order in terms of p- powers, since they are not all congruent to 1 mod q-1 now) works by Sheats, for each given y. Once y is large enough (compared to f, where $q = p^f$), the minimal is when the sum

has again low digits all $q-1$ (which maximizes the amount of lowest powers you can put in), So we reduce to consider large q^n-1's and as you increase n, eventually m_i's (for $i>0$) all stabilize, and only m_0 (for q^n-k) keeps on changing. Hence, we get unique best m_i's ($i>0$) and corresponding unique y. In particular, $s_d(k)$ for $k>0$, can be computed from suitable $s_d(k-q^n)$ (absolute values of these two add to dq^n) with $k-q^n<0$, so that this can be computed by Carlitz-Sheats greedy recipe with m_i's.

There is also an alternate proof of this relation between valuations, obtained for power sums by shifting the k's by multiples of large q^n's, in 3.8 of (55) by congruences: Since q^n-th power of one unit at infinity is congruent to 1 up to q^n-th power of the uniformizer at infinity, the term by term comparison of terms in $S_d(-k)/t^{dk}$ with those in $S_d(-k-mq^n)/t^{d(k+mq^n)}$ (with $k,m\in Z$), shows that when $q^n>|dk+s_d(-k)|$, if one is non-zero, then both are and both have the same v_∞-valuation, so that $|s_d(-k)|+|s_d(-k-mq^n)|=dmq^n$ under these conditions.

Note $k>0$, and $S_d(-k)\neq 0$, then $0\leq -s_d(-k)\leq dk$, then simplified $q^n>dk$, or since $d\leq \ell(k)/(q-1)$ by non-vanishing condition, even more simplified $q^n>k\ell(k)/(q-1)$ works.

Note also that in deducing theorem 2, $k>0$ and $k-q^n$ of this argument have been replaced by q^n-k and $-k$ respectively. Also, $s_d(k-q^n)$ finite implies $d(q-1)\leq \ell(q^n-k)$

34. (44) (Relations between multizeta) 2.7.1 : 'Drop the last term' should be 'drop the middle term'. (Thanks to George Todd for pointing this out.)

    Page 2333, first displayed formula, q^2-1 should be q^i-1.

    (Updates) Conjectures here have been generalized and mostly proved in Masters and PhD thesis (2013) of Alejandro Lara Rodriguez. There are more recent results jointly with him on zeta-like multizeta (see ArXive). (January 2014).

35. (45) (Tate PAMQ article) (Updates) Tate's treatment of residues has been found very useful in works by Anderson, Taguchi-Wan, Pink-Boeckle, Vincent Lafforgue and Lenny Taelman. (The last 3 are updates, the first two, I should have mentioned in the paper itself!).

36. (47) (Shuffle relations IMRN) (Updates) Alejandro Lara Rodriguea made this shuffle recipe more explicit giving formulas in his PhD thesis and Huei Jeng Chen proved simpler formulas directly. Jointly with Lara Rodriguez we have analyzed at non-rational infinite place case.

    (Correction) While the proof of Theorem 3 follows by induction, repeated use of depth 1 shuffle proved in Theorem 2 and splitting of iterated sums, as mentioned there, I goofed up in setting up the vector notation induction. The fourth equality in the display (page 1979) which is said to follow from shuffle is wrong and it should be replaced with what follows from shuffle etc. Here is an inductive proof by my student Shuhui Shi, who noticed and corrected this problem. Thanks to her.

    Theorem:
    $$S_d(a_1,\dots,a_m)S_d(b_1,\dots,b_n)=\sum_i f_iS_d(c_{i1},\dots,c_{iN_i}),$$
    where $f_i\in\mathbb{F}_p$, c_{ij}'s and $N_i\leq m+n$ are independent of d, and $\sum_j c_{ij}=\sum_i a_i+\sum_j b_j$.

    Proof. We prove the following two statements:
    $$S_d(a_1,\dots,a_m)S_{<d}(b_1,\dots,b_n)=\sum_i f_iS_d(\mathbf{C}_i), \qquad (*)$$
    $$S_d(a_1,\dots,a_m)S_d(b_1,\dots,b_n)=\sum_j g_jS_d(\mathbf{E}_j), \qquad (**)$$

where $f_i, g_j \in \mathbb{F}_p$, $\mathbf{C} = (c_{i1}, \dots, c_{iN_i})$, $\mathbf{E}_j = (e_{j1}, \dots, e_{jM_j})$ are independent of d, the lengths of $\mathbf{C}_i, \mathbf{E}_j$ are no greater than $m+n$, and their weights all equals $\sum_i a_i + \sum_j b_j$.

We prove by induction on $m+n$. For $m+n=2$, $m=n=1$, (*) follows by definition of S_d and (**) is Theorem 2. For a composition $\mathbf{A} = (a_1, \dots, a_m)$, we denote $\mathbf{A}_- = (a_2, \dots, a_m)$.

Assume (*) and (**) hold for $m+n \le K$, we show they hold for $m+n = K+1$. First prove (*). If $m=1$, $S_d(a)S_{<d}(b_1, \dots, b_K) = S_d(a, b_1, \dots, b_K)$ by definition of S_d. For $m > 1$,

$$\begin{aligned}
S_d(\mathbf{A})S_{<d}(\mathbf{B}) &= S_d(a_1)S_{<d}(\mathbf{A}_-)S_{<d}(\mathbf{B}) \\
&= S_d(a_1)\left(\sum_{d_1<d} S_{d_1}(\mathbf{A}_-)S_{<d_1}(\mathbf{B}) + \sum_{d_2<d} S_{d_2}(\mathbf{B})S_{<d_2}(\mathbf{A}_-) + \sum_{d_3<d} S_{d_3}(\mathbf{A}_-)S_{d_3}(\mathbf{B})\right) \\
&= S_d(a_1)\left(\sum_{d_1<d}\sum_i f_i S_{d_1}(\mathbf{C}_i) + \sum_{d_2<d}\sum_j g_j S_{d_2}(\mathbf{E}_j) + \sum_{d_3<d}\sum_k h_k S_{d_3}(\mathbf{U}_k)\right) \\
&= S_d(a_1)\left(\sum_i f_i S_{<d}(\mathbf{C}_i) + \sum_j g_j S_{<d}(\mathbf{E}_j) + \sum_k h_k S_{<d}(\mathbf{U}_k)\right) \\
&= \sum_i f_i S_d(a_1, \mathbf{C}_i) + \sum_j g_j S_d(a_1, \mathbf{E}_j) + \sum_k h_k S_d(a_1, \mathbf{U}_k),
\end{aligned}$$

where the third "=" follows from induction.

To prove (**), by symmetry of m and n, we assume $m > 1$, then

$$\begin{aligned}
S_d(\mathbf{A})S_d(\mathbf{B}) &= S_d(a_1)S_d(\mathbf{B})S_{<d}(\mathbf{A}_-) \\
&= S_d(a_1)\sum_i f_i S_d(\mathbf{C}_i) \\
&= \sum_i f_i S_d(a_1)S_d(\mathbf{C}_i) \\
&= \sum_i f_i S_d(a_1)S_d(c_{i1})S_{<d}(\mathbf{C}_{i-}) \\
&= \sum_i f_i \left(h_i S_d(a_1 + c_{i1}) + \sum_j g_{ij} S_d(v^i_{j1}, v^i_{j2})\right) S_{<d}(\mathbf{C}_{i-}) \\
&= \sum_{i,j} f_i h_i S_d(a_1 + c_{i1})S_{<d}(\mathbf{C}_{i-}) + f_i g_{ij} S_d(v^i_{j1}, v^i_{j2})S_{<d}(\mathbf{C}_{i-}),
\end{aligned}$$

where the second "=" follows from the assumption for (*), and the last line is an $\mathbb{F}_p$-linear sum of S_d's by (*) for $m+n \le K+1$. □

The zeta product shuffle identity follows from these power sums shuffle identity by summing, using shuffle on d and using these identities at each possible equality of indices d's. I will put the explicit recursive formula and details soon.

37. (48) (Higher diophantine exponents 2011) pa. 15 line 7 from bottom: 'ratio of the degrees of' should be 'ratio of the logarithms of heights of'.

Pa. 16 line 1, two 'deg' occurrences should be dropped.

(Update) Section 7 remark 2: I thank Cam Stewart for pointing out to me that Euler's e (real number) cannot be as well approximated by rationals as a typical real number. (For almost all real numbers, there are infinitely many approximations p/q with error less than $1/(q^2 \log(q) \log\log(q))$, whereas for e there is a positive c such that any p/q (with $q > 4$) approximation will give error worse than $c \log\log(q)/(\log(q)q^2)$. He asked about the comparison in our situation.

Note that for $q > 2$, the exponent for e here is q whereas classically it is 2, the exponent of 'typical real number' or the exponent of typical 'function field real number'. So for $q > 2$, function field e is much better approximated by rationals than a typical number.

For $q = 2$, using the explicit continued fraction in our paper, it is easy to see that the 'best 'approximations are $[1, [1], \cdots, [1], [n-1], [1], \cdots, [1]]$ which have denominator degree $d_n = (n-1)2^{n-1}$, as can be easily seen by recursion $d_n = 2^{n-1} + 2d_{n-1}$ apparent from the CF repeating structure. On the other hand, the degree of the next partial quotient is 2^n. Hence such approximation v/w will give error order better than $1/(|w|^2 c^{\log(|w|)/\log\log(|w|)})$, i.e. still much better than typical.

38. (49) (Hypergeometric Crelle paper) **Correction**: In Theorem 2 the hypothesis that '$F_q(T, \gamma)$ has less than q places above the infinite place of $F_q[T]$' (repeated in the abstract, introduction and Thm. 1 of (41)) should be replaced by 'the degree D of $F_q(T, \gamma)$ over $F_q(T)$ is less than q'. The corresponding changes in the proof are replacing '$v^{(i)}$' by $n_i v^{(i)}$ in the 6-th displayed formula on pa. 146 and 'd' by 'D' in the 7-th displayed formula. (Here n_i are the local degrees of $v^{(i)}$ normalised to extend the valuation at the base). (Sep. 12)

    (Update) The mistake above was due to mixing two common normalizations of valuations in the product formula. The correction above is mentioned in Yao's paper 'Special values of fractional Hypergeometric functions for function fields' which generalizes this result.

39. (50) (Bernoulli numerators) (Updates) Not exactly related to this paper, but Barry Mazur pointed out to me when he got my preprint that he had recently wrongly asserted in BAMS 48 (2011) article pa. 168 that B_{200} numerator had 204 digit prime factor, which he wrote down and then it was pointed out that it failed primality test. It was factored after a lot of effort only after a year or more! (2013)

    Thanks to Larry Washington for pointing out

    (1) Pollaczek F. –Math. Zeit 21 (1924), 1-38 Uber die irregularen Kreiskorper... where on pa. 31, it says (in today's notation) 37^2 divides B_{284}, 59^2 divides B_{914} and 67^2 divides B_{190}. (Note that this paper is even before Ramanujan's collected works and both of Chowla's papers!).

    (2) The last one is wrong and correct entry should be B_{3292} and that this was noticed by Wells Johnson and Bernd Kellner.

    (3) We might modify the conjecture to 'there are infinitely many n (log log x unto x) such that numerator is prime. For n up to 2000, only n=10, 14, 18, 42 yielded prime numerators.

    (4) As for my question about several primes at one, Larry Washington pointed out a few things, e.g., there are infinitely many p and n such that B_n is irregular for 59 and p (using Montgomery, Metsankyla etc.)

    (20 April 2014)

40. (51) (Infinite elliptic Carmichael) (Update) Thomas Wright (2018) Bulletin LMS 50, 791-800 proves our result unconditionally by removing the need of conjectural bound on least primes in arithmetic progressions.

41. (52) (Binomial congruences 2012) (Corrections) Thm. 4.1, proof third line '-1 and are' should be '-1 and 1 are' (thanks to Javier Diaz-Vargas for pointing out the misprint).

    In Section 9 (A), fourth para., 1st line, exponent $q^s - 1$ should be q^{s-1}. Fifth paragraph, $x - 1$ should be $1 - x$. Seventh paragraph second line, in the expression for G there is an extra F. (Thanks to George Todd for pointing these out. Feb. 2012). Also, there is unfortunate clash of G, the gcd and the expression G defined later.

In Thm. 3.2, 'm and n' should be replaced by 'a and n', inequality on $m_i(d)$ should be dropped and replaced by '$\deg a_i(d) < \deg \wp$' or equivalently, 'Norm $a_i(d) < q^d =$ Norm $\wp$' as mentioned in 3.3. Thanks to Alex Lara for pointing out this typo.

In Remark 7.2, -1 is missing after $(N\wp - 1)!$.

(Updates) For more about the distribution of refined Wilson, see Infinitude of Wieferich paper.

On 15 October 2015, the conjecture of Section 9 (A) was proved by Alex Borisov. I paste his nice argument below.

Suppose $[1]...[s-1]+1$ has a root in F_{q^s}. Then s is a multiple of p.

Proof. Suppose t is that root. Denote $x_i = t^{q^i}$ for $i = 0, 1, ..., (s-1)$.

Consider the Lagrange Interpolation formula for the constant polynomial 1 at the points $x_0, ..., x_{s-1}$:

$$1 = \sum_{i=0}^{s-1} \frac{1}{\prod_{j\neq i}(x_i - x_j)} \prod_{j\neq i}(x - x_j)$$

We are given that $\prod_{j=1}^{s-1}(x_0 - x_j) = (-1)^{s-1}(-1) = (-1)^s$.

Applying Frobenius, for any i we get $\prod_{j\neq i}(x_i - x_j) = ((-1)^s)^{q^i} = (-1)^s$.

Comparing the coefficients for x^{s-1} in the Lagrange Interpolation formula, we get $0 = \sum_{i=0}^{s-1}(-1)^s$. So s is a multiple of p.

42. (53) (SMF paper 2012) In 3.2(2) there is a typo 'and and' and in first paragraph section 9, a typo 'wwhich'.

43. (55) (Valuations paper 2013) The corrections already made in revised online journal version and on my webpage.

    (Thanks to Javier Diaz-Vargas for pointing this out. (October 2013))

44. (56) (Higher diophantine app. II) (Update) In theorem 1, the upper bound for degree $(q+1)$ is actually equality, as explained in Remarks 6.2 of SMF paper 2012 (and as was pointed out to me by Bugeaud whom I thank for this remark)

45. (57) (Differential Characterization) (Corrections) In proof of theorem 2.5, 'by (1)' should be 'by (i) and (ii)' and in Lemma 2.7, the 'r' in the two inequalities should be '$r+2$' . In definition 2.4 (3), $a \in A_\wp$ rather than $a \in A$.

    (Update) On 26 th April 2023 (private conversation), Felipe Voloch gave alternate derivation of 'infinitude of Wilson primes for $F_q[t]$' from my characterization theorem, when q is odd prime greater than 3: Using Thm. 3.75 of Lidl-Niedwriter book characterizing binomial primes: If $q \equiv 1 \mod 4$, $2^m \equiv 1 \mod q$ and a is a generator of F_q^*, then $t^{2^{km}} - a$ are binomial Wilson primes. If $q \equiv 3 \mod 4$, odd prime ℓ divides $q-1$ (exists if $q > 3$ as Fermat primes > 3 are $\equiv 1 \mod 4$, $t^{\ell^{km}} - a$ is binomial Wilson prime. These plus trinomials I provided in paper for $p = 2$ or $q = 3$ provides infinitely many short-length Wilson primes.

46. (58) (Zeta-like) (Corrections): Pa. 792, conjecture 4.4, the word 'primitive' should be dropped.

    (Updates) Remarks(0) after 4.3: After hearing of it at Lyon conference, M. Kaneko (july 16) conducted an extensive numerical search for all multizeta of weight at most 16 and found only the zeta-like classical multizeta values listed in this remark, where one needs to add 'duals' to get the full list of tuples, but those values are the same. (Thanks to M. Kaneko for this, Aug 16). Broadhurst (Sep 16) also tells me that he does not know of any other examples, and a long ago he had done weights up to 22 and that data can be checked.

The conjecture mentioned there due to BBB97 has been proved by Steven Charlton (Arxives) (and also independently by Francis Brown in letter to me (17 Oct. 2014) soon after I mentioned it to him).

Conjecture 4.6 has been proved by Huei Jeng Chen (2015 Arxives). (Sep. 16)

47. (60) (Fermat-Wilson) Pa. 199, second displayed equation, lower index of the sum should be 1 rather than 0.

    Pa. 205, notes added in the proof: Infinitude of non-wieferich primes for $q > 2$ result mentioned there (as due to Bamunoba) is already proved in [2] (Check: or at least its published version).

48. (65) (MZV survey) Theorem 10.1, 3rd line from end, $A[v^n]$ should be $A[\zeta_{v^n}]$. (9 June 18)

    In 5.2 end, associativity and coassociativity in the Hopf algebra mentioned is only conjectural, with some heuristics and experimentally verified.

49. (66) (Leading coefficient) (Update) Second paragraph after Hypothesis 6.4: Precise recipe for 4 digits has been conjectured by Lara Rodriguez. (May 2017).

50. (69) (MZV Banff survey) (Updates) JTNB survey contains more updated references and results.

51. (71??) (Surprising symmetries, Arxive version update/corrections: already made in my webpage version) Improve $q^n - 1$ to $2q^n - q^j - 1$ in conjectureC(iii) (verified a few $q = 8, 16, 32$ values, $q = 4$ corresponds to part (iii). Pa. 4 remark 3 generalized and expanded on. Add $k > 3$ in the last line of the paper (also k 'even' and not divisible by p) and add the word 'minus' before 'the logarithmic derivative' in pa. 4 remark 2. (14 Dec 15) Pa. 3 first $p(1)$ bound improved from 40 to 42. (22Dec. 15).

    (Updates) The conjecture A and parts of B and C have been proved by David Speyer after Terrence Tao put it on polymath. He also found right generalization for odd characteristic. (14 Jan 16).

    To do: update Preprints references to reprints.

The List of math corrections published in later papers

1. GammaOhio91 corrects GammaAnn misprints and references.
2. Hyp1995 end gives corrections for GaussJNT, IMRN93 and GammaOhio.
3. IMRN1998 end corrects GammaAnnals 96 missing ‘noin-trivial’ condition.
4. Dioph1999JNT end corrects ExpJNT98, Patterns JNT signs.
5. Power sums 2009 2.2.3 corrects Book sign misprints in 5.6
6. FermatWilson2015 end misprints in Binomial2012 corrected
7. MZVsurvey 2017 after Thm. 4, shuffle proof corrected from Shuffle IMRN paper.
8. Automata2020 end corrects Book misprints in Automata chapter.
9. Zetalike2021 end corrects Shtuka1993 normalization mistakes and 2017 Hopf algebra dropped ‘conjectural’ (now proved by Dac et al) and a misprint.