

Contents

<i>Preface</i>	vii
1. Number fields and Function fields	1
1.1 Global fields: Basic analogies and contrasts	1
1.2 Genus and Riemann-Roch theorem	9
1.3 Zeta function and class group	13
1.4 Class field theory and Galois group	20
2. Drinfeld modules	31
2.1 Carlitz module and related arithmetical objects	31
2.2 Drinfeld modules: Basic definitions	34
2.3 Torsion points	39
2.4 Analytic theory	41
2.5 Explicit calculations for Carlitz module	44
2.6 Reductions	48
2.7 Endomorphisms	50
2.8 Field of definition	52
2.9 Points on Drinfeld modules	53
2.10 Adjoints and duality	54
2.11 Useful tools in non-archimedean or finite field setting	56
(a) Properties of $k\{\tau\}$	57
(b) Moore determinant	58
(c) q -resultants	58
(d) Non-archimedean calculus	59
(e) Dwork's trace formula	61

3.	Explicit class field theory	63
3.1	Torsion of rank one Drinfeld modules	64
3.2	Sign normalization of the top coefficient	68
3.3	Normalizing Field as a class field	70
3.4	Smallest field of definition as a class field	73
3.5	Ring of definition	74
3.6	Cyclotomic fields	76
3.7	Moduli approach	77
3.8	Summary	79
3.9	Maximal abelian extension	80
3.10	Cyclotomic theory of $\mathbb{F}_q[t]$	82
3.11	Cyclotomic units and conjectures of Brumer and Stark	83
3.12	Some contrasts and open questions	85
4.	Gauss sums and Gamma functions	87
4.1	Gauss and Jacobi sums: Definitions	88
4.2	Gauss and Jacobi sums: $\mathbb{F}_q[t]$ case	91
4.3	Gauss and Jacobi sums: General A	94
4.4	Sign of Gauss sums for $\mathbb{F}_q[t]$	99
4.5	Arithmetic Factorial and Gamma: Definitions	101
(a)	$\mathbb{F}_q[t]$ case	101
(b)	General A	103
4.6	Functional equations in arithmetic case	105
4.7	Special values for arithmetic Γ_∞	109
(a)	Periods: $\mathbb{F}_q[t]$ case	109
(b)	Periods: General A	110
4.8	Special values of arithmetic Γ_v	113
(a)	$\mathbb{F}_q[t]$ case: Analog of Gross-Koblitz .	113
(b)	General A	114
4.9	Geometric Factorial and Gamma: Definitions	116
4.10	Functional equations in geometric case	118
4.11	Special values of geometric Γ and Γ_v : $\mathbb{F}_q[t]$ case	122
4.12	Comparisons and uniform framework	124
4.13	More analogies for $\mathbb{F}_q[t]$: Divisibilities	130
4.14	Binomial coefficients	132
(a)	Binomial coefficients as nice basis .	134
(b)	Difference and differentiation operators	136
4.15	Relations between the two notions of binomials	138

Contents

xiii

4.16 Bernoulli numbers and polynomials	142
4.17 Note on finite differences and q -analogs	149
5. Zeta functions	153
5.1 Zeta values at integers: Definitions	154
5.2 Values at positive integers	158
5.3 Values at non-positive integers	162
5.4 Multiplicities of trivial zeros	166
5.5 Zeta function interpolation on character space	170
(a) ∞ -adic interpolation	170
(b) \wp adic interpolation	173
5.6 Power sums	174
5.7 Zeta measure	176
5.8 Zero distribution	178
5.9 Low values and multi-logarithms	182
5.10 Multizeta values	185
(a) Complex valued multizeta	186
(b) Finite characteristic variants	187
(c) Interpolations	195
5.11 Analytic properties of zeta and Fredholm determinant . . .	197
5.12 Note on classical interpolations	199
6. Higher rank theory	205
6.1 Elliptic modules	205
6.2 Modular forms	209
6.3 Galois representations	213
6.4 DeRham Cohomology	218
(a) Elliptic curves case: Motivation . . .	219
(b) Drinfeld modules case	221
6.5 Hypergeometric functions	226
(a) The first analog	227
(b) The second analog	233
7. Higher dimensions and geometric tools	237
7.1 t -modules and t -motives	239
7.2 Torsion	244
7.3 Purity	244
7.4 Exponential, period lattice and uniformizability	247

7.5 Cohomology realizations	253
7.6 Example: Carlitz-Tate twist $C^{\otimes n}$	254
7.7 Drinfeld dictionary in the simplest case	256
7.8 Krichever/Drinfeld dictionary in more generality	260
 8. Applications to Gauss sums, Gamma and Zeta values	265
8.1 $C^{\otimes n}$ and $\zeta(n)$	266
8.2 Shtuka and Jacobi sums	267
(a) Gauss sums and Theta divisor	269
(b) Examples and applications	269
(c) The case $g = d_\infty = 1$	271
8.3 Another Gamma function	273
(a) Analog of Gross-Koblitz	276
(b) Interpolation at ∞ for new Gamma .	276
8.4 Fermat motives and Solitons	280
8.5 Another approach to solitons	286
8.6 Analog of Gross-Koblitz for Geometric Gamma: $\mathbb{F}_q[t]$ case .	290
8.7 What is known or expected in general case?	290
8.8 Gamma values to Periods connection via solitons: Sketch .	293
8.9 Log-algebraicity, Cyclotomic module and Vandiver conjecture	295
8.10 Explicit Log-Algebraicity formulas	298
 9. Diophantine approximation	303
9.1 Approximation exponents	304
9.2 Good approximations: Continued fractions	306
9.3 Range of exponents : Frobenius	312
9.4 Range of exponents: Differentiation	319
9.5 Connection with deformation theory	322
(a) Height inequalities for algebraic points	323
(b) Exponent hierarchy	324
(c) Approximation by algebraic functions	325
9.6 Note on connection with Diophantine equations	326
 10. Transcendence results	329
10.1 Approximation techniques and irrationality	330
10.2 Transcendence results on Drinfeld modules	331
10.3 Application to Zeta and Gamma values	334
10.4 Transcendence results in higher dimensions	335

Contents

xv

10.5 Application to Zeta and Gamma values	337
11. Automata and algebraicity: Applications	341
11.1 Automata and algebraicity	341
11.2 Some useful automata tools	347
11.3 Applications to transcendence of gamma values and monomials	348
11.4 Applications to transcendence: periods and modular functions	353
11.5 Classifying finite characteristic numbers	355
11.6 Computational classes and basic tools	357
11.7 Algebraic properties of computational classes	360
11.8 Applications to refined transcendence	364
<i>Note on the Notation</i>	371
<i>Bibliography</i>	373