MULTIZETA SHUFFLE RELATIONS FOR FUNCTION FIELDS WITH NON RATIONAL INFINITE PLACE

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Abstract. In contrast to the 'universal' multizeta shuffle relations, when the chosen infinite place of the function field over $\mathbb{F}_q$ is rational, we show that in the non-rational case, only certain interesting shuffle relations survive, and the $\mathbb{F}_q$-linear span of the multizeta values does not form an algebra. This is due to the subtle interactions between the larger finite field $\mathbb{F}_\infty$, the residue field of the completion at infinity where the signs live and $\mathbb{F}_q$, the field of constants where the coefficients live. We study the classification of these special relations which survive.

1. Introduction

In 1775, more than 30 years after Euler introduced the zeta values $\zeta(s)$, he introduced and studied the multizeta values $\zeta(s_1, \cdots, s_r)$. They have resurfaced with renewed interest because of their connections with several other parts (see e.g., introduction of [T2009] for references) of mathematics, for example, in the Grothendieck-Ihara program to study the absolute Galois group of $\mathbb{Q}$ through the algebraic fundamental group of the projective line minus three points. Thus the understanding of the structure of relations between them is quite important. The simple trichotomy $n_i > n_{i+1}$ or $n_i = n_{i+1}$ or $n_i < n_{i+1}$ applied to the sum definition of multizeta (which is over such ordered tuples of natural numbers) shows (the so-called sum shuffle relations) that the product of two multizeta values is a linear combination of multizeta values, and thus the $\mathbb{Q}$-span of the multizeta values is an algebra.

While the trichotomy approach and these sum shuffle relations completely fail for the multizeta values of [T2004, Section 5.10], [T2009, T2010, AT2009] for function fields over $\mathbb{F}_q$, it was shown in [T2010] that a different mechanism leads to a different kind of combinatorially involved shuffle identities (Theorem 3.1.1), which are 'universal' in the sense that they work for any function field together with a rational (i.e., degree one) place at infinity corresponding to the ring of integers A. In particular, the $\mathbb{F}_q$-span of all multizeta values is an algebra in this case.

We focus on this aspect, but point out in passing that, as in the classical case, these multizeta values also have connections with the absolute Galois group through the analogue of Ihara power series, in a work by G. Anderson with the second author, and with the periods of Carlitz-Tate-Anderson mixed t-motives [A1986, AT2009], at least in the simplest case $A = \mathbb{F}_q[t]$.

Date: March 12, 2015.

2010 *Mathematics Subject Classification.* 11M32, 11M38, 11R58, 12Y05, 11T55.

Key words and phrases. Multizeta, Drinfeld modules, t-motives, periods.

The authors were supported in part by PROMEP grant F-PROMEP-36/Rev-03 SEP-23- 006 and NSA grant H98230-14-1-0162 respectively.

In this paper, we look at the shuffle relations in the case where the place at infinity is not rational, so that there are more choices of signs (in the finite residue field of the completion) than those available in the finite residue field of the function field. This leads to two different natural approaches to define multizeta, and we show that in each approach, certain kind of (different for the two approaches) fundamental relations (Theorems 3.3.1, 3.4.1) survive! In Section 4, we discuss the results and conjectures on the classification of these relations, and in the last section we briefly mention higher depth situation.

The interesting form of the surviving relations as well as the numerical experimentation, admittedly quite limited, which suggests that these might be the only ones which survive, make us wonder if there is any deeper reason behind this.

2. Notation, Background and Definitions

$\mathbb{Z}$	$\{$integers$\}$
$\mathbb{Z}_+$	$\{$positive integers$\}$
q	a power of a prime p
$\mathbb{F}_q$	a finite field of q elements
K	a function field of one variable with field of constants $\mathbb{F}_q$
∞	a place of K of any degree
d_∞	the degree of ∞
K_∞	the completion of K at ∞
$\mathbb{F}_\infty$	the residue field of K_∞
A	the ring of elements of K with no poles outside ∞
A_+	'monics' or 'positives' in A, to be defined below
A_d	elements of A of degree d
$A_{<d}$	elements of A of degree less than d
A_{d+}	$A_d \cap A_+$
$A_{<d+}$	$A_{<d} \cap A_+$
$[n]$	$= t^{q^n} - t$
ℓ_n	$= \prod_{i=1}^{n} (t - t^{q^i})$
"even"	multiple of $q-1$

For $A = \mathbb{F}_q[t]$, with a chosen t (or equivalently a sign), Carlitz considered the set of 'monic' polynomials A_+ in t as replacement of $\mathbb{Z}_+$ and investigated the (Carlitz) zeta values $\zeta_A(s) = \sum_{a \in A_+} a^{-s} \in K_\infty$ for $s \in \mathbb{Z}_+$, in a parallel way to the (Euler-Riemann) zeta values $\zeta_{\mathbb{Z}}(s) = \sum_{a \in \mathbb{Z}_+} a^{-s}$. Note that while $\mathbb{Z}_+$ is closed under multiplication and addition, A_+ is only closed under multiplication.

Note $K_\infty := \mathbb{F}_\infty((u))$ and that $\mathbb{F}_\infty$ is a finite field with q^{d_∞} elements. From the expansion of non-zero element $x \in K_\infty$ in a Laurent series in (fixed generator) u, we define the degree $\deg x = -d_\infty v_\infty(x)$ as usual, so that for a in A, $\deg a = \dim_{\mathbb{F}_q}(A/aA)$, and we define the sign $\operatorname{sgn}(x) \in \mathbb{F}_\infty^*$ to be the leading coefficient in the expansion of x.

If $d_\infty = 1$, the elements with sign 1 are called *monic* or *positive*. When the degree $d_\infty > 1$, there are two somewhat natural naive approaches generalizing this.

In the *approach 1*, the monic elements are elements of some fixed sign $\theta \in \mathbb{F}_\infty^*$. We put $S = \{\theta\}$ in this case. (We will take it to be 1 in practice, so that monics are multiplicatively closed).

In the *approach 2*, we can let $S \subset \mathbb{F}_\infty^*$ be a fixed set of representatives of $\mathbb{F}_\infty^*/\mathbb{F}_q^*$ and define a monic element to be an element of sign in S. (When $q = 2$, all signs are then monic, and monics are multiplicatively closed, but in general, it is not possible to choose S to be multiplicatively closed).

Let S be a subset of $\mathbb{F}_\infty^*$ chosen in either of the two ways explained above. Let A_+ be the set of 'monic' elements, that is,

$$A_+ := \{a \in A : \mathrm{sgn}(a) \in S\}.$$

For $s \in \mathbb{Z}_+$ and $d \in \mathbb{Z}_{\geq 0}$, put

$$S_d(s) := \sum_{a \in A_{d+}} \frac{1}{a^s} \in K, \quad \zeta(s) := \sum_{d=0}^{\infty} S_d(s) \in K_\infty.$$

Given integers $s_i \in \mathbb{Z}_+$ and $d \geq 0$, put

$$S_d(s_1, \ldots, s_r) := S_d(s_1) \sum_{d > d_2 > \cdots > d_r \geq 0} S_{d_2}(s_2) \cdots S_{d_r}(s_r) \in K.$$

In particular, with an obvious extension of notation, we have $S_d(s_1, s_2) = S_d(s_1) S_{<d}(s_2)$.

For $s_i \in \mathbb{Z}_+$, define the multizeta value $\zeta(s_1, \ldots, s_r)$ by using the partial order on A_+ given by the degree, and grouping the terms according to it:

$$\zeta(s_1, \ldots, s_r) := \sum_{d \geq 0} S_d(s_1, \cdots s_r) = \sum_d \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in K_\infty,$$

where the second sum is over all $a_i \in A_+$ of degree d_i such that $d = d_1 > \cdots > d_r \geq 0$. We say that this multizeta value, or rather the tuple $(s_1, \ldots, s_r)$, has depth r and weight $\sum s_i$.

We focus on depth 2. For $a, b \in \mathbb{Z}_+$, we define

$$\Delta_d(a, b) := S_d(a) S_d(b) - S_d(a + b), \quad \Delta(a, b) := \Delta_{d_\infty}(a, b).$$

Remarks 2.0.1.

(1) Since $\Delta_d(a, b) = \Delta_d(b, a)$, we can assume without loss of generality, when needed, that $a \leq b$.

(2) Since we are in characteristic p, we have

$$S_d(sp^n) = S_d(s)^{p^n}, \quad S_d(s_1 p^n, s_2 p^n) = S_d(s_1, s_2)^{p^n}.$$

So we can often restrict to s_i's not all divisible by p without loss of generality. We call such tuples *primitive*.

(3) If we choose the usual place at infinity, for $K = \mathbb{F}_q(t)$, with uniformizer $1/t$ and the usual definition of monic, then the ring of integers A is $\mathbb{F}_q[t]$ and the zeta values coincide with the Carlitz zeta values.

(4) When $d_\infty > 1$, we have used [T2004] the second approach while dealing with the gamma or zeta values.

3. Shuffle relations in depth 2

3.1. **When the place at infinity is rational.** First we recall [T2010] the shuffle relations result when $d_\infty = 1$.

Theorem 3.1.1. *(1) If for $A = \mathbb{F}_q[t]$ there exist $f_i \in \mathbb{F}_p$ and $a_i \in \mathbb{Z}_+$ such that*

$$\Delta_d(a,b) = \sum f_i S_d(a_i, a+b-a_i) \tag{1}$$

holds for $d = 1$, then for any A with a rational place at infinity, (1) *holds for all $d \geq 0$.*

(2) For $\mathbb{F}_q[t]$, given $a, b \in \mathbb{Z}_+$, there always exist $f_i \in \mathbb{F}_p$ and $a_i \in \mathbb{Z}_+$, so that (1) *holds for $d = 1$.*

3.2. **Differences when the place at infinity is not rational.** Calculation (using open-source SAGE software) of several examples, when $d_\infty > 1$ trying to express Δ_d as linear combination of S_d's suggested that only certain relations survive. We prove those relations below.

The numerical evidence suggests that independently of which convention for signs we take, there exist a, b such that $\zeta(a)\zeta(b)$ can not be written as a $\mathbb{F}_q$-linear combination of multizetas. We first examine differences at the finite level.

Example 3.2.1. Let $K = \mathbb{F}_2(t)$, let ∞ be the only place of K of degree two. Then $K_\infty = \mathbb{F}_\infty((u))$, with $\mathbb{F}_\infty = \{0, 1, \theta, \theta + 1\}$ and $\theta^2 + \theta + 1 = 0$.

Let $a = 1$ and $b = 2$. For $\mathbb{F}_2[t]$, we have $\Delta_d(a, b) = S_d(b, a)$ for all $d \geq 0$.

(i) With singleton $S = \{\theta\}$ of $\mathbb{F}_\infty^*$, we have that $\Delta_2(a, b) \neq 0$ and $S_2(a, b) = S_2(b, a) = 0$. Therefore, $\Delta_d(a, b)$ is not a $\mathbb{F}_2$-linear combination of $S_d(a_i, b_i)$'s, with weight $a_i + b_i = 3$, when $d = d_\infty = 2$.

(ii) If we take $S = \{1\}$, then $\Delta_2(a, b) = S_2(b, a)$ but $\Delta_d(a, b) \neq S_d(b, a)$ for $d = 2d_\infty, 3d_\infty, 4d_\infty$. Thus, $\Delta_{d_\infty}(a, b) = S_{d_\infty}(b, a)$ does not imply $\Delta_d(a, b) = S_d(b, a)$ for all $d \geq 0$.

(iii) With $S = \mathbb{F}_\infty^*$, $\Delta(a, b) = 0$. None of the $\mathbb{F}_2$-linear combinations of $S_d(1, 2)$ and $S_d(2, 1)$ matches $\Delta_d(a, b)$, when $d = 2d_\infty, 3d_\infty, 4d_\infty$.

For $S = \mathbb{F}_\infty^*$ and (a, b) any of the pairs $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(1, 6)$, $(1, 7)$, $(1, 8)$, $(2, 5)$, $(2, 7)$, $(3, 4)$, $(3, 5)$, $(3, 6)$, $(3, 7)$, $(3, 8)$, $(4, 5)$, $(5, 6)$, $(5, 7)$, $(5, 8)$, it is possible to write $\Delta(a, b)$ as a linear combination of $S_{d_\infty}(a_i, b_i)$'s, but the corresponding linear combination does not hold when we take $d = 2d_\infty = 4$.

(I) $(a, b) = (1, 3)$. Then, $\Delta(1, 3) = 0$, but there are no $f_1, f_2, f_3 \in \mathbb{F}_2$ such that $\Delta_d(1, 3) = f_1 S_d(1, 3) + f_2 S_d(2, 2) + f_3 S_d(3, 1)$ when $d = 4, 6, 8$. On the other hand, for $\mathbb{F}_2[t]$, $\Delta_d(a, b) = S_d(2, 2) + S_d(3, 1)$ holds for all $d \geq 0$.

(II) The equality $\Delta_d(1, 4) = S_d(2, 3) + S_d(3, 2)$ holds for $d = d_\infty$, but not for $d = 2d_\infty, 3d_\infty, 4d_\infty$. Furthermore, when $d = 2d_\infty$, $\Delta_d(1, 4)$ is not a $\mathbb{F}_2$-linear combination of $S_d(a_i, b_i)$'s of weight 5.

Example 3.2.2 (S is a set of representatives of $\mathbb{F}_\infty^*/\mathbb{F}_q^*$)**.** Let $K = \mathbb{F}_3(t)$, let P_∞ the degree two place corresponding to $t^2 + 1$, so that $K = \mathbb{F}_\infty((u))$, with $\mathbb{F}_\infty = \mathbb{F}_3(i)$, where $i^2 = -1$. Let $S = \{1, 2i, i + 1, i + 2\}$. It is not possible to write $\Delta_d(a, b)$ as a linear combination of $S_d(1, a + b - 1), \ldots, S_d(a + b - 1, 1)$ for $d = d_\infty = 2$ when (a, b) is one of the following: $(1, 1)$, $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(1, 6)$, $(1, 7)$, $(1, 8)$, $(2, 3)$, $(2, 5)$, $(3, 4)$, $(3, 5)$, $(4, 4)$. If $(a, b) = (1, 3)$, no linear combination of $S_d(1, 3), S_d(2, 2), S_d(3, 1)$ matches $\Delta_d(a, b)$, $d = \nu d_\infty = 2, 4, 6$.

Example 3.2.3. Next we give an example (one in each approach) showing that the product of zetas needs not to be a sum of multizetas, respecting weights or not (i.e., the $\mathbb{F}_p$-span of the multizetas is not an algebra). In fact, we can use the first examples above, such as $K = \mathbb{F}_2(t)$ and the degree 2 prime at infinity.

Then with the approach 2, i.e., with the full set of signs, to see that $\zeta(1)\zeta(2)$ is not linear combination of $\zeta(3), \zeta(1,2), \zeta(2,1), \zeta(1,1,1)$, it is enough to note that $\zeta(1)\zeta(2) - \zeta(3)$ has the maximum agreement as Laurent series in u with linear combination $\zeta(2,1) + \zeta(1,2)$, but they differ in u^{33} place. (This corresponds to $\Delta_2(1,2) = 0$, but $\Delta_4(1,2)$ having valuation 33 as mentioned above). Ignoring all the multizeta values with valuation at infinity more than 33, we have checked that no other combination works, even if we ignore weights. But we omit the details.

The same can be said with approach 1, using just one sign θ, when $\zeta(1)\zeta(2)-\zeta(3)$ has valuation 4, but valuations of $\zeta(2,1), \zeta(1,2), \zeta(1,1,1)$ are $22, 14, 54$ respectively, and even without weight restriction, no combination is possible.

3.3. **Situation in approach 1 to the signs.**

Theorem 3.3.1. *Consider 'monics' and multizeta defined by approach 1. If $a, b \in \mathbb{Z}_+$ are such that*

$$\Delta_d(a,b) = S_d(a)S_d(b) - S_d(a+b) = 0 \tag{2}$$

holds for $\mathbb{F}_q[t]$ and $d = 1$, then, (2) holds for all $d \geq 0$ for any A with ∞ of any degree. In this case, we have the 'classical sum shuffle' identity

$$\zeta(a)\zeta(b) = \zeta(a+b) + \zeta(a,b) + \zeta(b,a). \tag{3}$$

Proof. (Compare with the first part of the proof of Theorem 2 in [T2010].)

First note that if d_∞ does not divide d, then A_{d+} is empty (but $A_{<d+}$ could be non empty) and thus $\Delta_d(a,b) = 0$. Note that it is possible that A_{d+} be non empty but $A_{<d+} = \emptyset$.

Consider $n, n' \in A_{d+}$, $m, m' \in A_{<d}$. Define

$$S_{n,m} := \{(n + \theta m, n + \mu m) \colon \theta, \mu \in \mathbb{F}_q, \theta \neq \mu\},$$
$$n \sim_m n' \leftrightarrow n' = n + \theta m, \text{ for some } \theta \in \mathbb{F}_q.$$

Then, the sets $S_{n,m}$ are equal or disjoint with $S_{n',m'}$, they are the equivalence classes of $\sim$ and partition the set $\{(n_1, n_2) \colon n_i \in A_{d+}, n_1 \neq n_2\}$. Fix $d > 0$ divisible by d_∞. Writing $t = n/m$, we have

$$\begin{aligned}
\sum_{(n_1,n_2)\in S_{n,m}} \frac{1}{n_1^a n_2^b} &= \sum_{\theta\neq\mu\in\mathbb{F}_q} \frac{1}{(n+\theta m)^a (n+\mu m)^b} \\
&= \frac{1}{m^{a+b}} \sum_{\theta\neq\mu\in\mathbb{F}_q} \frac{1}{(t+\theta)^a (t+\mu)^b} \\
&= \frac{1}{m^{a+b}}(0) \\
&= 0.
\end{aligned}$$

The third equality follows from (2). By summing over the partition, we obtain $\Delta_d(a,b) = 0$. Since also $\Delta_d(a,b) = 0$ for $d = 0$, the theorem follows. □

3.4. **Situation in approach 2 to the signs.**

Theorem 3.4.1. *Consider 'monics' and multizeta defined by approach 2. If $a, b \in \mathbb{Z}_+$ are such that*

$$\Delta_d(a,b) = -S_d(a,b) - S_d(b,a) \tag{4}$$

holds for $\mathbb{F}_q[t]$ *and* $d = 1$*, then,* (4) *holds for all* $d \geq 0$ *and for any* A *with* ∞ *of any degree. In this case, we have the 'zeta product' identity*

$$\zeta(a)\zeta(b) = \zeta(a+b). \tag{5}$$

Proof. Consider $n, n' \in A_{d+}$, $m, m' \in A_{<d+}$, such that $\mathrm{sgn}(n) = \mathrm{sgn}(n')$. Define

$$S_{n,m} := \{(n+\theta m, n+\mu m) \colon \theta, \mu \in \mathbb{F}_q, \theta \neq \mu\},$$
$$S'_{n,m} := \{(n+\theta m, m) \colon \theta \in \mathbb{F}_q\},$$
$$n \sim_m n' \leftrightarrow n' = n + \theta m, \text{ for some } \theta \in \mathbb{F}_q.$$

Since $\deg m < d$ and $\deg n = d$, we have $\mathrm{sgn}(n+\theta m) = \mathrm{sgn}(n)$. It follows that the sets $S_{n,m}$'s (equal or disjoint as n, m vary), which are the equivalence classes of $\sim_m$'s, partition the set $\{(n_1, n_2) \colon n_i \in A_{d+}, n_1 \neq n_2, \mathrm{sgn}(n_1) = \mathrm{sgn}(n_2)\}$. On the other hand, the sets $S'_{n,m}$ partition the set $\{(n_1, m_1) \colon n_1 \in A_{d+}, m_1 \in A_{<d+}\}$.

Note that $\Delta_d(a,b) = \mathcal{S}_1 + \mathcal{S}_2$, where

$$\mathcal{S}_1 = \sum_{\substack{n_i \in A_{d+} \\ \mathrm{sgn}(n_1) \neq \mathrm{sgn}(n_2)}} \frac{1}{n_1^a n_2^b}, \qquad \mathcal{S}_2 = \sum_{\substack{n_1 \neq n_2, n_i \in A_{d+} \\ \mathrm{sgn}(n_1) = \mathrm{sgn}(n_2)}} \frac{1}{n_1^a n_2^b}.$$

We claim that $\mathcal{S}_1 = 0$.

Let $d > 0$ divisible by d_∞. Letting $t = n/m$, we have

$$\begin{aligned}
\sum_{(n_1,n_2) \in S_{n,m}} \frac{1}{n_1^a n_2^b} &= \sum_{\theta \neq \mu \in \mathbb{F}_q} \frac{1}{(n+\theta m)^a (n+\mu m)^b} \\
&= \frac{1}{m^{a+b}} \sum_{\theta \neq \mu \in \mathbb{F}_q} \frac{1}{(t+\theta)^a (t+\mu)^b} \\
&= \frac{1}{m^{a+b}} \left(-\sum_{\theta \in \mathbb{F}_q} \frac{1}{(t+\theta)^a} - \sum_{\theta \in \mathbb{F}_q} \frac{1}{(t+\theta)^b} \right) \\
&= -\sum_{\theta \in \mathbb{F}_q} \frac{1}{(n+\theta m)^a m^b} - \sum_{\theta \in \mathbb{F}_q} \frac{1}{(n+\theta m)^b m^a} \\
&= -\sum_{(x,y) \in S'_{n,m}} \frac{1}{x^a y^b} - \sum_{(x,y) \in S'_{n,m}} \frac{1}{x^b y^a}.
\end{aligned}$$

The third equality results from the hypothesis. By summing over the partition, we obtain $\mathcal{S}_2 = -S_d(a,b) - S_d(b,a)$. Since $\Delta_d(a,b) = 0$ and $S_d(a,b) = S_d(b,a) = 0$ for d not dividing d_∞, we get the relation we want, assuming the claim.

Note that this part of the proof is parallel to proof of Theorem 2 in [T2010], except that we restrict to the same sign part. The argument so far has not used any particular form of the relation in the hypothesis. But now having proved the relation

$$\sum_{\theta \neq \mu \in \mathbb{F}_q} \frac{1}{(n+\theta m)^a (n+\mu m)^b} + \sum_{\theta \in \mathbb{F}_q} \frac{1}{(n+\theta m)^a m^b} + \sum_{\theta \in \mathbb{F}_q} \frac{1}{(n+\theta m)^b m^a} = 0,$$

formally without using restrictions on n, m, we reuse it by replacing n, m by n_1, n_2 of the same degree, but of different signs. Our claim that $\mathcal{S}_1 = 0$ follows by summing over the resulting relations.

In more details, first note that by [L2012, Theorem 5.1] or [L2011, Corollary 6.4] (4) implies that both a, b are "even" (unless $a = b$ and $p = 2$ when the right side of the equation vanishes, but in this case, for the same reason our claimed relation works!). So $(q-1)^2\mathcal{S}_1$ gives the same sum as $\mathcal{S}_1$, except we replace A_{d+} in the condition by A_d, making the choice of representatives irrelevant. Now the sets $\{(n_1 + \theta n_2, n_1 + \mu n_2), (n_1 + \theta n_2, n_2), (n_2, n_1 + \theta n_2) : \theta \neq \mu \in \mathbb{F}_q\}$, when both components of the tuples are allowed to be multiplied by elements in $\mathbb{F}_q^*$, correspond to sets containing $(q^2+q)(q-1)^2 = (q^2-1)(q^2-q)$ elements representing non-zero vector in two dimensional $\mathbb{F}_q$-vector space by two fixed such n_1, n_2 together with another $\mathbb{F}_q$-linearly independent vector (meaning different sign). They partition exactly the set over which we sum, giving the result. □

4. Tuples satisfying the special relations

The numerical evidence with small parameters data calculated suggests that for each approach, there are no other relations of the type 'the product of two zeta values equals linear combination of multizetas with $\mathbb{F}_p$-coefficients' than the ones given in the corresponding Theorems above that survive for infinite places of higher degrees.

We now try to understand all tuples which satisfy these classical sum shuffle or zeta product relations. It is enough to restrict to primitive tuples (a, b) with $a \leq b$.

First observe that if $p = 2$, $\zeta(a)^2 = \zeta(2a) = \zeta(2a) + 2\zeta(a, a)$, so that (a, a) tuples satisfy both special relations!

4.1. **Tuples satisfying classical sum shuffle.** By Theorem 8 in [T2009a], we know that for $\mathbb{F}_2[t]$, $\Delta_d(a, b) = 0$ if and only if $a = b$.

For general q, we have [T2009, Theorem 1, p. 2324] a non-exhaustive list of such tuples in general. We now try to extend it.

The following formulas, which are consequence of Theorems 1 and 3 in [LT2015], will be used in the proof of the theorems in this and next section.

(1) For $1 \leq s < q$ and $0 \leq k_i < k$ with $1 \leq i \leq s$, we have

$$S_d(q^k - \sum_{i=1}^{s} q^{k_i}) = \ell_d^{(s-1)q^k} S_d(q^k - q^{k_1}) \cdots S_d(q^k - q^{k_s}). \tag{6}$$

(2) For $1 \leq s \leq q$, and any $0 \leq k_i \leq k$, with $1 \leq i \leq s$, we have

$$S_{<d}(\sum_{i=1}^{s}(q^k - q^{k_i})) = \prod_{i=1}^{s} S_{<d}(q^k - q^{k_i}). \tag{7}$$

We also recall Carlitz' evaluations (see e.g., [T2009, 3.3.1, 3.3.2])

$$S_d(a) = 1/\ell_d^a, \quad (a \leq q) \tag{8}$$

$$S_d(q^j - 1) = \ell_{d+j-1}/\ell_{j-1}\ell_d^{q^j} \tag{9}$$

$$S_{<d}(q^j - 1) = \ell_{d+j-1}/\ell_j\ell_{d-1}^{q^j}, \tag{10}$$

Theorem 4.1.1. *Let q be any prime power. Consider the following cases.*
(1) $a + b \leq q$.

(2) $n \geq 0$, $a = s_1 q^n$ *and* $b = q^{n+1} - s_2 q^n - 1$, *with* $1 \leq s_1 \leq s_2 \leq q-2$.
(3) $n \geq 1$, $1 \leq s \leq q-1$, *and* $a = q^n - s$, $b = q^n - (q-s)$.

If $a, b \in \mathbb{Z}_+$ *satisfy either (1) or (2), then for* $A = \mathbb{F}_q[t]$ *we have the classical sum shuffle*

$$\zeta(a)\zeta(b) = \zeta(a+b) + \zeta(a,b) + \zeta(b,a).$$

Proof. (1) Follows [T2004, Thm. 5.10.6] from equation 8. By Theorem 3.1.1, in each case, it is enough to prove $\Delta(a,b) = 0$. Now, since $s_2 - s_1 + 1 \leq q-2 < q$ and $s_2 + 1 \leq q - 1 < q$, the formula 6 can be applied; keeping in mind that $S_d(q^{n+1} - q^n) = \ell_d^{q^n} / \ell_d^{q^{n+1}}$, by a straight calculation we obtain

$$S_d(q^{n+1} - (s_2 - s_1)q^n - 1) = S_d(s_1 q^n) S_d(q^{n+1} - s_2 q^n - 1),$$

and (2) is proved. In a similar way, a straight calculation shows that $S_d(q^n - s)S_d(q^n - (q-s))$ and $S_d(2q^n - q)$ are equal, proving the claim (3). □

Remark 4.1.2. The numerical evidence suggests that Theorem 4.1.1 completely characterizes the primitive tuples (a,b) with $a \leq b$, such that (3) holds, when $q = 3$.

Numerical evidence suggests the following conjectures.

Conjecture 4.1.3. *(i) Let* q *be any prime power and* $n \geq 0$. *If* $a = a_{s_a+1} q^{s_a+1} + \cdots + a_n q^n$, $b = b_{s_b+1} q^{s_b+1} + \cdots + b_n q^n$, *with* $-1 \leq s_a, s_b \leq n-1$, $1 \leq a_i, b_j \leq q-1$,

$$q + j(q-1) \leq a_n + a_{n-1} + \cdots + a_{n-j-1}, \qquad (0 \leq j \leq n - s_a - 2),$$
$$q + j(q-1) \leq b_n + b_{n-1} + \cdots + b_{n-j-1}, \qquad (0 \leq j \leq n - s_b - 2),$$

and

$$a_n + b_n \leq \begin{cases} q & \text{if } s_a = s_b = n-1, \\ q-2 & \text{if } s_a, s_b < n-1, \\ q-1 & \text{otherwise} \end{cases}$$

then $\zeta(a)\zeta(b) = \zeta(a+b) + \zeta(a,b) + \zeta(b,a)$ *holds for* $\mathbb{F}_q[t]$.

(ii) Let q *be a prime. If* (a,b) *satisfies the classical sum shuffle above, then the base* q *expansions of* a *and* b *have the same length.*

Remarks 4.1.4.

(i) By Theorem 1 [T2009, p. 2324], it follows that if $a = a_n q^n$ and $b = b_n q^n$, then the classical sum shuffle identity holds if $a_n + b_n \leq q$. Therefore, the part of Conjecture 4.1.3 (i) corresponding to $s_a = s_b = n-1$ is actually proved.

(ii) The numerical evidence suggests that when q is 3 or 5, Theorem 4.1.1 together with Conjecture 4.1.3 (i) characterize primitive tuples (a,b) with $a \leq b$ such that the classical sum shuffle holds. For $q = 7$, the tuple $(8, 10)$ is neither accounted by Theorem 4.1.1 nor by Conjecture 4.1.3.

(iii) Conjecture 4.1.3 (ii) does not generalize to non-prime q. For example, take $q = 4$ with $(a,b) = (2,4), (2,5)$ or $q = 9$, with $(a,b) = (3,9), (3,10)$.

4.2. **Tuples satisfying zeta product relation.** Since $S_1(k) = S_{<2}(k) - 1 = S_{\leq 1}(k) - 1$, a straight calculation shows that the condition is equivalent to

$$\Delta(a,b) = -S_1(a) - S_1(b) \Leftrightarrow S_{\leq 1}(a+b) = S_{\leq 1}(a)S_{\leq 1}(b). \tag{11}$$

As mentioned above, for tuples satisfying this relation, a, b must be "even". Next, we try to understand them.

Theorem 4.2.1. *Let $q = 2$. If $a = b$, or $a = 2^{\beta+1} - 2^{\alpha}$, $b = 2^{\beta+1} - 1$, for $0 \leq \alpha \leq \beta$, then* (4) *holds for $\mathbb{F}_2[t]$, and therefore $\zeta(a)\zeta(b) = \zeta(a+b)$.*

Proof. It is enough to prove $\Delta_1(a,b) = S_1(a) + S_1(b)$.

If $a = b$, by [T2009a, Thm. 8], $S_1(a)S_1(b) = S_1(a+b)$, so that $S_1(a)S_1(b) = S_1(a+b) + S_1(a) + S_1(b)$.

For $a = 2^{\beta+1} - 2^{\alpha}$ and $b = 2^{\beta+1} - 1$, we apply [L2011, Corollary 7.2]. We have

$$a - 1 = 1 + 2 + \cdots + 2^{\alpha-1} + 0 \cdot 2^{\alpha} + 2^{\alpha+1} + \cdots + 2^{\beta}.$$

Since $2^{\beta+2} - b = 1 + 2^{\beta+1}$ and $2^{\beta+2} - a = 2^{\alpha} + 2^{\beta+1}$, from Lucas theorem, we conclude that $\binom{2^{\beta+2}-b}{i}$ and $\binom{2^{\beta+2}-a}{j}$ are nonzero for $i = 0, 1$ and $j = 0, 2^{\alpha}$. Hence,

$$\Delta_1(a,b) = S_1(a) + S_1(a-1) + S_1(b) + S_1(b - 2^{\alpha}).$$

Since $a - 1 = b - 2^{\alpha}$, the theorem follows. □

Remark 4.2.2. The numerical evidence suggests that for $\mathbb{F}_2[t]$ the primitive tuples (a, b) with $a \leq b$ satisfying relation (4) are characterized by Theorem 4.2.1.

We have the following generalization.

Theorem 4.2.3. *Let q be any prime power. Consider the following cases.*

(1) Let $n \geq 0$, $0 \leq k_i < n$, $1 \leq s \leq q$. Let $s_1, s_2 \geq 1$ such that $s = s_1 + s_2$ and let $a = \sum_{i=1}^{s_1}(q^n - q^{k_i})$, $b = \sum_{i=s_1+1}^{s_2}(q^n - q^{k_i})$.

(2) Let

$$a = s(q^n - q^{k_1}) + (q - s)(q^n - q^{k_2}),$$
$$b = (q - s)(q^n - q^{k_1}) + s(q^n - q^{k_2}),$$

where $n \geq 0$, $1 \leq s \leq q$, $0 \leq k_1, k_2 < n$.

If $a, b \in \mathbb{Z}_+$ satisfy either (1) or (2), then (4) *holds for $\mathbb{F}_q[t]$, and therefore $\zeta(a)\zeta(b) = \zeta(a+b)$.*

Proof. (1) By Theorem 3.1.1, it is enough to prove that $\Delta(a,b) = -S_1(a) - S_1(b)$; or equivalently to prove that $S_{<2}(a+b) = S_{<2}(a)S_{<2}(b)$, but the latter follows immediately from (7).

(2) By (7), we have

$$\begin{aligned} S_{<2}(a)S_{<2}(b) &= S_{<d}(q^n - q^{k_1})^s S_{<2}(q^n - q^{k_2})^{q-s} S_{<2}(q^n - q^{k_1})^{q-s} S_{<2}(q^n - q^{k_2})^s \\ &= S_{<2}(q^n - q^{k_1})^q S_{<2}(q^n - q^{k_2})^q \\ &= S_{<2}(a+b). \end{aligned}$$

□

Numerical evidence suggests the following conjectures.

Conjecture 4.2.4. *(i) Let* $a = (q-1)\sum_0^m a_i q^i$, $b = (q-1)\sum_0^m b_i q^i$, *with* $a_i, b_i < q$ *monotonically increasing and* $a_m + b_m \leq q$. *Then* (4) *holds for* $\mathbb{F}_q[t]$, *and therefore* $\zeta(a)\zeta(b) = \zeta(a+b)$.

(ii) Let q *be a prime. If* (a, b) *satisfies the zeta product relation above, then* $a/(q-1)$ *and* $b/(q-1)$ *have the base* q *expansion of the same length and the sum of digits base* q *of* $a-1$ *and of* $b-1$ *is the same.*

Remarks 4.2.5. When q is 3 or 5, Conjecture 4.2.4 (i) and Theorem 4.2.3 account for all the primitive tuples (a, b), with $a \leq b$, that we know for which (4) holds. For $q = 7$, $(54, 66)$ seems to be the only tuple not accounted for, in the range $1 \leq a \leq 250, a \leq b \leq 300$. Conjecture 4.2.4 (ii) does not generalize to non-prime q. For example, both parts fail for $q = 4$ and $(a, b) = (6, 12)$ or $6, 15)$.

5. Higher depth

We have not yet investigated higher depth situation much numerically or otherwise, so will be content to list a couple of almost formal consequences of the depth 2 situation, omitting the details.

5.1. **Approach 1.** The stuffle product of two tuples $\underline{a} = (a_1, \ldots, a_{r_1})$ and $\underline{b} = (b_1, \ldots, b_{r_2})$ of positive integers, denoted by $\mathrm{st}(\underline{a}, \underline{b})$, is the union of all the tuples $\underline{c} = (c_1, \ldots, c_r)$ obtained by inserting, in any required position, some 0 in the string $(a_1, \ldots, a_{r_1})$ as well as in the string $(b_1, \ldots, b_{r_2})$ (this may be made even before the first term or after the last one), so that the two new strings have the same length r, with $\max\{r_1, r_2\} \leq r \leq r_1 + r_2$, and by adding the two sequences term by term. The 0's are inserted so that no c_i is zero [W2000].

Theorem 5.1.1. *Let* $\underline{a} = (a_1, \ldots, a_{r_1}) \in \mathbb{Z}_+^{r_1}$ *and* $\underline{b} = (b_1, \ldots, b_{r_2}) \in \mathbb{Z}_+^{r_2}$ *such that* $\Delta(a_i, b_j) = 0$ *for* $1 \leq i \leq r_1$, $1 \leq j \leq r_2$ *holds for* $\mathbb{F}_q[t]$, *then for any* A *with the place at infinity of any degree and using sign convention A1, we have*

$$\zeta(\underline{a})\zeta(\underline{b}) = \sum_{c \in \mathrm{st}(\underline{a}, \underline{b})} \zeta(\underline{c}).$$

Proof. (Sketch) Follows from Theorem 3.3.1 and straightforward calculations. □

5.2. **Approach 2.**

Theorem 5.2.1. *Let* q *be general. Let* $a_1, a_2, b \in \mathbb{Z}_+$ *such that* $\Delta(a_1, a_2) = -S_1(a_1) - S_1(a_2)$ *and* $\Delta(a_1, b) = -S_1(a_1) - S_1(b)$ *hold for* $\mathbb{F}_q[t]$, *then for any* A *with the place at infinity of any degree and using sign convention A2, we have*

$$\zeta(a_1, a_2)\zeta(b) = \zeta(b, a_1, a_2) + \zeta(a_1 + b, a_2) - \zeta(b, a_1 + a_2).$$

Proof. (Sketch) This follows by straight manipulations using Theorem 3.4.1. □

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